FORMULE DI POISSON

Sia $x(t)$ un segnale a durata limitata $\mathcal{C}$-tratti, per es. $x(t)=0$ se $|t|>T_o/2$

$(T_o>0)$

Esso è dunque F trasformabile con trasformata

$$X(f)=\int_{-T_o/2}^{T_o/2} e^{-j2\pi ft}x(t)\,dt$$

A partire da esso costruiamo il segnale di periodo T_0

$$y(t) = \sum_{k=-\infty}^{+\infty} x(t - kT_o)$$

i cui coefficienti di Fourier sono:

$$Y_k = \frac{1}{T_o} \int_{-T_o/2}^{T_o/2} e^{-j2k\pi f_o t} y(t)\,dt =$$

$$f_o = 1/T_o$$

$$= \frac{1}{T_o} \int_{-T_o/2}^{T_o/2} e^{-j2\pi \frac{k}{T_o} t} x(t)\,dt$$

confrontando :

$$X(f) = \int_{-T_o/2}^{T_o/2} e^{-j2\pi ft} x(t)\,dt$$

$$Y_k = \frac{1}{T_o} \int_{-T_o/2}^{T_o/2} e^{-j2\pi \frac{k}{T_o} t} x(t)\,dt$$

si deduce che:

$$Y_k = \frac{1}{T_o} X\left(\frac{k}{T_o}\right)$$

Dunque:
i coefficienti di Fourier di un segnale $y(t)$ periodico di periodo T_o e $\mathcal{C}$-tratti ottenuto a partire dal segnale “base” $x(t)$, $x(t)=0$ per $|t|>T_0/2$

$$y(t)=\sum_{k=-\infty}^{+\infty} x(t-kT_o)$$

possono essere ottenuti dalla trasformata di Fourier $X(f)$ di $x(t)$ ponendo:

$$Y_k=\frac{1}{T_o}X\left(\frac{k}{T_o}\right)$$

Se $y(t)$ è $\mathcal{C}^1$-tratti e normalizzato, per il Teorema di Dirichlet vale:

$$y(t) = \sum_{k=-\infty}^{+\infty} Y_k e^{j2\pi \frac{k}{T_o} t}$$

e dunque:

$$\sum_{k=-\infty}^{+\infty} x(t - kT_o) = \frac{1}{T_o} \sum_{k=-\infty}^{+\infty} X\left(\frac{k}{T_o}\right) e^{j2\pi \frac{k}{T_o} t}$$

1ª formula (sommatoria) di Poisson

o di *campionamento in frequenza*.

$$\sum_{k=-\infty}^{+\infty} x\left(t-kT_o\right)=\frac{1}{T_o}\sum_{k=-\infty}^{+\infty} X\left(\frac{k}{T_o}\right)e^{j2\pi\frac{k}{T_o}t}$$

Applicando la proprietà di dualità:

$$\sum_{k=-\infty}^{+\infty} X\left(t-kT_o\right)=\frac{1}{T_o}\sum_{k=-\infty}^{+\infty} x\left(-\frac{k}{T_o}\right)e^{j2\pi\frac{k}{T_o}t}$$

e scambiando, al secondo membro, k con $-k$

$$\sum_{k=-\infty}^{+\infty} X\left(t-kT_o\right)=\frac{1}{T_o}\sum_{k=-\infty}^{+\infty} x\left(\frac{k}{T_o}\right)e^{-j2\pi\frac{k}{T_o}t}$$

$$\sum_{k=-\infty}^{+\infty} X\left(t-kT_o\right)=\frac{1}{T_o}\sum_{k=-\infty}^{+\infty} x\left(\frac{k}{T_o}\right)e^{-j2\pi\frac{k}{T_o}t}$$

Poiché $T_0 = 1/f_0$ e scambiando t con f :

$$\sum_{k=-\infty}^{+\infty} X\left(f-\frac{k}{f_0}\right)= f_0\sum_{k=-\infty}^{+\infty} x\left(f_0 k\right)e^{-j2\pi k f_0 t}$$

2ª formula (sommatoria) di Poisson
o di *campionamento nei tempi*.

