SINGLE-DEGREE-OF-FREEDOM-SYSTEMS

Frequency domain analysis

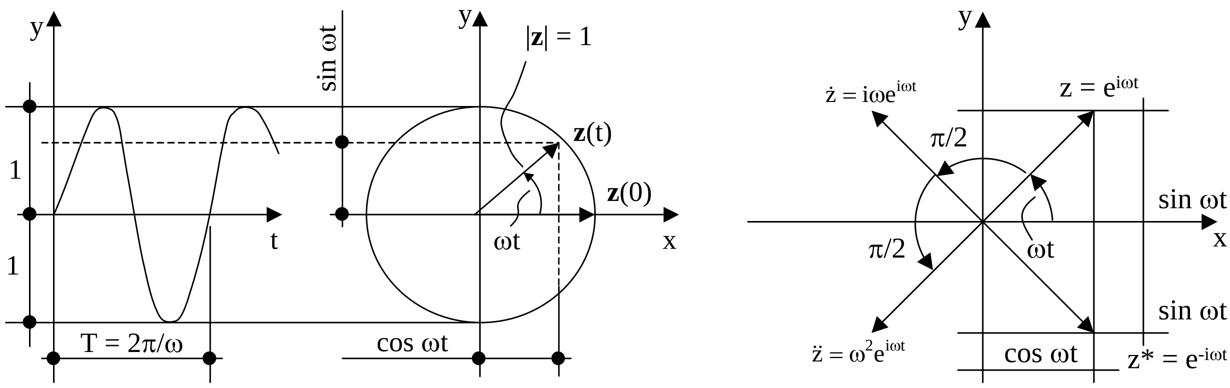
Elementary harmonic force

A harmonic force f(t) is defined as elementary when it has a unit amplitude. This condition is satisfied by the real expression $f(t) = \sin\omega t$ and by the complex expression $f(t) = e^{i\omega t}$:

$$f(t) = e^{i\omega t} = \cos\omega t + i\sin\omega t \Rightarrow$$
$$\left|e^{i\omega t}\right| = \sqrt{(\sin\omega t + i\cos\omega t)\ (\sin\omega t - i\cos\omega t)} = \sqrt{\sin^2\omega t + \cos^2\omega t} = 1$$

Observation: the elementary harmonic function $\sin\omega t$ may be regarded as the projection on the axis y of the ordinates of a vector **Z** with unit modulus, rotating around the origin of a Cartesian reference system (x, y) with uniform angular velocity ω and nil initial phase.

Interpreting (x, y) as an Argand-Gauss plane, the vector **Z** is associated with a complex number z whose real and imaginary parts are respectively the projections on x and y of **Z** : z = x + iy, x = Re(z) = cos ωt, y = Im(z) = sin ωt; thus, z = cos ωt + isinωt. Using Euler's formula $z = e^{i\omega t}$.



Let us consider the equation of motion:

$$\boxed{\begin{aligned}&\ddot{q}(t) + 2\xi\omega_0\dot{q}(t) + \omega_0^2 q(t) = \frac{1}{m}f(t) = \frac{1}{m}e^{i\omega t} = \frac{1}{m}(\cos\omega t + i\sin\omega t)\\ &q(0) = q_0\ ;\ \dot{q}(0) = \dot{q}_0\end{aligned}} \qquad (1)$$

Since f(t) is a complex quantity, thus also q(t) is complex and may be written as:

$$q(t) = x(t) + i\,y(t) \qquad (2)$$

where the real functions x(t) = Re[q(t)] and y(t) = Im[q(t)] are, respectively, the solutions of the two problems:

$$\boxed{\begin{aligned}&\ddot{x}(t)+2\xi\omega_0\dot{x}(t)+\omega_0^2 x(t)=\frac{1}{m}\cos\omega t\\&x(0)=x_0=\mathrm{Re}(q_0)\ ;\ \dot{x}(0)=\dot{x}_0=\mathrm{Re}(\dot{q}_0)\end{aligned}} \tag{3}$$

$$\boxed{\begin{aligned}&\ddot{y}(t)+2\xi\omega_0\dot{y}(t)+\omega_0^2 y(t)=\frac{1}{m}\sin\omega t\\&y(0)=y_0=\mathrm{Im}(q_0);\ \dot{y}(0)=\dot{y}_0=\mathrm{Im}(\dot{q}_0)\end{aligned}} \tag{4}$$

The response y(t) to the real elementary harmonic force f(t) = sin ωt is the imaginary part of the response q(t) to the complex elementary harmonic force $f(t)=e^{i\omega t}$:

$$\begin{aligned}&\ddot{q}(t)+2\xi\omega_0\dot{q}(t)+\omega_0^2 q(t)=\frac{1}{m}f(t)=\frac{1}{m}e^{i\omega t}=\frac{1}{m}(\cos\omega t+i\sin\omega t)\\&q(0)=q_0\ ;\ \dot{q}(0)=\dot{q}_0\end{aligned} \tag{5}$$

The solution of Eq. (5) is the sum of the integral $q'(t)$ of the homogeneous associated equation and of any particular integral $q''(t)$ of the complete equation:

$$q(t)=q'(t)+q''(t) \tag{6}$$

From Eq. (2) it results:

$$\begin{aligned}&q'(t)=x'(t)+iy'(t) \qquad && x(t)=x'(t)+x''(t)\\&q''(t)=x''(t)+iy''(t) && y(t)=y'(t)+y''(t)\end{aligned}$$

where $x'(t)$ and $y'(t)$ are the integrals of the homogeneous equations associated with Eqs. (3) and (4), respectively:

$$\begin{aligned}&x'(t)=e^{-\xi\omega_0 t}\left(a_{x1}\cos\omega_0\sqrt{1-\xi^2}\,t+a_{x2}\sin\omega_0\sqrt{1-\xi^2}\,t\right)\\&y'(t)=e^{-\xi\omega_0 t}\left(a_{y1}\cos\omega_0\sqrt{1-\xi^2}\,t+a_{y2}\sin\omega_0\sqrt{1-\xi^2}\,t\right)\end{aligned}$$

The integration constants depend on the initial conditions.

