University of Ferrara

DIPARTIMENTO DI ECONOMIA E MANAGEMENT

Stefano Bonnini & Valentina Mini

Matrix algebra for multivariate problems

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Matrix algebra: linearly independent vectors

Vectors $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k$ are called **linearly dependent**

if there exist numbers $\lambda_1,\ldots, \lambda_K$ not all zero such that

$$\lambda_1\boldsymbol{x}_1+\ldots+\lambda_K\boldsymbol{x}_k=\mathbf{0}.$$

Otherwise the *k* vectors are linearly independent.

Matrix algebra: linearly independent vectors

Let $\boldsymbol{W}$ be a subspace of $\boldsymbol{R}^n$.

Then a basis of $\boldsymbol{W}$ is a maximal linearly independent set of vectors.

Every basis of $\boldsymbol{W}$ contains the same (finite) number of elements.

This number is the dimension of $\boldsymbol{W}$.

If $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k$ is a basis for $\boldsymbol{W}$ then every element $\boldsymbol{x}$ in $\boldsymbol{W}$ can be expressed as a linear combination of $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k$.

Matrix algebra: linearly independent vectors

Example:

The dimension of **W**=$\boldsymbol{R}^3$ is 3.

A basis for $\boldsymbol{R}^3$ is $\boldsymbol{x}_1=(1,0,0)'$, $\boldsymbol{x}_2=(0,1,0)'$ and $\boldsymbol{x}_3=(0,0,1)'$.

As a matter of fact $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ are linearly independent and every vector

$\boldsymbol{a}=(a_1,a_2,a_3)'$ can be expressed as linear combination of

$\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$: $\boldsymbol{a}= a_1\boldsymbol{x}_1 + a_2\boldsymbol{x}_2 + a_3\boldsymbol{x}_3$

Matrix algebra: the rank

The **rank** of a $n \times k$ matrix $\boldsymbol{A}$ is defined as the maximum number of linearly independent columns (rows) in $\boldsymbol{A}$.

The following properties hold for the rank of $\boldsymbol{A}$, denoted with $r(\boldsymbol{A})$:

1. $r(\boldsymbol{A})$ is the largest order of those submatrices of $\boldsymbol{A}$ with non null determinants.
2. $0 \leq r(\boldsymbol{A}) \leq min(n,k)$
3. $r(\boldsymbol{A})=r(\boldsymbol{A}')$
4. $r(\boldsymbol{A}'\boldsymbol{A})= r(\boldsymbol{A}\boldsymbol{A}')= r(\boldsymbol{A})$
5. If $n=k$ then $r(\boldsymbol{A})=k$ if and only if A is non-singular

Example:

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 3 \\ -2 & 2 & 5 \\ 1 & -1 & 9 \end{pmatrix} \quad \det(\mathbf{A}) = 0 \quad \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} = 0 \quad \begin{vmatrix} -1 & 3 \\ 2 & 5 \end{vmatrix} = -11 \neq 0$$

Thus the r($\boldsymbol{A}$)=2

Matrix algebra: eigenvalue and eigenvector

If ***A*** is a square matrix of order *n*, in some problems we are interested in finding a vector x and a scalar λ which satisfy the following property:

eigenvalue

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \Rightarrow (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

eigenvector

A trivial solution is $\boldsymbol{x}=\boldsymbol{0}$, any $\lambda \in \boldsymbol{R}$

Matrix algebra: the eigenvalues

The n ***eigenvalues*** of ***A*** $\lambda_1,\ldots,\lambda_n$ are the n solutions of the characteristic equation

$$|\mathbf{A} - \lambda \mathbf{I}| = \mathbf{0}$$

Properties of the eigenvalues of ***A***:

1. $|\boldsymbol{A}| = \Pi_i \lambda_i$
2. $tr(\boldsymbol{A}) = \Sigma_i \lambda_i$
3. $r(\boldsymbol{A})$ equals the number of non-zero eigenvalues
4. The set of all eigenvectors for an eigenvalue λ_i is called the eigenspace of ***A*** for λ_i
5. Any symmetric $n \times n$ matrix ***A*** can be written as $\boldsymbol{A} = \Gamma \Lambda \Gamma' = \Sigma_i \lambda_i \gamma_{(i)} \gamma_{(i)}'$ where Λ is a diagonal matrix of eigenvalues of ***A*** and Γ is an orthogonal matrix whose columns are eigenvectors with $\gamma_{(i)}' \gamma_{(i)} = 1$

Matrix algebra: examples

Example:

$$\mathbf{A} = \begin{pmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}$$

The characteristic equation is: $\Rightarrow$

$$\begin{vmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$$

By computing the determinant we have:

$$(2-\lambda)\cdot\begin{vmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{vmatrix} - 1\cdot\begin{vmatrix} -2 & 3 \\ 3 & -1-\lambda \end{vmatrix} + 1\cdot\begin{vmatrix} -2 & 3 \\ 1-\lambda & 1 \end{vmatrix} =$$

$$= (2-\lambda)\left[-1+\lambda^2-3\right] - 2 - 2\lambda + 9 - 2 - 3 + 3\lambda = (\lambda+2)(\lambda-1)(3-\lambda) = 0$$

The solutions represent the 3 eigenvalues of ***A***:

$$\lambda_1 = 1 \qquad \lambda_2 = -2 \qquad \lambda_3 = 3$$

Matrix algebra: the dominant eigenvalue

The eigenvalue with maximum absolute value $\lambda_3=3$ is called dominant

There is an infinite number of eigenvectors ***x*** which satisfy (***A***-3***I***)***x***= ***0***

$$\begin{pmatrix} -1 & -2 & 3 \\ 1 & -2 & 1 \\ 1 & 3 & -4 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Lab using R: matrix

Please, open R and
follow the professor's instruction