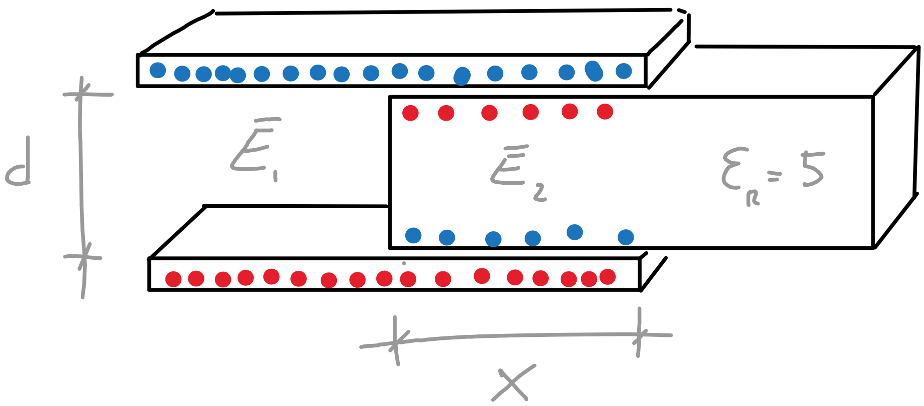
A flat plate capacitor has a surface of 100 cm2 (square plates) the distance between the plates is 1 cm.
a glass slab ($\varepsilon_R=5$) is partially inserted in the capacitor by an amount of 5 cm. a charge of 10 uC is stored in the capacitor.
the fringe field effects can be considered negligible.

1) calculate the fields $\bar{D}, \bar{E}, \bar{P}$ between the two plates of the capacitor (inside and outside) the dielectric slab.

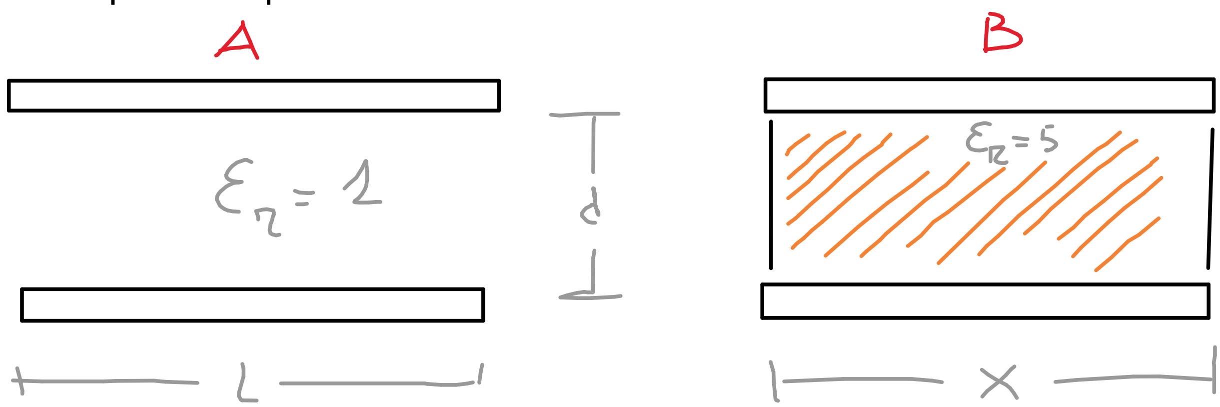
2) calculate the capacity of the capacitor

3) calculate the energy stored in the capacitor

4) calculate the force acting on the slab and indicate the direction of the force



1) neglecting the fringe field we can consider the system as two separate capacitors

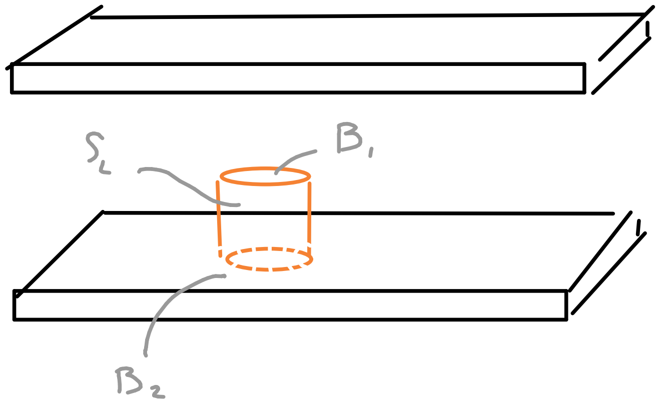


$S = L^2$

(A) Calculation of The $\bar{D}$ field in The A region ($\varepsilon_r = 1$)