It is easy to demonstrate that an expression of $q''(t)$ is given by:

$$q''(t)=H(\omega)e^{i\omega t} \tag{7}$$

Substituting Eq. (7) into Eq. (5):

$$-\omega^2 H(\omega)\cancel{e^{i\omega t}}+2\xi\omega_0\cdot i\omega H(\omega)\cancel{e^{i\omega t}}+\omega_0^2 H(\omega)\cancel{e^{i\omega t}}=\frac{1}{m}\cancel{e^{i\omega t}}\Rightarrow$$

$$\boxed{H(\omega)=\frac{1}{m\,\omega_0^2}\frac{1}{1-\frac{\omega^2}{\omega_0^2}+2i\xi\frac{\omega}{\omega_0}}} \tag{8}$$

$H(\omega)$ is the complex frequency response function and may be rewritten as:

$$H(\omega)=R(\omega)+i\,I(\omega) \tag{9}$$

$$H(\omega)=|H(\omega)|e^{i\psi(\omega)} \tag{10}$$

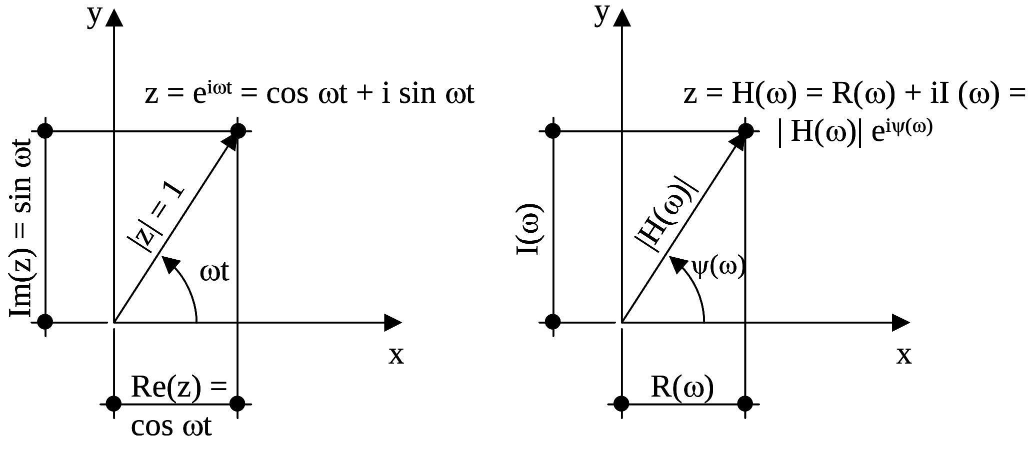
where:

$$R(\omega)=\mathrm{Re}[H(\omega)]=\frac{1}{m\,\omega_0^2}\frac{1-\omega^2/\omega_0^2}{\left(1-\omega^2/\omega_0^2\right)+4\xi\,\omega^2/\omega_0^2} \tag{11}$$

$$I(\omega)=\mathrm{Im}[H(\omega)]=\frac{1}{m\,\omega_0^2}\frac{-2\zeta\,\omega/\omega_0}{\left(1-\omega^2/\omega_0^2\right)^2+4\xi^2\,\omega^2/\omega_0^2} \tag{12}$$

$$|H(\omega)|=\sqrt{R^2(\omega)+I^2(\omega)}=\frac{1}{m\omega_0^2}\frac{1}{\sqrt{\left(1-\omega^2/\omega_0^2\right)^2+4\xi^2\omega^2/\omega_0^2}} \tag{13}$$

$$\psi(\omega)=\mathrm{arctg}\frac{I(\omega)}{R(\omega)}=\mathrm{arctg}\left(-\frac{2\zeta\omega/\omega_0}{1-\omega^2/\omega_0^2}\right) \tag{14}$$



Neglecting the initial transient stage of the motion, i.e. $q'(t)=0 \Rightarrow q(t)=q''(t) \Rightarrow$

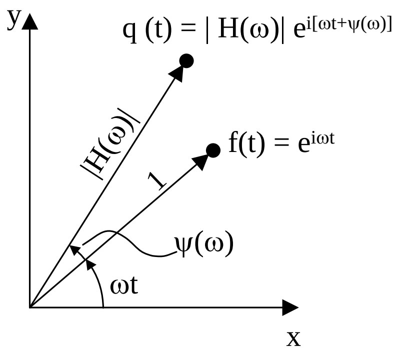
$$\boxed{q(t)=H(\omega)e^{i\omega t}} \tag{15}$$

Eq. (15) provides the steady-state response of a S.D.O.F. subjected to the complex elementary harmonic force $f(t) = e^{i\omega t}$; $H(\omega)$ is the ratio between the time-dependent response and force. Thus it has the meaning of the inverse of a dynamic stiffness.

Substituting Eq. (10) into Eq. (15):

$$\boxed{q(t) = |H(\omega)| e^{i[\omega t + \psi(\omega)]}} \tag{16}$$

Thus, $|H(\omega)|$ is the amplitude of the dynamic response, $\psi(\omega)$ is the phase delay of the response q(t) with respect to the force $f(t) = e^{i\omega t}$.



$f(t) = e^{i\omega t} \Rightarrow$

$$q(t) = |H(\omega)| e^{i[\omega t + \psi(\omega)]}$$

$$|H(\omega)| = \frac{1}{m\,\omega_0^2} \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + 4\xi^2 \frac{\omega^2}{\omega_0^2}}}$$

$$\psi(\omega) = \text{arctg}\left\{-\frac{2\xi \frac{\omega}{\omega_0}}{1 - \frac{\omega^2}{\omega_0^2}}\right\}$$

for $\omega = 0 \Rightarrow$ $f(t) = 1 =$ static force with unit amplitude

$$\psi(0) = 0; \quad |H(0)| = H(0) = \frac{1}{m\omega_0^2} = \frac{1}{k} \Rightarrow$$

$$q(t) = \frac{1}{m\omega_0^2} = \text{ static response to a unit static force}$$

The magnification factor $N(\omega)$ is defined as the ratio between the amplitude $|H(\omega)|$ of the dynamic response and the amplitude $H(0)$ of the static response:

$$\boxed{N(\omega)=\frac{|H(\omega)|}{H(0)}\frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2+4\xi^2\frac{\omega^2}{\omega_0^2}}}} \qquad (17)$$

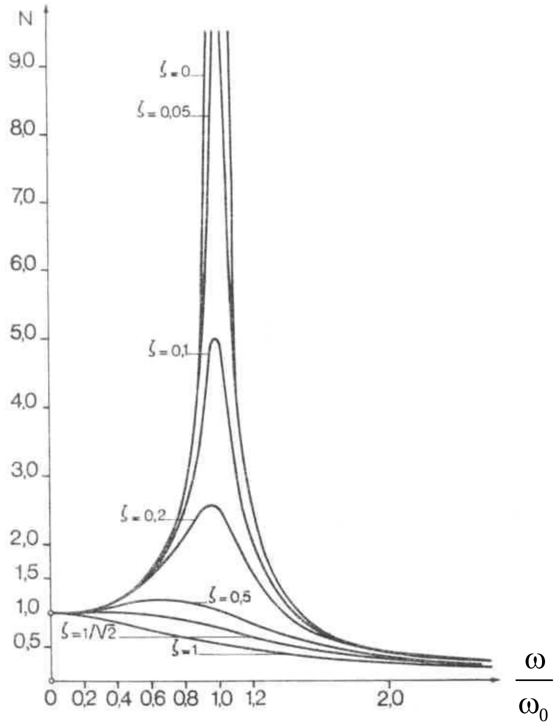
In conclusion: $q(t)=N(\omega)\frac{1}{m\,\omega_0^2}e^{i\omega t}e^{i\psi(\omega)}$

