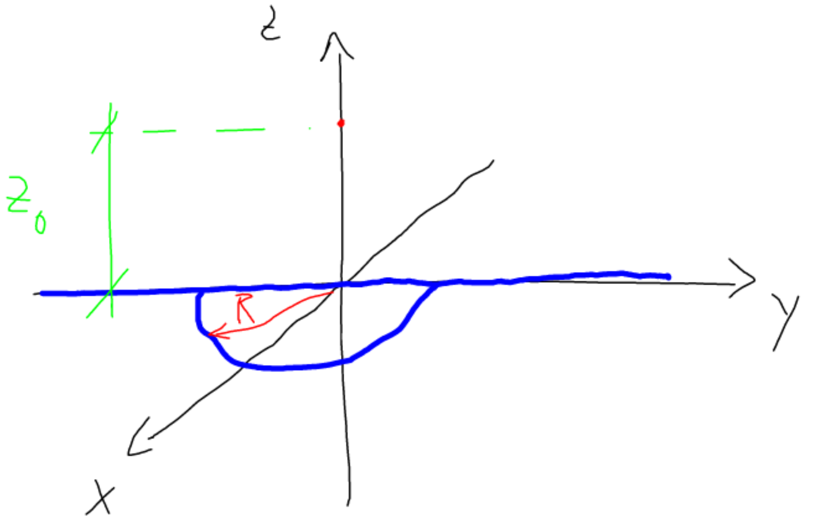
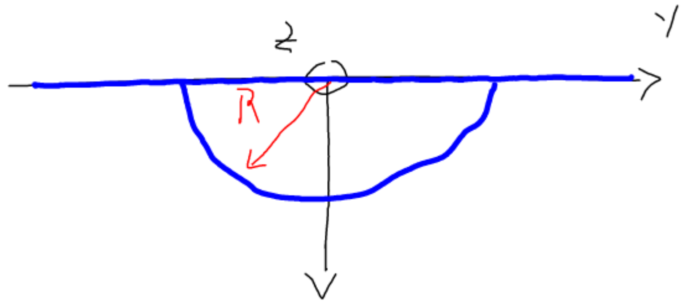
A thin conductive wire of infinite lenght is charged with a linear charge distribution lambda = 1 uC/cm and lays on the Y axis.
An half ring of negligible thickness and radius R = 5 cm is soldered to the wire is depicted in figure. The origin of the XYZ reference frame is centered in the center of the ring (see figure) .

i) Calculate the electric field at a distance Z0 from the XY plane (suggestion : use the superposition principle)

ii) A negative charge of 1uC is then placed at the center of the ring. Calculate the electric field in the same position Z0.

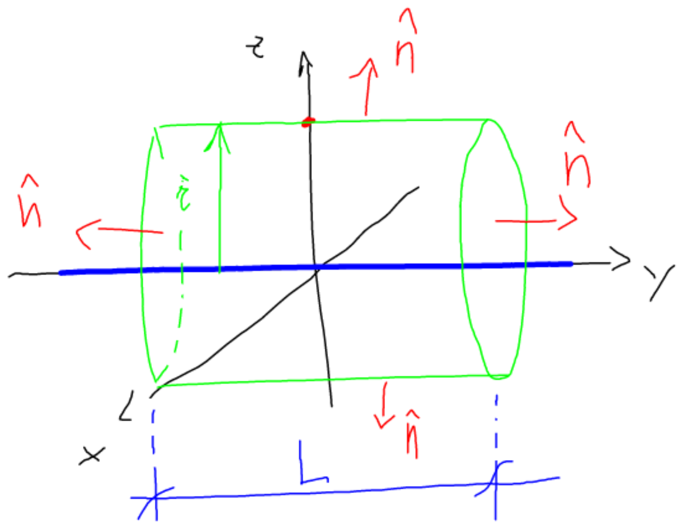
iii) Calculate the electric potential in the same position Z0 and explain the result.





I use the superposition principle to calculate the total electric field. The electric field generated by the wire is carried out by the Gauss law (cylindrical symmetry) , and the contribution of the half ring by direct integration.

Let's start from the electric field generated by the wire



$$|\bar{r}| = z_0$$

$$\phi_E = \iint_{S_g} \bar{E} \cdot \hat{n} \, ds$$

$$\phi_E = \iint_{B_1} \underbrace{\cancel{\bar{E} \cdot \hat{n}}}_{\bar{E} \perp \hat{n}} \, ds + \iint_{B_2} \underbrace{\cancel{\bar{E} \cdot \hat{n}}}_{\bar{E} \perp \hat{n}} \, ds + \iint_{S_L} \bar{E} \cdot \hat{n} \, ds$$