Esempio 27.

Il segnale

$$y(t) = \sum_{k=-\infty}^{+\infty} \mathrm{rect}\left(\frac{t - kT_o}{T_0 / 2} \right)$$

(vedi Esercizio 14) è $\mathcal{C}$-tratti, normalizzato, periodico di periodo T_o

E' ottenuto dal segnale "base"

$$x(t) = \mathrm{rect}\left(\frac{t}{T_0 / 2} \right)$$

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che è $\mathcal{C}$-tratti e tale che $x(t) = 0$ per $|t| > \frac{T_0}{4}$

Poiché

$$\mathrm{rect}\left(\frac{t}{T_0/2}\right) \xrightarrow{\mathcal{F}} \frac{T_0}{2}\mathrm{sinc}\left(f\frac{T_0}{2}\right)$$

possiamo dedurre che

$$Y_k = \frac{1}{T_o} X\left(\frac{k}{T_o}\right) = \frac{1}{\cancel{T_o}} \frac{\cancel{T_0}}{2}\mathrm{sinc}\left(\frac{k}{\cancel{T_o}}\frac{\cancel{T_0}}{2}\right) = \frac{1}{2}\mathrm{sinc}\left(\frac{k}{2}\right)$$

Cosicché l'equazione di sintesi

$$y(t) = \sum_{k=-\infty}^{+\infty} Y_k e^{j2k\pi f_0 t}$$

diventa:

$$\sum_{k=-\infty}^{+\infty} \mathrm{rect}\left(\frac{t-kT_0}{T_0/2}\right) = \sum_{k=-\infty}^{+\infty} \frac{1}{2}\mathrm{sinc}\left(\frac{k}{2}\right) e^{j2\pi\frac{k}{T_o}t}$$

Poiché

$$\sum_{k=-\infty}^{+\infty} \frac{1}{2}\operatorname{sinc}\left(\frac{k}{2}\right) e^{j2k\pi f_0 t} = \underbrace{\frac{1}{2}\operatorname{sinc}\left(\frac{0}{2}\right)}_{=1} +$$

$$+\sum_{k=1}^{+\infty} \frac{1}{2}\operatorname{sinc}\left(\frac{-k}{2}\right) e^{-j2k\pi f_0 t} + \sum_{k=1}^{+\infty} \frac{1}{2}\operatorname{sinc}\left(\frac{k}{2}\right) e^{j2k\pi f_0 t} =$$

$$= \frac{1}{2} + \sum_{k=1}^{+\infty} \operatorname{sinc}\left(\frac{k}{2}\right) \underbrace{\left(\frac{e^{j2k\pi f_0 t} + e^{-j2k\pi f_0 t}}{2}\right)}_{=\cos(2k\pi f_0 t)}$$

(la funzione sinc è pari)

si ottiene ancora (vedi Esempio 14):

$$\sum_{k=-\infty}^{+\infty} \mathrm{rect}\left(\frac{t-kT_0}{T_0/2}\right) = \frac{1}{2} + \sum_{k=1}^{+\infty} \mathrm{sinc}\left(\frac{k}{2}\right)\cos\left(2k\pi f_o t\right)$$

Esempio 27bis.