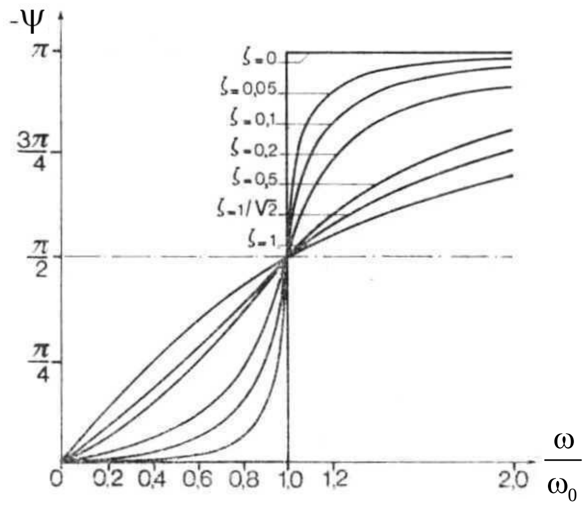
force

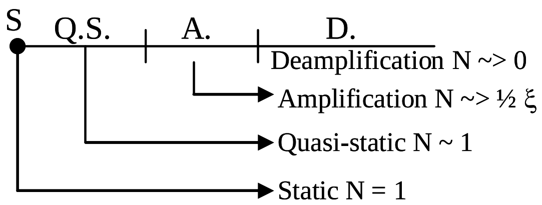
dynamic phase delay

dynamic magnification of response amplitude

static response







$$\omega_m=\omega_0\sqrt{1-2\xi^2} \quad ; \quad N_m=\frac{1}{2\xi\sqrt{1-\xi^2}} \Rightarrow \omega_m\approx\omega_0\,;\, N_m\approx 1/2\xi \Rightarrow \boxed{\xi=\frac{1}{2N_m}}$$

Let $\omega=\omega_a,\omega_b$ be values such as $N=N_m/\sqrt{2}$. It follows that $\boxed{\xi\approx\frac{\omega_b-\omega_a}{2\omega_m}}$ $\left(\omega_b>\omega_a\right)$

Real elementary harmonic force f(t) = sinωt

$$\ddot{q}(t)+2\xi\omega_0\dot{q}(t)+\omega_0^2 q(t)=\frac{1}{m}f(t)\ ;\ q(0)=q_0\ ,\ \dot{q}(0)=\dot{q}_0$$

$$\boxed{f(t)=e^{i\omega t}} = \cos\omega t + i\sin\omega t$$

$$q(t)=q'(t)+q''(t)$$

$$q(t)=x'(t)+iy'(t)$$

$$x'(t)=e^{-\xi\omega_0 t}\left(a_{x1}\cos\omega_0\sqrt{1-\xi^2}\ t+a_{x2}\sin\omega_0\sqrt{1-\xi^2}\ t\right)$$

$$y'(t)=e^{-\xi\omega_0 t}\left(a_{y1}\cos\omega_0\sqrt{1-\xi^2}\ t+a_{y2}\sin\omega_0\sqrt{1-\xi^2}\ t\right)$$

$$q''(t)=H(\omega)e^{i\omega t}$$

$$\boxed{f(t)=\sin\omega t}$$

$$q(t)=q'(t)+q''(t)$$

$$q'(t)=y'(t)$$

$$q''(t)=\mathrm{Im}\left[H(\omega)e^{i\omega t}\right]=$$

$$=\mathrm{Im}\left[|H(\omega)|e^{i\{\omega t+\psi(\omega)\}}\right]=$$

$$=|H(\omega)|\sin\left[\omega t+\psi(\omega)\right]$$

In steady-state conditions: $q'(t)=0\Rightarrow q(t)=q''(t)\Rightarrow$

$$\boxed{q(t)=H(\omega)e^{i\omega t}}$$

$$\boxed{q(t)=|H(\omega)|\sin\left[\omega t+\psi(\omega)\right]}$$

$$\boxed{f(t)=F\sin\omega t}\ \Rightarrow$$

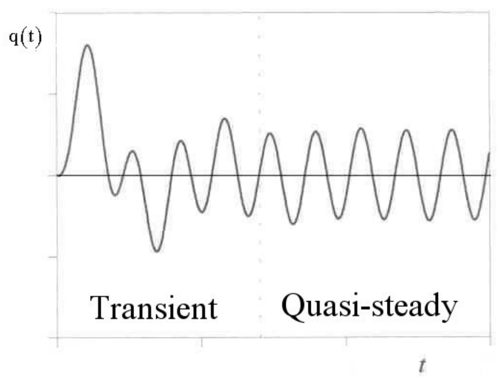
$$q(t)=F|H(\omega)|\sin\left[\omega t+\psi(\omega)\right]$$

$$N(\omega)=\frac{|H(\omega)|}{H(0)}=m\omega_0^2|H(\omega)|=k|H(\omega)|\Rightarrow$$

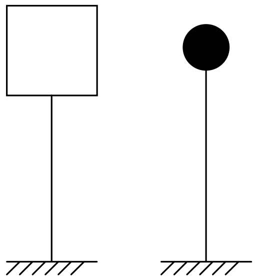
$$q(t)=\frac{F}{k}N(\omega)\sin\left[\omega t+\psi(\omega)\right]\Rightarrow$$

$$\boxed{q(t)=q_{st}\,N(\omega)\sin\left[\omega t+\psi(\omega)\right]}$$

The following figure shows how the steady-state regime is approached after a transient vibration.

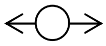


Example: Arc lamp



k = 21063 N/m
m = 10671 kg

$\omega_0 = 1.405$ rad/s
$n_0 = 0.223$ Hz

Vibrodyne  $f(t) = F \sin \omega t$ F = 100 N

$$q(t) = \underbrace{q_{st} N(\omega)}_{Q = \frac{F}{k} N(\omega)} \sin[\omega t + \psi(\omega)]$$

$$\omega \approx 0 \Rightarrow N(\omega) = 1 \Rightarrow Q = 0.00475\,\text{m} = 4.75\,\text{mm}$$

$$\omega = \omega_0 \Rightarrow N(\omega) = 1/2\xi \Rightarrow \quad \xi = 0.01 \Rightarrow N = 50 \Rightarrow Q = 0.237\,\text{m}$$

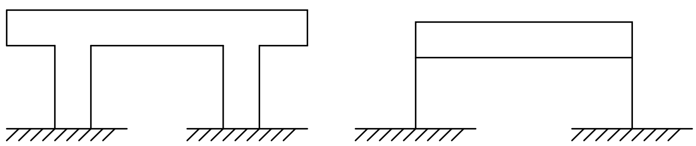
$$\xi = .005 \Rightarrow N = 100 \Rightarrow Q = 0.475\,\text{m}$$

$$\xi = .002 \Rightarrow N = 250 \Rightarrow Q = 1.187\,\text{m}$$

$$\xi = .001 \Rightarrow N = 500 \Rightarrow Q = 2.37\,\text{m}$$

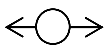
$$\omega = 10\,\text{rad/s} \Rightarrow N(\omega) \approx 0.142 \Rightarrow Q = 6.74 \times 10^{-4}\,\text{m} = 0.674\,\text{mm}$$