$$\phi_E = \iint_{S_L} E(r)\, ds = E(r) \iint_{S_L} ds = E(r)\, 2\pi r L$$

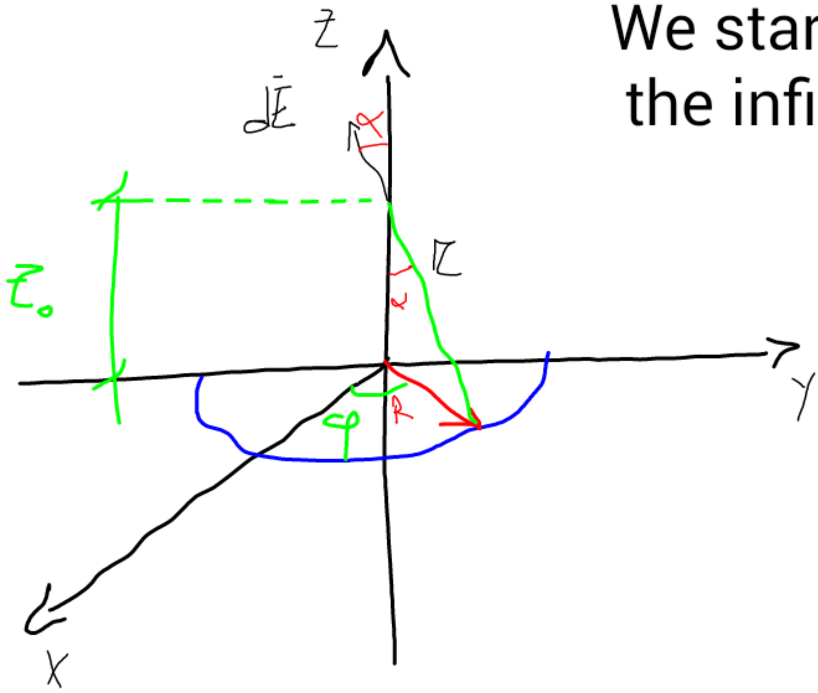
$$= \frac{Q_{enc}}{\varepsilon_0} \quad \text{GAUSS LAW}$$

$$= \frac{\lambda \cdot L}{\varepsilon_0} \Rightarrow E\, 2\pi r \cancel{L} = \frac{\lambda \cancel{L}}{\varepsilon_0}$$

$$E = \frac{\lambda}{2\pi \varepsilon_0} \frac{1}{r} \Bigg|_{r = z_0} = \frac{\lambda}{2\pi \varepsilon_0} \frac{1}{z_0}$$

This was just the electric field generated by the wire. let's now calculate the field generated by the half ring..

We start from the calculation of the infinitesimal contribution dE



$$dq = \lambda\, d\ell = \lambda R\, d\varphi$$

$$d\bar{E} = \frac{1}{4\pi\varepsilon_0} \frac{dq}{r^2}\hat{r} = \frac{1}{4\pi\varepsilon_0} \frac{\lambda R\, d\varphi}{r^2}\hat{r}$$

We now project the vector dE along the Z axis and on the XY plane

$$dE_z = d\bar{E}\cdot\hat{z} = \frac{1}{4\pi\varepsilon_0} \frac{\lambda R\, d\varphi}{r^2}\cos\alpha$$

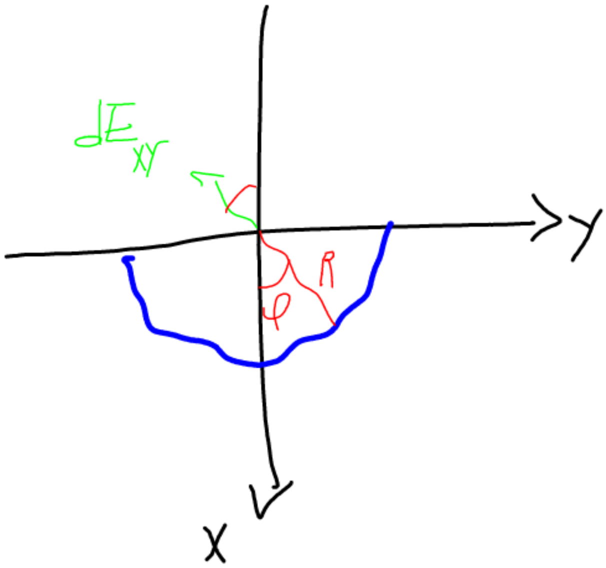
$$= \frac{1}{4\pi\varepsilon_0} \frac{\lambda R\, d\varphi}{r^2} \frac{z_0}{r}$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{\lambda R z_0}{r^3} d\varphi = \frac{1}{4\pi\varepsilon_0} \frac{\lambda R z_0\, d\varphi}{\left(z_0^2 + R^2\right)^{3/2}}$$