Let's start with Gauss theorem for the $\bar{D}$ field

$$\oiint_S \bar{D}\cdot\hat{n}\, ds = q_{FREE}$$



We split the surface in B_1, B_2 and S_L

$$\oiint_S \bar{D}\cdot\hat{n}\, ds = \iint_{B_1} \bar{D}\cdot\hat{n}\, ds + \cancel{\iint_{B_2} \bar{D}\cdot\hat{n}\, ds} + \cancel{\iint_{S_L} \bar{D}\cdot\hat{n}\, ds}$$

$\bar{D} = 0$ inside the metal plate

$\bar{D} \perp \hat{n}$ on the lateral surface

$$\iint_{B_1} \bar{D}\cdot\hat{n}\, ds = \iint_{B_1} D\, ds = D \iint_{B_1} ds = D\cdot B_1 \overset{\text{GAUSS}}{=} \sigma_A B_1$$

$$\boxed{D = \sigma_A}$$

Just depend on the free surface charge distribution σ_A

$$E = \frac{D}{\varepsilon_0 \varepsilon_r} = \frac{\sigma_A}{\varepsilon_0}$$

in air $\varepsilon_r = 1$ $\quad E = \frac{D}{\varepsilon_0} = \frac{\sigma_A}{\varepsilon_0}$

$$\bar{P} = \varepsilon_0 \chi \bar{E} \quad \text{where} \quad \chi = \varepsilon_R - 1$$

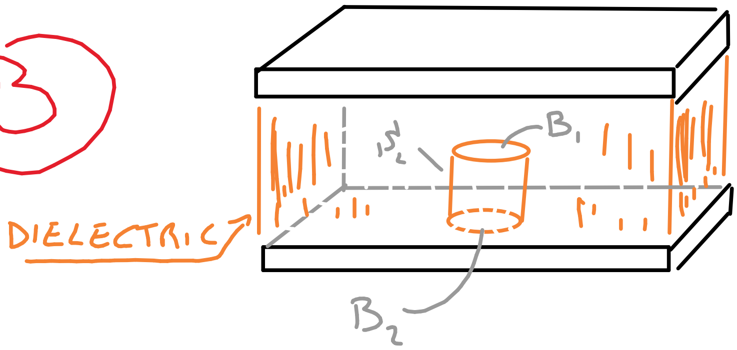
in air $\varepsilon_R = 1 \Rightarrow \chi = \emptyset$

$$\bar{P} = \emptyset$$

there is no dielectric and no dipoles can be created

NO DIPOLES $\Rightarrow$ NO POLARIZATION

B



We start to calculate $\bar{D}$

$$\iint_S \bar{D} \cdot \hat{n} \, ds = \sigma_{FREE}$$

We split S into B_1, B_2 and S_L

$$\iint_S \bar{D} \cdot \hat{n} \, ds = \iint_{B_1} \bar{D} \cdot n \, ds = \iint_{B_1} D \, ds = D \iint_{B_1} ds = D B_1$$

$$= q_{free} = \sigma_B \cdot B_1$$

Gauss theorem for $\bar{D}$

$$\boxed{D = \sigma_B}$$

Doesn't depend on polarization

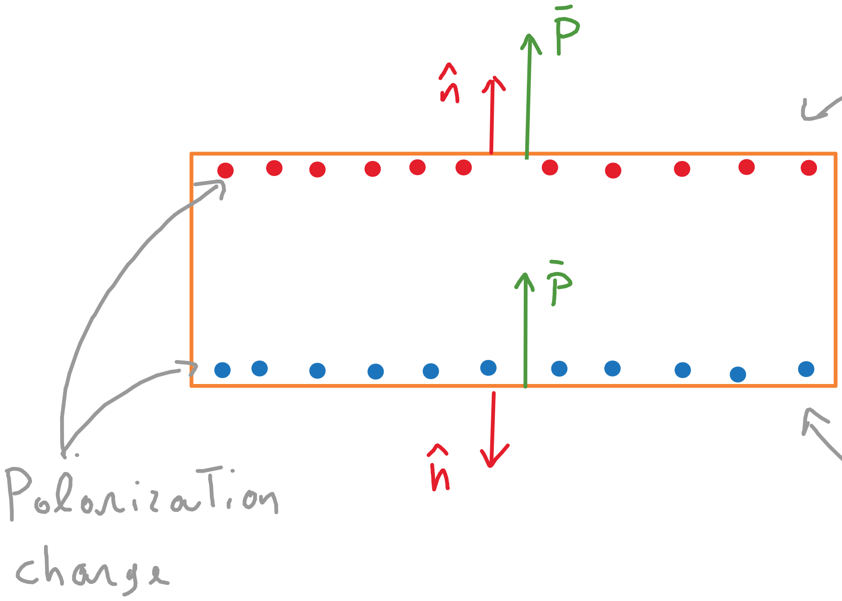
$$\bar{E} = \frac{\bar{D}}{\varepsilon_0 \varepsilon_R} \Rightarrow E = \frac{\sigma_B}{\varepsilon_0 \varepsilon_R}$$