Example: Single-storey R.C. building



$k = 0.8333 \times 10^8$ N/m
m = 88087.5 kg

$\omega_0 = 30.76$ rad/s
$n_0 = 4.9$ Hz

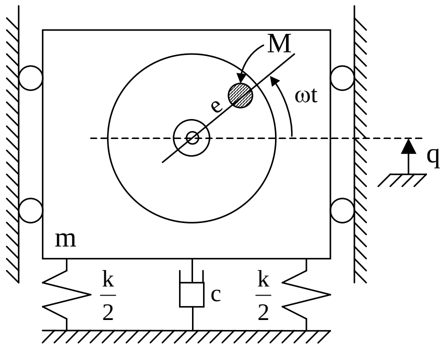
 $f(t) = F \sin \omega t$ F = 1000 N

$$\omega \approx 0 \Rightarrow N(\omega) = 1 \Rightarrow Q = 1.2 \times 10^{-5}\,\text{m}$$

$$\omega = \omega_0 \Rightarrow \quad \xi = 0.05 \Rightarrow N = 10 \Rightarrow Q = 1.2 \times 10^{-4}\,\text{m}$$

$$\xi = 0.02 \Rightarrow N = 25 \Rightarrow Q = 3.0 \times 10^{-4}\,\text{m}$$

Rotating mass with an eccentricity



m –M = non-rotating part of the mass
M = rotating mass

$q + e\sin\omega t$ = rotating mass displacement

$$(m - M)\ddot{q}(t) + M\frac{d^2}{dt^2}\left[q(t) + e\sin\omega t\right] = -Kq(t) - c\dot{q}(t)$$

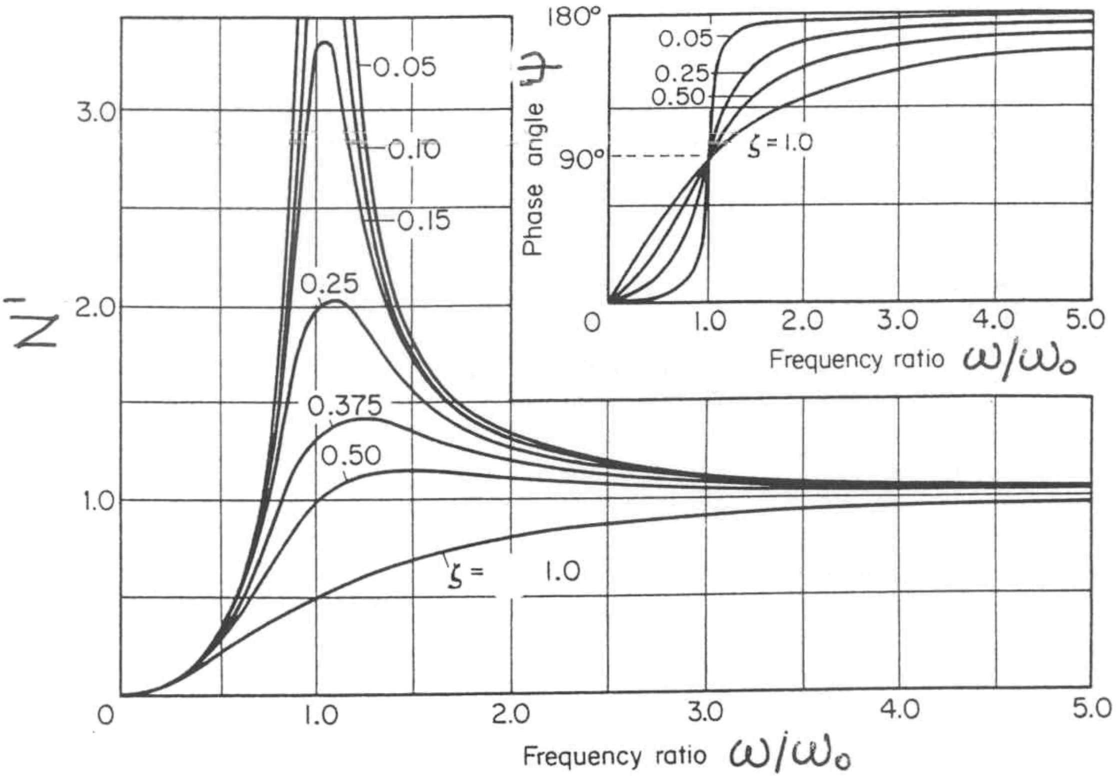
$$m\ddot{q}(t) - M\ddot{q}(t) + M\ddot{q}(t) - Me\omega^2\sin\omega t +$$
$$Kq(t) + c\dot{q}(t) = 0 \Rightarrow$$
$$m\ddot{q}(t) + \dot{c}q(t) + Kq(t) = \underbrace{Me\omega^2}_{F}\sin\omega t$$

$$q(t) = \frac{Me\omega^2}{K}N(\omega)\sin\left[\omega t + \psi(\omega)\right]\cdot\frac{m}{m} \Rightarrow$$

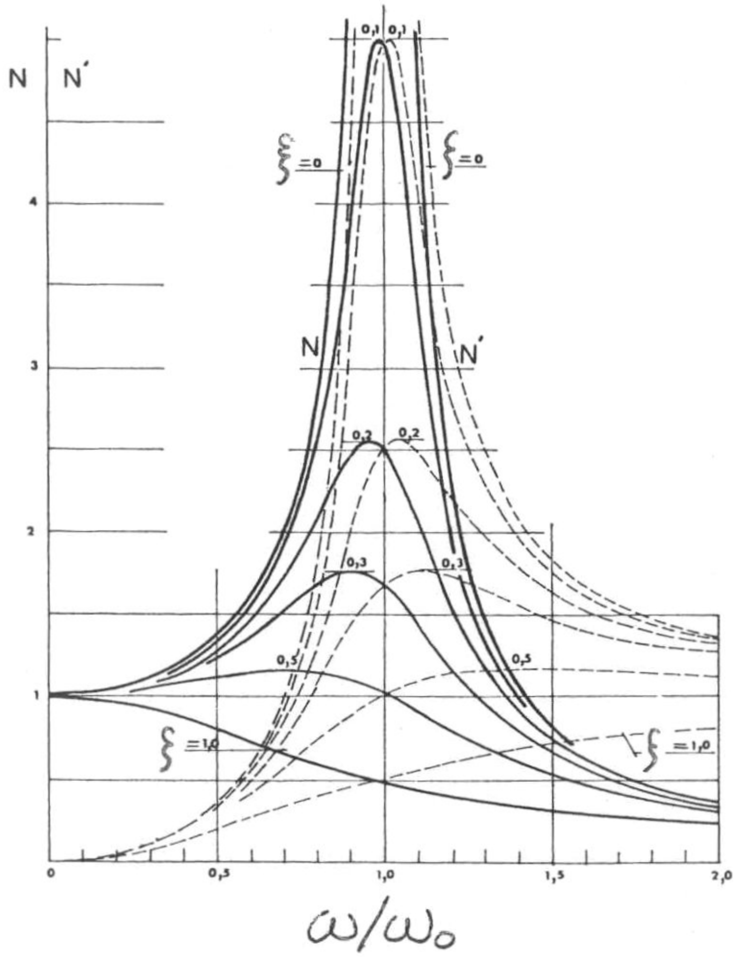
$$q(t) = \frac{Me}{m}\,\frac{\omega^2}{\omega_0^2}N(\omega)\sin\left[\omega t + \psi(\omega)\right]$$