$$dE_{xy} = \frac{1}{4\pi\varepsilon_0} \frac{\lambda R \, d\varphi}{r^2} \sin\alpha$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{\lambda R \, d\varphi}{r^2} \frac{R}{r}$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{\lambda R^2 \, d\varphi}{r^3} = \frac{1}{4\pi\varepsilon_0} \frac{\lambda R^2 \, d\varphi}{(z_0^2 + R^2)^{3/2}}$$



$$dE_x = -dE_{xy} \cos\varphi$$

$$dE_y = -dE_{xy} \sin\varphi$$

Now we have to integrate dEx, dEy, and dEz. let's start from dEz

$$E_z = \int_{\varphi=-\pi/2}^{\pi/2} dE_z = \int_{-\pi/2}^{\pi/2} \frac{1}{4\pi\varepsilon_0} \frac{\lambda R z_0 \, d\varphi}{(z_0^2 + R^2)^{3/2}}$$

$$E_z = \frac{1}{4\pi\varepsilon_0}\frac{\lambda R z_0}{(z_0^2+R^2)^{3/2}}\int_{-\pi/2}^{\pi/2} d\varphi = \frac{1}{4\varepsilon_0}\frac{\lambda R z_0}{(z_0^2+R^2)^{3/2}}$$

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$$E_x = \int_{-\pi/2}^{\pi/2} dE_x = \int_{-\pi/2}^{\pi/2} \frac{-1}{4\pi\varepsilon_0}\frac{\lambda R^2 d\varphi}{(z_0^2+R^2)^{3/2}}\cos\varphi$$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{\lambda R^2}{(z_0^2+R^2)^{3/2}}\int_{-\pi/2}^{\pi/2}\cos\varphi\, d\varphi$$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{\lambda R^2}{(z_0^2+R^2)^{3/2}}\Big[\sin\varphi\Big]_{-\pi/2}^{\pi/2}$$

$$= -\frac{1}{2\pi\varepsilon_0}\frac{\lambda R^2}{(z_0^2+R^2)^{3/2}}$$

$$E_y = \int_{\varphi=-\pi/2}^{\pi/2} dE_y = -\int_{-\pi/2}^{\pi/2} dE_{xy} \sin\varphi$$

The sine function is odd, so its integral from $-\frac{\pi}{2} \to \pi/2$ is equal to 0

$$E_x = -\frac{1}{2\pi\varepsilon_0}\frac{\lambda R^2}{(z_0^2+R^2)^{3/2}}$$

HALF RING
X - COMPONENT

$$E_y = \phi$$

HALF RING
Y - COMPONENT

$$E_z = \underbrace{\frac{\lambda}{2\pi\varepsilon_0}\frac{1}{z_0}}_{\text{WIRE}} + \underbrace{\frac{1}{4\varepsilon_0}\frac{\lambda R z_0}{(z_0^2+R^2)^{3/2}}}_{\text{HALF RING z - COMPONENT}}$$

ii) $$E_x = -\frac{1}{2\pi\varepsilon_0}\frac{\lambda R^2}{(z_0^2+R^2)^{3/2}}$$ the same as before

$$E_y = \phi$$ same as before

$$E_z = \underbrace{\frac{\lambda}{2\pi\varepsilon_0}\frac{1}{z_0}}_{\text{WIRE}} + \underbrace{\frac{1}{4\varepsilon_0}\frac{\lambda R z_0}{(z_0^2+R^2)^{3/2}}}_{\text{HALF RING}} - \underbrace{\frac{1}{4\pi\varepsilon_0}\frac{q}{z_0^2}}_{\text{CHARGE } q}$$

iii) $$V(z_0) = \int_{z_0}^{\infty} \bar{E}\cdot d\ell = \underbrace{\int_{z=z_0}^{\infty} E(z)\,dz}_{\text{INTEGRATION PATH ALONG } z\text{-AXIS}}$$

WE WRITE E AS A FUNCTION OF Z

$$E_z(z) = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{z} + \frac{1}{4\varepsilon_0}\frac{\lambda R z}{(z^2+R^2)^{3/2}} - \frac{1}{4\pi\varepsilon_0}\frac{q}{z^2}$$

1 2 3

VARIABLE

NB: $V(z_0) = \int_{z_0}^{\infty} (1 + 2 - 3)\, dz$

$$= \int_{z_0}^{\infty} \frac{\lambda}{2\pi\varepsilon_0} \frac{dz}{z} + \int_{z_0}^{\infty} 2 - \int_{z_0}^{\infty} 3$$

$$V(z_0) = \frac{\lambda}{2\pi\varepsilon_0} \Big[\ln z \Big]_{z_0}^{\infty} + \int 2 - \int 3$$

$$= \frac{\lambda}{2\pi\varepsilon_0} \Big[\ln \infty - \ln z_0 \Big] + \int 2 - \int 3$$

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THE ELECTRIC POTENTIAL IS INFINITE BECAUSE THE WIRE HAS INFINITE CHARGE