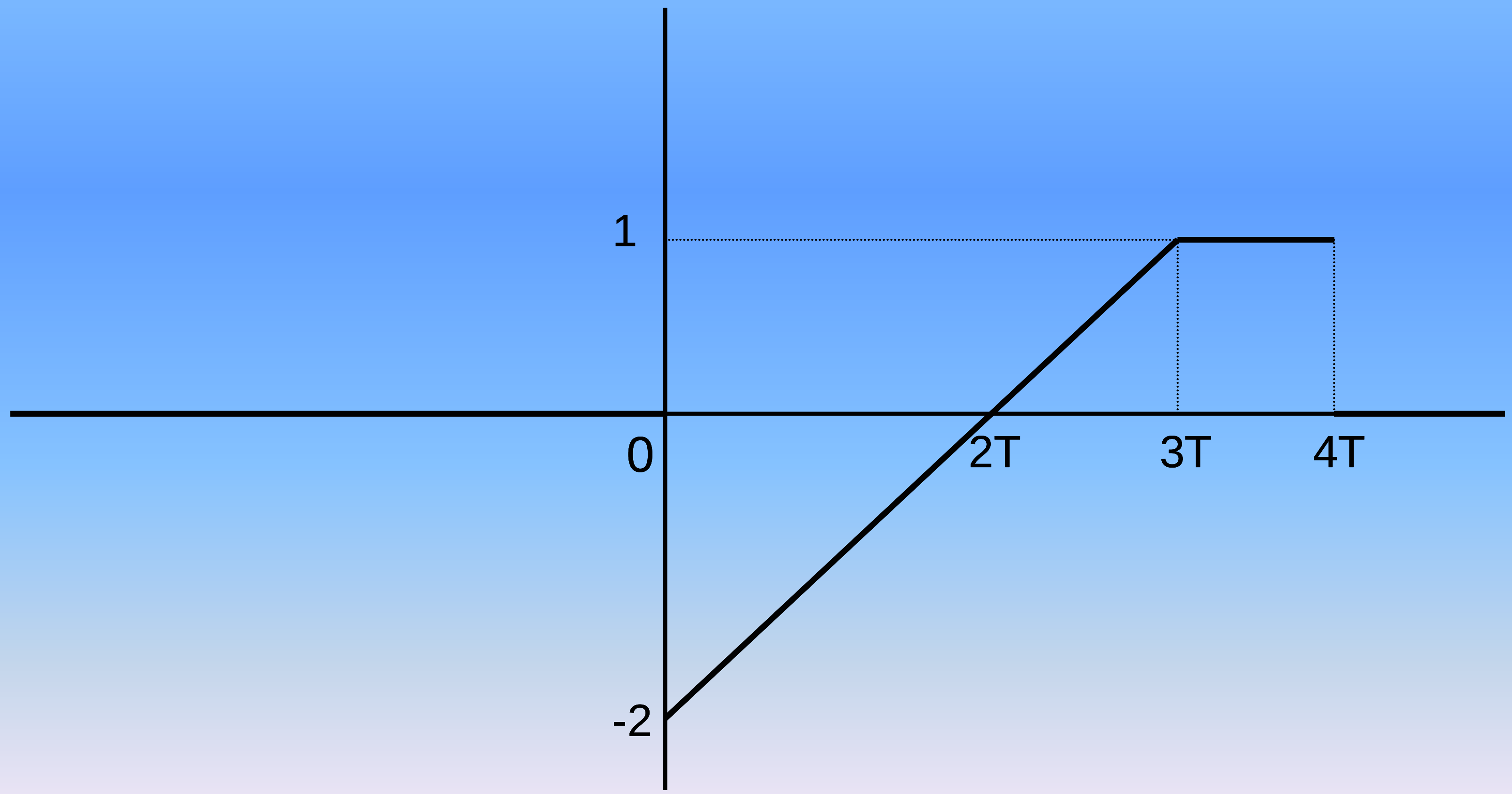
Sia

$$y(t)=\sum_{k=-\infty}^{+\infty} x(t-4kT)$$

Il segnale ottenuto ripetendo con periodicità 4T il segnale “base”:

$$x(t)=\left[1-3\ triang\left(\frac{t}{3T}\right)\right]rect\left(\frac{t-2T}{4T}\right)$$

$$x(t) = \left[1 - 3\,triang\left(\frac{t}{3T}\right)\right] rect\left(\frac{t-2T}{4T}\right)$$



$x(t)$ è $\mathcal{C}^1$-tratti e a durata limitata 4T.