$$N'(\omega) = \frac{\omega^2}{\omega_0^2}N(\omega) = \frac{\left(\dfrac{\omega^2}{\omega_0^2}\right)}{\sqrt{\left(1 - \dfrac{\omega^2}{\omega_0^2}\right)^2 + 4\xi^2\dfrac{\omega^2}{\omega_0^2}}} \Rightarrow$$

$$q(t) = \underbrace{\frac{Me}{m}N'(\omega)}_{Q}\sin\left[\omega t + \psi(\omega)\right]$$

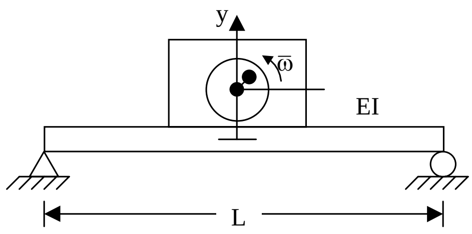


N'
3.0
2.0
1.0
0
1.0
2.0
3.0
4.0
5.0
0.05
0.10
0.15
0.25
0.375
0.50
ζ = 1.0
Frequency ratio ω/ω₀
180°
90°
Phase angle Ψ
0.05
0.25
0.50
ζ = 1.0
Frequency ratio ω/ω₀



N
N'
4
3
2
1
0
0,5
1,0
1,5
2,0
ξ = 0
ξ = 0
0,1
0,1
0,2
0,2
0,3
0,3
0,5
0,5
ξ = 1,0
ξ = 1,0
ω/ω₀

Example: Rotating mass over a beam (with negligible mass)



L = 4 m
E = .21 × 10^{12} N/m^2
J = 5 × 10^{-5} m^4
m = 8.000 kg
W = 80.000 N

$$K = \frac{48EJ}{L^3} = 7875000\,N/m \Rightarrow q_S = W/K = 0.01\,m$$
$$\omega_0 = \sqrt{K/m} = 31.375\,rad/s \Rightarrow n_0 = 4.99\,Hz$$
$$\xi = 0.02$$

a) M = 20 kg; e = 0.25 m; ω = 25 rad/s ⇒ $\omega/\omega_0 = 0.80$

$$N' = \frac{\frac{\omega^2}{\omega_0^2}}{\sqrt{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + 4\xi^2 \frac{\omega^2}{\omega_0^2}}} = 1.7325 \Rightarrow$$
$$Q = \frac{Me}{m} N' = \frac{20 \times 0.25}{8.000} \times 1.7325 = 1.0828 \times 10^{-3}\,m$$

Amplitude of oscillation around the equilibrium condition ⇒
Maximum displacement 0.01 m + 0.001 m = 0.011 m

b) ω = 31.375 rad/s ⇒ $\omega/\omega_0 = 1$

$$N' = 25 \Rightarrow Q = 0.0156\,m \Rightarrow S_{max} = q_S + Q = 0.0256\,m$$

c) How to obtain ξ by measuring Q ?

Periodic force

A function f(t) is defined as periodic with period T when f(t) = f(t + T) for $\forall t \in \mathbf{R}$, with T > 0. The minimum period, or simply the period, is the minimum value of T for which above condition is satisfied.

Under very general conditions, a periodic function f(t) can be expanded according to the following Fourier series:

$$f(t) = \frac{a_0}{2} + \sum_{1}^{\infty}{}_k \left(a_k \cos \omega_k t + b_k \sin \omega_k t \right) \tag{18}$$

where:

$$\begin{aligned} a_k &= \frac{2}{T}\int_{-T/2}^{T/2} f(t) \cos \omega_k t \, dt \quad k = 0, 1, 2, \ldots \\ b_k &= \frac{2}{T}\int_{-T/2}^{T/2} f(t) \sin \omega_k t \, dt \quad k = 1, 2, \ldots \\ \omega_k &= k\frac{2\pi}{T} \quad k = 0, 1, 2, \ldots \infty \end{aligned} \tag{19}$$

The mean value of f(t) is $a_0/2$:

The Fourier series:

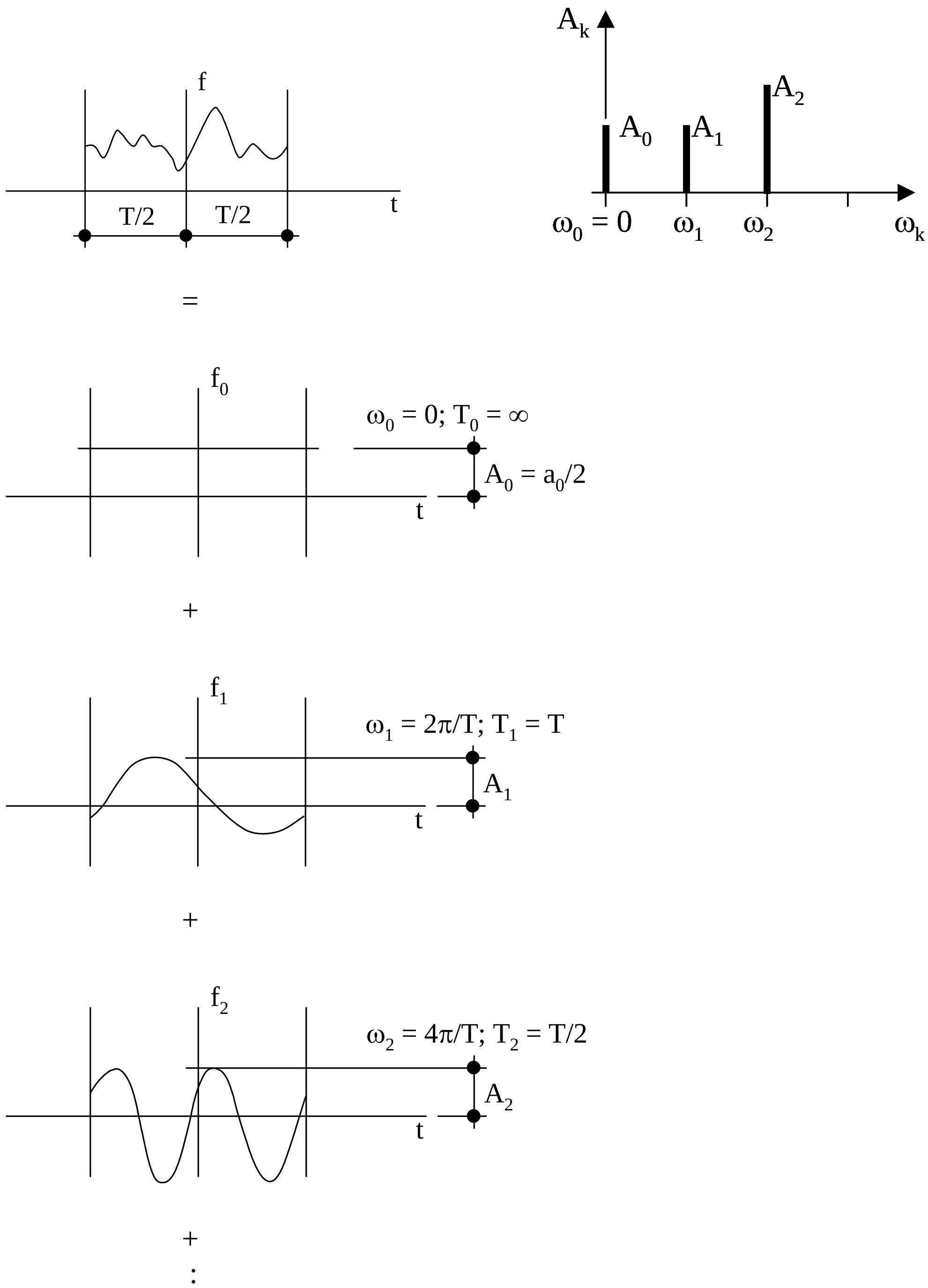
$$f(t) = \frac{a_0}{2} + \sum_{1}^{\infty}{}_k \left(a_k \cos \omega_k t + b_k \sin \omega_k t \right)$$