E feels the effect of the polarization change through ε_R

$$\bar{P} = \varepsilon_0 \chi \bar{E} \Rightarrow P = \frac{(\varepsilon_R - 1)}{\varepsilon_R} \sigma_B$$

If there is POLARIZATION there is σ_P

Let's focus just on the DIELECTRIC

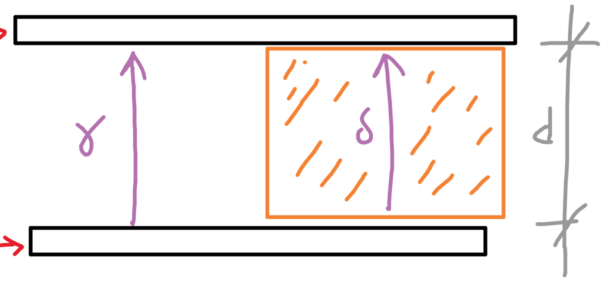


$$\sigma_P = \bar{P} \cdot \hat{n} = +P$$

$$\sigma_P = \bar{P} \cdot \hat{n} = -P$$

Let's calculate how the difference of potential between the plates

$$\Delta V = \int_{\text{POSITIVE PLATE}}^{\text{NEG. PLATE}} \bar{E} \cdot d\ell$$



$$= \int_\gamma \bar{E}_A \cdot d\bar{\ell} = \int_\delta \bar{E}_B \cdot d\bar{\ell}$$

$$\int_\gamma \bar{E}_a \cdot d\bar{\ell} = \int_\gamma \frac{\sigma_A}{\varepsilon_0} d\ell = \frac{\sigma_A}{\varepsilon_0} d = \Delta V$$

$$\int_\delta \bar{E}_B \cdot d\bar{\ell} = \int_\delta \frac{\sigma_B}{\varepsilon_0 \varepsilon_r} d\ell = \frac{\sigma_B}{\varepsilon_0 \varepsilon_r} d = \Delta V$$

$$\frac{\sigma_A \cancel{d}}{\cancel{\varepsilon_0}} = \frac{\sigma_B \cancel{d}}{\cancel{\varepsilon_0}\,\varepsilon_R} \qquad \sigma_A = \sigma_B / \varepsilon_R$$

NB: σ_A and σ_B ARE DIFFERENT

so the charge is NOT uniformly distributed on the plate of the capacitor

$$Q = \sigma_A \cdot L(L-x) + \sigma_B L \cdot x$$

$\hookrightarrow$ TOTAL CHARGE

$\hookrightarrow \sigma_B = \sigma_A \cdot \varepsilon_R$

$$Q = \sigma_A L(L-x) + \sigma_A \varepsilon_R L \cdot x = \sigma_A L\left((L-x) - \varepsilon_R x\right)$$

$$\Rightarrow \quad \sigma_A = \frac{Q}{L\left[(L-x) - \varepsilon_R x\right]}$$

$$\sigma_B = \frac{Q\,\varepsilon_R}{L\left[(L-x) - \varepsilon_R x\right]}$$

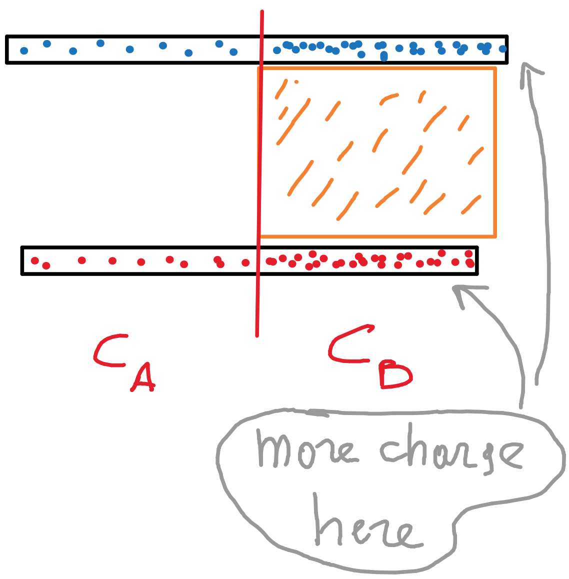
$$Q_A = L\cdot(L-x)\,\sigma_A \qquad Q_B = L\,x\,\sigma_B$$