Indicata con $X(f)$ la sua trasformata di Fourier, l'equazione di sintesi di $y(t)$ è

$$y(t) = \sum_{k=-\infty}^{+\infty} Y_k e^{j2k\pi f_0 t}$$

con $f_0 = \frac{1}{4T}$ e $Y_k = \frac{1}{4T} X\left(\frac{k}{4T}\right)$

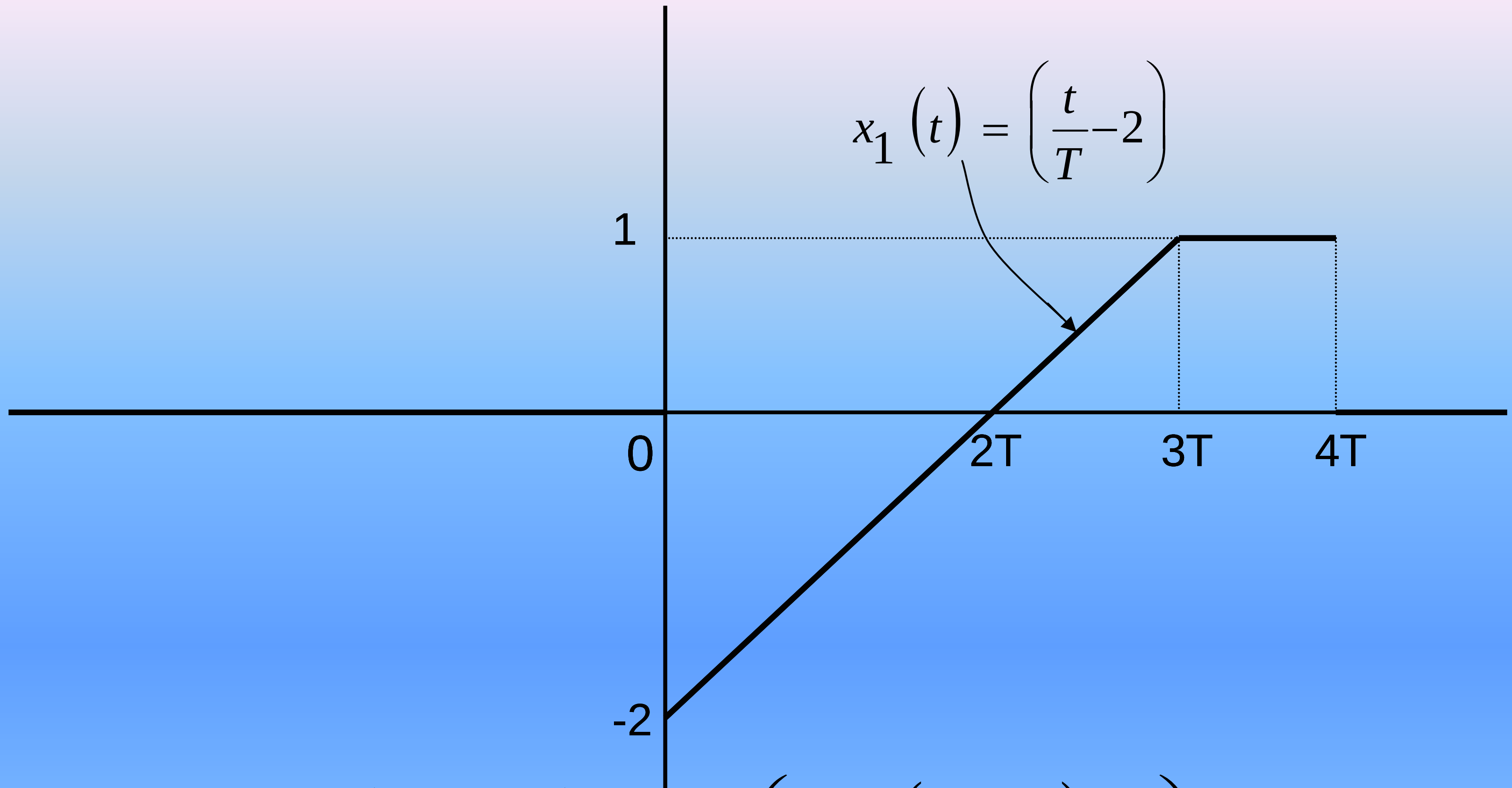
La trasformata di Fourier di $x(t)$ può essere determinata osservando che:

$$x(t) = \left[1 - 3\ triang\left(\frac{t}{3T}\right)\right] rect\left(\frac{t-2T}{4T}\right)$$

può anche essere scritto:

$$x(t) = \left(\frac{t}{T} - 2\right) rect\left(\frac{t-(3/2)T}{3T}\right) +$$

$$+ rect\left(\frac{t-(7/2)T}{T}\right)$$



$$x(t) = \left(\frac{t}{T} - 2\right) rect\left(\frac{t-(3/2)T}{3T}\right) +$$

$$+ rect\left(\frac{t-(7/2)T}{T}\right)$$

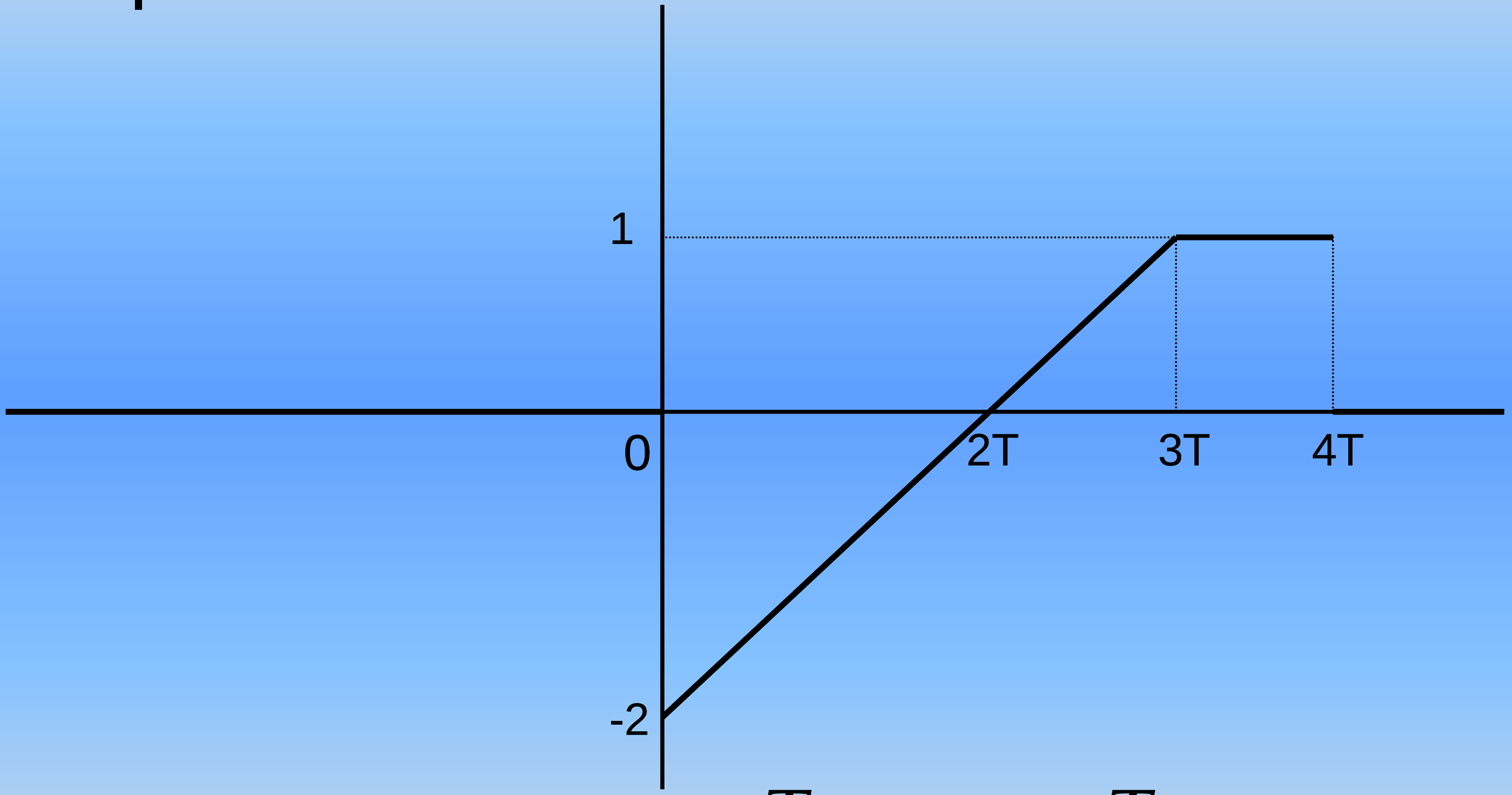
$$x(t)=\left(\frac{t}{T}-2\right)rect\left(\frac{t-(3/2)T}{3T}\right)+$$