may be rewritten as:

$$f(t) = \frac{a_0}{2} + \sum_{1}^{\infty}{}_k A_k \sin\left(\omega_k t + \varphi_k \right) \tag{20}$$

where:

$$\begin{aligned} A_k &= \sqrt{a_k^2 + b_k^2} \\ \varphi_k &= \operatorname{arctg}\left(\frac{a_k}{b_k} \right) \end{aligned} \tag{21}$$



Moreover, the Fourier series may be rewritten using the following exponential complex notation:

$$f(t) = \sum_{1}^{\infty}{}_{k}\, c_k e^{i\omega_K t} \qquad \omega_k = k\frac{2\pi}{T} \tag{22}$$

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-i\omega_K t} dt \tag{23}$$

<u>Demonstration</u>: Starting from Eq. (23) $\Rightarrow$

$c_0 = a_0/2;\ c_k = (a_k - ib_k)/2,\ c_{-k} = (a_k + ib_k)/2$ (k > 1);

$a_0 = 2c_0;\ a_k = c_k + c_{-k},\ b_k = i\ (c_k - c_{-k})$ (k > 1)

$$\Rightarrow a_k \cos\omega_k t + b_k \sin\omega_k t = (c_k + c_{-k})\ \left(e^{i\omega_k t} + e^{-i\omega_k t}\right)/2 +$$
$$i(c_k - c_{-k})\ \left(e^{i\omega_k t} - e^{-i\omega_k t}\right)/2i = c_k e^{i\omega_k t} + c_{-k} e^{-i\omega_k t} \Rightarrow (78).$$

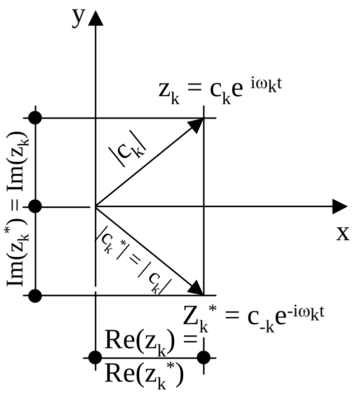
Rewriting Eq. (22) as:

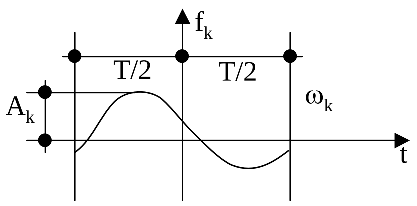
$$f(t) = c_0 + \sum_1^{\infty}{}_k \left(c_k e^{i\omega_k t} + c_{-k} e^{-i\omega_k t}\right) \tag{24}$$

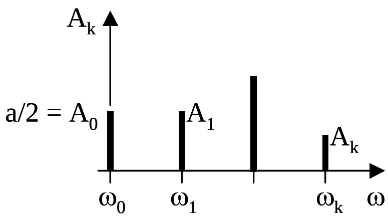
the correspondence with Eq. (20) is apparent. Each real harmonic term in Eq. (20) corresponds to a couple of complex harmonic terms in Eq. (24). In particular:

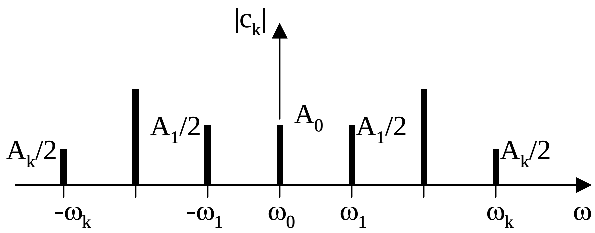
$c_0 = a_0/2$; $|c_k| = |c_{-k}| = A_k/2 = \sqrt{a_k^2 + b_k^2}/2$;
$c_k e^{i\omega_k t}$ is the complex conjugate of $c_{-k}e^{-i\omega_k t}$.

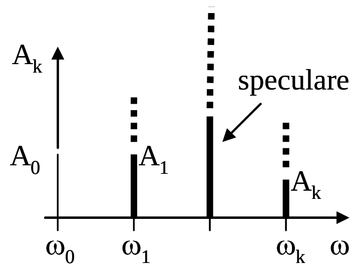
$$f(t) = \underbrace{\frac{a_o}{2}}_{f_o=\bar{f}} + \sum_1^{\infty}{}_k \underbrace{A_k \sin(\omega_k t + \varphi_k)}_{f_k(t)} = \underbrace{c_o}_{f_o=\bar{f}} + \sum_1^{\infty}{}_k \underbrace{\left(c_k e^{i\omega_k t} + c_{-k} e^{-i\omega_k t}\right)}_{f_k(t)}$$