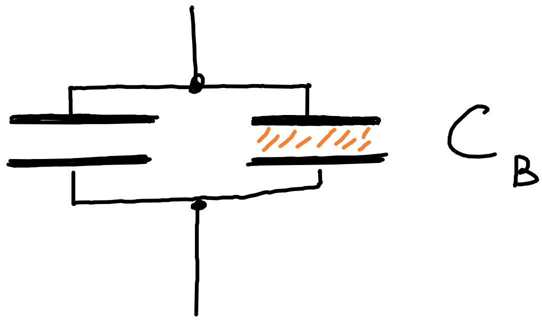
$$Q_A = \cancel{4}\cdot(L-x)\,\frac{Q}{\cancel{4}\left[(L-x)-\varepsilon_r x\right]} = \frac{\cancel{(L-x)}}{\cancel{(L-x)}}\;\frac{Q}{1-\dfrac{\varepsilon_r x}{(L-x)}}$$

$$Q_3 = \cancel{4}\cdot x\,\frac{Q\,\varepsilon_r}{\cancel{4}\left[(L-x)-\varepsilon_r x\right]} = \frac{Q\,\varepsilon_r\,x}{(L-x)-\varepsilon_r x}$$

$$= \frac{Q\,\cancel{\varepsilon_r}\,\cancel{x}}{\cancel{\varepsilon_r}\cdot\cancel{x}\left[\dfrac{(L-x)}{\varepsilon_r\cdot x}-1\right]} = \frac{Q}{\dfrac{(L-x)}{\varepsilon_r\cdot x}-1}$$

② Calculation of C_A and C_B





CAPACITORS CONNECTED IN PARALLEL

C_A is a flat plate capacitor in air

$$C_A = \frac{\varepsilon_0\,S_A}{d} = \frac{\varepsilon_0\,L\,(L-x)}{d}$$

Let's verify it:

$$C_A = \frac{Q_a}{V} = \frac{\cancel{\sigma_a} \cdot L \cdot (L-x)}{\frac{\cancel{\sigma_a}\, d}{\varepsilon_0}} = \frac{\varepsilon_0 L (L-x)}{d} \quad \text{OK!!!}$$

C_B is filled with dielectric $\boxed{\text{NB}}$

$$C_B = \frac{Q_B}{V} = \frac{\cancel{\sigma_B}\, L \cdot x}{\frac{\cancel{\sigma_B} \cdot d}{\varepsilon_0 \varepsilon_R}} = \underbrace{\varepsilon_0 \varepsilon_R \frac{L \cdot x}{d}}$$

CAPACITY IS INCREASED BY ε_R

$$C_{EQ} = C_A + C_B = \frac{\varepsilon_0 L (L-x)}{d} + \frac{\varepsilon_0 \varepsilon_R L \cdot x}{d}$$

$$= \frac{\varepsilon_0 L}{d} \left[L - x + \varepsilon_R \cdot x \right]$$

3 ENERGY STORED

$$U = \frac{1}{2} \frac{Q^2}{C_{EQ}}$$

Q is known and C_{eq} has just been calculated

$$U = \frac{1}{2}\frac{Q^2}{\frac{\varepsilon_0 L}{d}\left[L - x + \varepsilon_r \cdot x\right]} = \frac{Q^2 d}{\varepsilon_0 L\left[L - x(1+\varepsilon_r)\right]}$$

as x increases the electrostatic energy of the capacitor decreases

If the system is ISOLATED, it evolves NATURALLY towards the minimum potential energy

4 Force acting on the glass slab

$$F = -\left(\frac{\partial U}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{Q^2 d}{\varepsilon_0 L\left[L - x(1+\varepsilon_r)\right]}\right)$$

$$= -\frac{Q^2 d}{\varepsilon_0 L}\frac{\partial}{\partial x}\left(\frac{1}{L - x(1+\varepsilon_r)}\right)$$

$$= -\frac{Q^2 d}{\varepsilon_0 L}\frac{-(\varepsilon_r+1)}{\left[L - x(1+\varepsilon_r)\right]^2} = \frac{Q^2 d\,(\varepsilon_r+1)}{\left[L - x(1+\varepsilon_r)\right]^2}$$

the force is positive if the system evolves in the direction of increasing x coordinate.

the capacitor naturally attract the glass slab towards the center of the capacitor because this is the configuration of lower potential energy