$$+rect\left(\frac{t-(7/2)T}{T}\right)$$

cosicché

$$X(f)=\int_0^{3T} e^{-j2\pi ft}\left(\frac{t}{T}-2\right)dt+T\,\mathrm{sinc}(fT)\,e^{-j7\pi fT}$$

Considerazioni di tipo geometrico permettono di concludere subito che:



$$X(0) = -2T + \frac{T}{2} + T = -\frac{T}{2}$$

Se $f \neq 0$, integrando per parti,

$$\int_0^{3T} e^{-j2\pi ft}\left(\frac{t}{T}-2\right)dt =$$

$$=\left[\frac{e^{-j2\pi ft}}{-j2\pi f}\left(\frac{t}{T}-2\right)\right]_0^{3T} - \int_0^{3T}\frac{e^{-j2\pi ft}}{-j2\pi f}\frac{1}{T}dt =$$

$$=-\frac{1}{j2\pi f}\left\{\left(e^{-j6\pi fT}+2\right)-\frac{1}{T}\left[\frac{e^{-j2\pi ft}}{-j2\pi f}\right]_0^{3T}\right\}$$

$$\int_0^{3T} e^{-j2\pi ft}\left(\frac{t}{T}-2\right)dt =$$

$$= -\frac{1}{j2\pi f}\left\{\left(e^{-j6\pi fT}+2\right)-\frac{1}{T}\left[\frac{e^{-j2\pi ft}}{-j2\pi f}\right]_0^{3T}\right\} =$$

$$= -\frac{1}{j2\pi f}\left\{e^{-j6\pi fT}+2+\frac{1}{Tj2\pi f}\left(e^{-j6\pi fT}-1\right)\right\} =$$

$$= -\frac{1}{j2\pi f}\left\{e^{-j6\pi fT}+2-\frac{e^{-j3\pi fT}}{T\pi f}\left(\frac{e^{j3\pi fT}-e^{-j3\pi fT}}{2j}\right)\right\} =$$

$$\int_0^{3T} e^{-j2\pi ft}\left(\frac{t}{T}-2\right)dt =$$

$$= -\frac{1}{j2\pi f}\left\{e^{-j6\pi fT}+2-\frac{e^{-j3\pi fT}}{T\pi f}\left(\frac{e^{j3\pi fT}-e^{-j3\pi fT}}{2j}\right)\right\} =$$

$$= -\frac{1}{j2\pi f}\left\{e^{-j6\pi fT}+2-e^{-j3\pi fT}\,\frac{3\sin\left(3\pi fT\right)}{3\pi fT}\right\} =$$

$$= -\frac{1}{j2\pi f}\left\{e^{-j6\pi fT}+2-3e^{-j3\pi fT}\operatorname{sinc}\left(3fT\right)\right\}$$