Dynamic response to a periodic force

$$\ddot{q}(t) + 2\xi\omega_0\dot{q}(t) + \omega_0^2 q(t) = \frac{1}{m} f(t)$$

The steady-state response q(t) to a periodic force $f(t) = \sum_k f_k(t)$ may be expressed as the superposition of the responses $q_k(t)$ to the component elementary harmonic responses $f_k(t)$: $q(t) = \sum_k q_k(t)$:

$$f_k(t) = e^{i\omega_k t} \Rightarrow \qquad q_k(t) = H(\omega_k)\ e^{i\omega_k t}$$
$$f_k(t) = c_k e^{i\omega_k t} \Rightarrow \qquad q_k(t) = c_k H(\omega_k)\ e^{i\omega_k t}$$

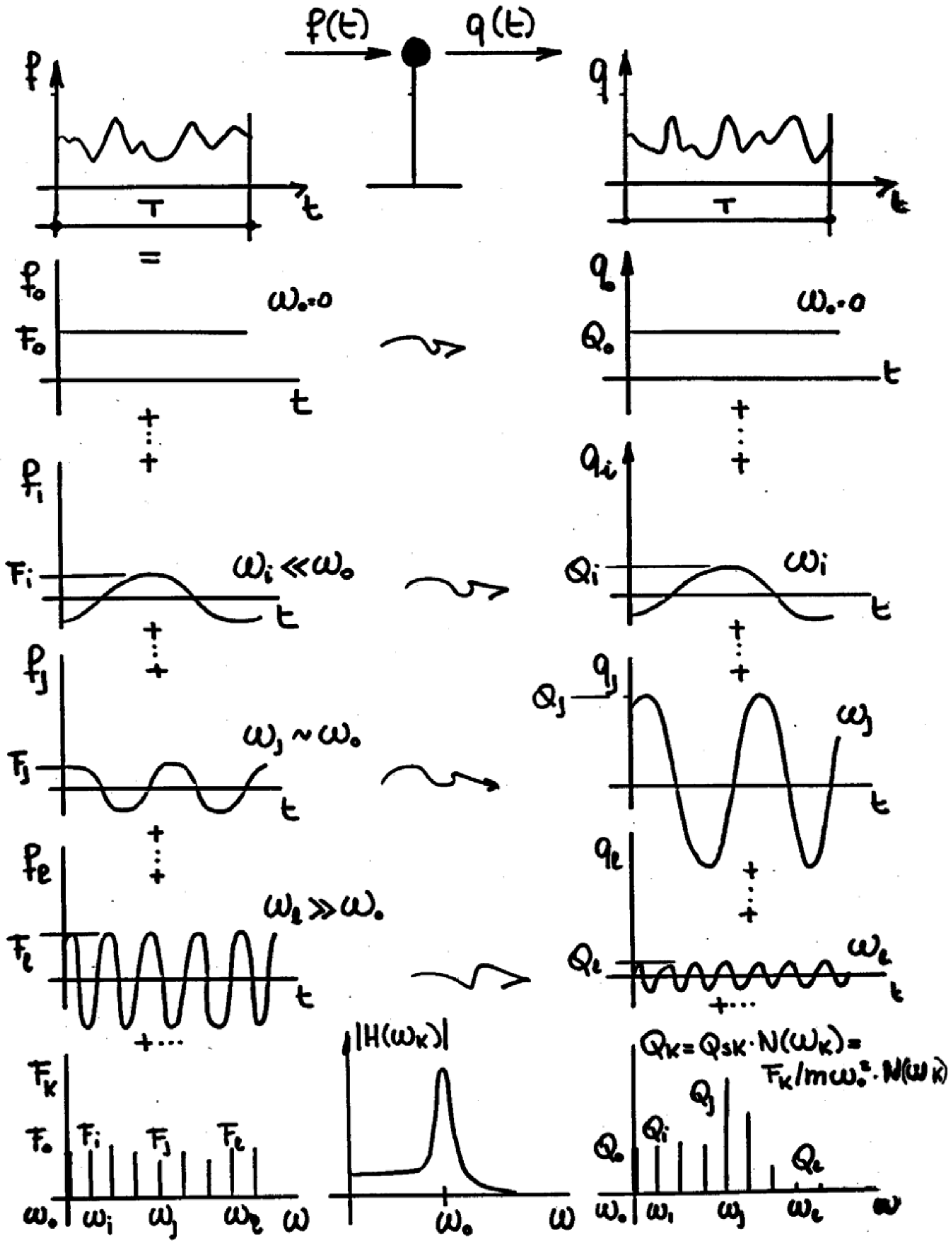
$$f_k(t) = \sum_{-\infty}^{\infty}{}_k f_k(t) = \sum_{-\infty}^{\infty}{}_k c_k e^{i\omega_k t} \Rightarrow \qquad (25)$$

$$q(t) = \sum_{-\infty}^{\infty}{}_k q_k(t) = \sum_{-\infty}^{\infty}{}_k c_k H(\omega_k) e^{i\omega_k t} \qquad (26)$$

The structural system operates a filtering effect related to its complex frequency response function. $F_k = 2|c_k|$ is the amplitude of the k-th component harmonic force $f_k(t)$. The amplitude of the k-th component harmonic response $q_k(t)$ to $f_k(t)$ is given by:

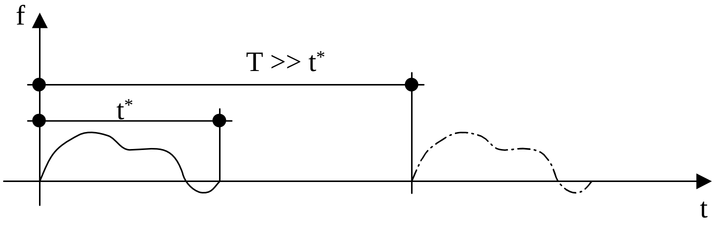
$$Q_k = 2|c_k||H(\omega_k)| = F_k|H(\omega_k)| = F_k H(0)\, N(\omega_k) =$$
$$F_k / m\omega_o^2 \cdot N(\omega_k) \Rightarrow Q_k = Q_{sk}\ N(\omega_k)$$

where $Q_{sk} = F_k / m\omega_o^2$ is the amplitude of the static response to a static force F_k.


f(t)
q(t)
f
q
T
t
$\omega_0 = 0$
F_0
Q_0
$\omega_i \ll \omega_0$
ω_i
F_i
Q_i
$\omega_j \sim \omega_0$
ω_j
F_j
Q_j
$\omega_l \gg \omega_0$
ω_l
F_l
Q_l
$|H(\omega_K)|$
$Q_K = Q_{sK} \cdot N(\omega_K) = F_K / m\omega_0^2 \cdot N(\omega_K)$
F_K
ω_0
ω

Generic force

A generic force $f(t)$ may be dealt with as a periodic force with period $T \to \infty$.