Ne segue:

$$X(f) = \int_0^{3T} e^{-j2\pi ft}\left(\frac{t}{T} - 2\right)dt + T\,\mathrm{sinc}(fT)\,e^{-j7\pi fT} =$$

$$= -\frac{1}{j2\pi f}\left\{e^{-j6\pi fT} + 2 - 3e^{-j3\pi fT}\mathrm{sinc}(3fT)\right\} +$$

$$+T\,\mathrm{sinc}(fT)\,e^{-j7\pi fT}$$

Osservando poi che:

$$T\,\mathrm{sinc}(fT)e^{-j7\pi fT} = Te^{-j7\pi fT}\frac{\sin(\pi fT)}{\pi fT} =$$

$$= e^{-j7\pi fT}\frac{1}{\pi f}\left(\frac{e^{j\pi fT} - e^{-j\pi fT}}{2j}\right) =$$

$$= \frac{1}{j2\pi f}\left(e^{-j6\pi fT} - e^{-j8\pi fT}\right)$$

risulta:

$$X(f) = -\frac{1}{j2\pi f}\left\{e^{-j6\pi fT} + 2 - 3e^{-j3\pi fT}\operatorname{sinc}(3fT)\right\} +$$

$$+T\operatorname{sinc}(fT)e^{-j7\pi fT} =$$

$$= -\frac{1}{j2\pi f}\left\{\cancel{e^{-j6\pi fT}} + 2 - 3e^{-j3\pi fT}\operatorname{sinc}(3fT)\right\} +$$

$$+\frac{1}{j2\pi f}\left(\cancel{e^{-j6\pi fT}} - e^{-j8\pi fT}\right) =$$

$$= \frac{1}{j2\pi f}\left\{-2 + 3e^{-j3\pi fT}\operatorname{sinc}(3fT) - e^{-j8\pi fT}\right\}$$

Dunque, se $f = 0$, $X(0) = -\frac{T}{2}$

se $f \neq 0$,

$$X(f) = \frac{1}{j2\pi f}\left\{-2 + 3e^{-j3\pi fT}\operatorname{sinc}(3fT) - e^{-j8\pi fT}\right\}$$

Ne seguono:

$$Y_0 = \frac{1}{4T}X(0) = -\frac{1}{4\cancel{T}}\frac{\cancel{T}}{2} = -\frac{1}{8}$$

mentre se $k \neq 0$,essendo

$$X(f) = \frac{1}{j2\pi f}\left\{-2 + 3e^{-j3\pi fT}\,\mathrm{sinc}(3fT) - e^{-j8\pi fT}\right\}$$

$$f \neq 0$$

$$Y_k = \frac{1}{4T} X\left(\frac{k}{4T}\right) =$$

$$= \frac{1}{\cancel{4T}} \frac{1}{j2\pi \frac{k}{\cancel{4T}}}\left\{-2 + 3e^{-j3\pi \frac{k}{4\cancel{T}}\cancel{T}}\,\mathrm{sinc}\left(3\frac{k}{4\cancel{T}}\cancel{T}\right) + \right.$$

$$\left. - e^{-j8\pi \frac{k}{4\cancel{T}}\cancel{T}} \right\}$$

$$k \neq 0$$

$$Y_k = \frac{1}{j2\pi k}\left\{-2+3e^{-j\frac{3}{4}k\pi}\operatorname{sinc}\left(\frac{3}{4}k\right)-e^{-j2k\pi}\right\}=$$

$$=\frac{3}{j2\pi k}\left\{-1+e^{-j\frac{3}{4}k\pi}\operatorname{sinc}\left(\frac{3}{4}k\right)\right\}$$

$$e^{-j2k\pi}=1$$

In particolare, per $k=4$

$$Y_4=\frac{3}{j8\pi}\left\{-1+e^{-j3\pi}\operatorname{sinc}(3)\right\}=-\frac{3}{j8\pi}=j\frac{3}{8\pi}$$

$$\operatorname{sinc}(3)=0$$

Cosicché l'equazione di sintesi

$$y(t) = \sum_{k=-\infty}^{+\infty} Y_k e^{j2k\pi f_0 t}$$

diventa:

$$y(t) = \sum_{k=-\infty}^{+\infty} Y_k e^{j2k\pi\frac{1}{4T}t} = \sum_{k=-\infty}^{+\infty} Y_k e^{j\frac{k}{2T}\pi t} =$$

$$= -\frac{1}{8} + \sum_{k=1}^{+\infty} Y_{-k} e^{-j\frac{k}{2T}\pi t} + \sum_{k=1}^{+\infty} Y_k e^{j\frac{k}{2T}\pi t}$$

$$Y_0 = -\frac{1}{8}$$

$$Y_k = \frac{3}{j2\pi k}\left\{-1 + e^{-j\frac{3}{4}k\pi}\,\mathrm{sinc}\left(\frac{3}{4}k\right)\right\} \qquad k \neq 0$$

$$y(t) = -\frac{1}{8} + \sum_{k=1}^{+\infty} Y_{-k} e^{-j\frac{k}{2T}\pi t} + \sum_{k=1}^{+\infty} Y_k e^{j\frac{k}{2T}\pi t} =$$

$$= -\frac{1}{8} - \sum_{k=1}^{+\infty} \frac{3}{j2\pi k}\left\{-1 + e^{j\frac{3}{4}k\pi}\,\mathrm{sinc}\left(-\frac{3}{4}k\right)\right\} e^{-j\frac{k}{2T}\pi t} +$$

$$+ \sum_{k=1}^{+\infty} \frac{3}{j2\pi k}\left\{-1 + e^{-j\frac{3}{4}k\pi}\,\mathrm{sinc}\left(\frac{3}{4}k\right)\right\} e^{j\frac{k}{2T}\pi t}$$

sinc è pari

$$y(t)=-\frac{1}{8}-\frac{3}{2\pi j}\sum_{k=1}^{+\infty}\frac{1}{k}\left\{-1+e^{j\frac{3}{4}k\pi}\operatorname{sinc}\left(\frac{3}{4}k\right)\right\}e^{-j\frac{k}{2T}\pi t}+$$

$$+\frac{3}{2\pi j}\sum_{k=1}^{+\infty}\frac{1}{k}\left\{-1+e^{-j\frac{3}{4}k\pi}\operatorname{sinc}\left(\frac{3}{4}k\right)\right\}e^{j\frac{k}{2T}\pi t}=$$

$$=-\frac{1}{8}+\frac{3}{2\pi j}\sum_{k=1}^{+\infty}\frac{1}{k}\left(e^{-j\frac{k}{2T}\pi t}-e^{j\frac{k}{2T}\pi t}\right)+$$

$$+\frac{3}{2\pi j}\sum_{k=1}^{+\infty}\frac{1}{k}\operatorname{sinc}\left(\frac{3}{4}k\right)\left(e^{-j\frac{3}{4}k\pi}e^{j\frac{k}{2T}\pi t}-e^{j\frac{3}{4}k\pi}e^{-j\frac{k}{2T}\pi t}\right)$$

$$y(t)=-\frac{1}{8}+\frac{3}{\cancel{2j}\pi}\sum_{k=1}^{+\infty}\frac{\cancel{2j}}{k}\left(\frac{e^{-j\frac{k}{2T}\pi t}-e^{j\frac{k}{2T}\pi t}}{2j}\right)+$$

$$+\frac{3}{\cancel{2j}\pi}\sum_{k=1}^{+\infty}\frac{\cancel{2j}}{k}\operatorname{sinc}\left(\frac{3}{4}k\right)\left(\frac{e^{j\left(\frac{k}{2T}\pi t-\frac{3}{4}k\pi\right)}-e^{-j\left(\frac{k}{2T}\pi t-\frac{3}{4}k\pi\right)}}{2j}\right)=$$

$$=-\frac{1}{8}+$$

$$-\frac{3}{\pi}\sum_{k=1}^{+\infty}\frac{1}{k}\left[\sin\left(\frac{k}{2T}\pi t\right)-\operatorname{sinc}\left(\frac{3}{4}k\right)\sin\left(\frac{k}{2T}\pi t-\frac{3}{4}k\pi\right)\right]$$