Let us consider the complex exponential Fourier series:

$$f(t) = \sum_{-\infty}^{\infty} {}_k c_k e^{i\omega_k t}$$
$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{i\omega_k t} dt$$
$$\omega_k = k \cdot 2\pi / T$$

Assuming $\Delta\omega = \omega_{k+1} - \omega_k = 2\pi / T \Rightarrow 1/T = \Delta\omega / 2\pi \Rightarrow$

$$f(t) = \sum_{-\infty}^{\infty} {}_k e^{i\omega_k t} \frac{1}{T}\int_{-T/2}^{T/2} f(\eta) e^{-i\omega_k \eta} d\eta =$$
$$= \frac{1}{2\pi}\sum_{-\infty}^{\infty} {}_k e^{i\omega_k t}\Delta\omega \int_{-T/2}^{T/2} f(\eta) e^{-i\omega_k \eta} d\eta$$

For $T \to \infty$, $\Delta\omega \to 0$, ω tends to become a continuos variable $\Rightarrow$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} \left[\int_{-\infty}^{\infty} f(\eta) e^{i\omega\eta} d\eta\right] d\omega$$

and the exponential Fourier series tends to become the Fourier integral:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{i\omega t}\, d\omega \qquad (27)$$

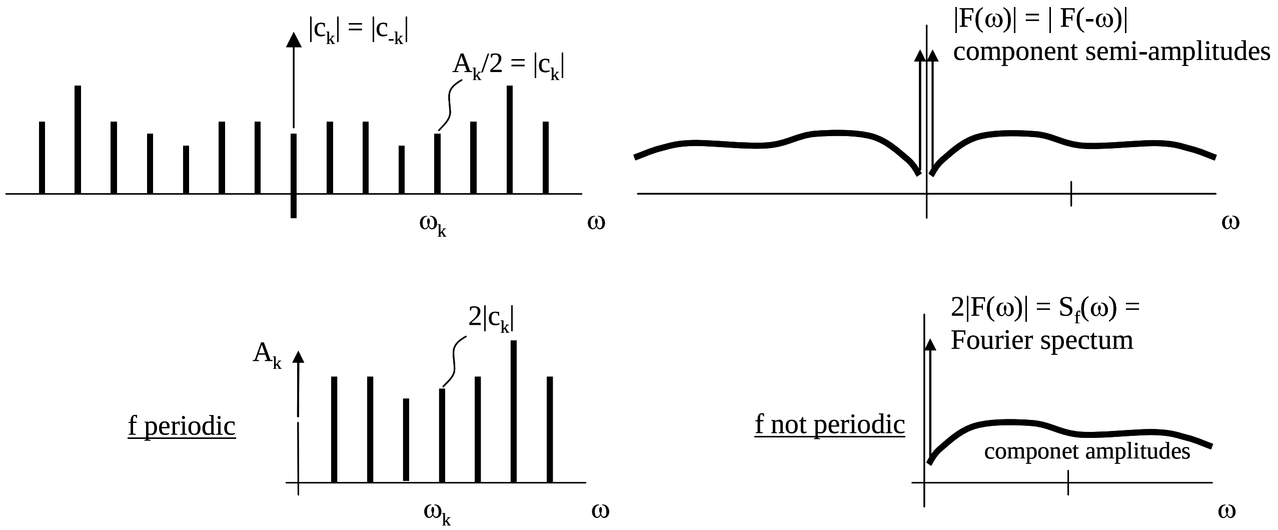
$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt \qquad (28)$$

$F(\omega)$ is a complex function called Fourier transform; $f(t)$ is consequently called inverse Fourier transform. The uniqueness of a Fourier couple, $f(t)$ and $F(\omega)$, is demonstrated under wide conditions. $F(\omega)$ exists provided that:

$\int_{-\infty}^{\infty} |f(t)| dt$ is finite.

It can be shown that:

$$F(\omega_k) = \lim_{T \to \infty} c_k ; \; f(t) = f_0 \Rightarrow F(\omega) = f_0 \delta(\omega)$$



$$\ddot{q}(t) + 2\xi\omega_0 \dot{q}(t) + \omega_0^2 q(t) = \frac{1}{m} f(t)$$

The steady-state response q(t) to a generic force $f(t)$ can be expressed as the integral of the elementary component responses to the elementary component harmonic forces:

$$f(t) = e^{i\omega t} \Rightarrow \qquad q(t) = H(\omega) e^{i\omega t}$$
$$f(t) = F(\omega) e^{i\omega t} \Rightarrow \qquad q(t) = F(\omega) H(\omega) e^{i\omega t}$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega \Rightarrow \tag{29}$$

$$q(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) H(\omega) e^{i\omega t} d\omega \tag{30}$$

Moreover, using the definition of Fourier transform and inverse Fourier response of the response:

$$q(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Q(\omega) e^{i\omega t} d\omega \tag{31}$$

$$Q(\omega) = \int_{-\infty}^{\infty} q(t) e^{-i\omega t} dt \tag{32}$$

Comparing Eqs. (30) and (32):

$$\boxed{Q(\omega) = H(\omega)\,F(\omega)} \qquad (33)$$

Eq. (33) is the basic relationship between $f(t)$ and $q(t)$ in the frequency domain.

Summarising, the frequency domain analysis consists of 4 steps:

(1) Starting from $f(t)$ its Fourier transform is calculated $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\,dt$;

(2) The structural system is characterised by its complex frequency response function:
$H(\omega) = \frac{1}{m\,\omega_o^2}\,\frac{1}{\left(1-\omega^2/\omega_o^2\right) + 2i\xi\omega/\omega_o}$;

(3) The Fouriem of $q(t)$ is determined: $Q(\omega) = H(\omega)\,F(\omega)$;

(4) The inverse Fourier transform of $Q(\omega)$ is calculated: $q(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Q(\omega) e^{i\omega t} d\omega$.

It is easy to demonstrate that:

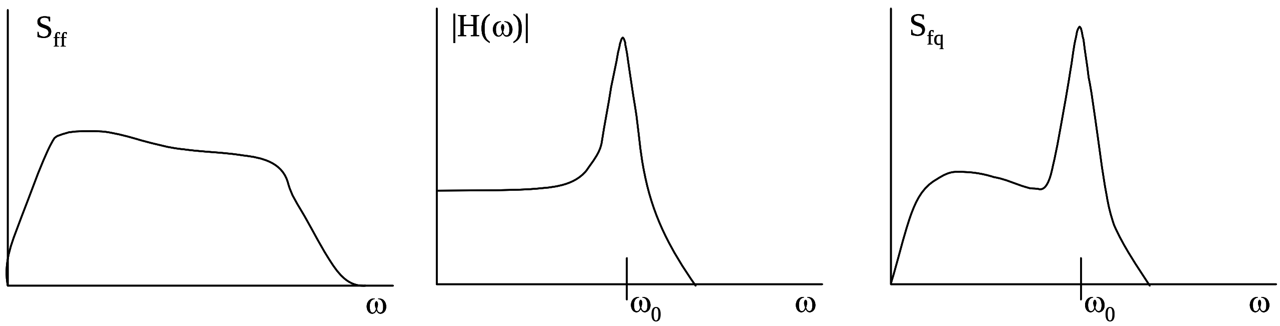
$$\boxed{|Q(\omega)| = |H(\omega)|\,|F(\omega)|} \qquad (34)$$

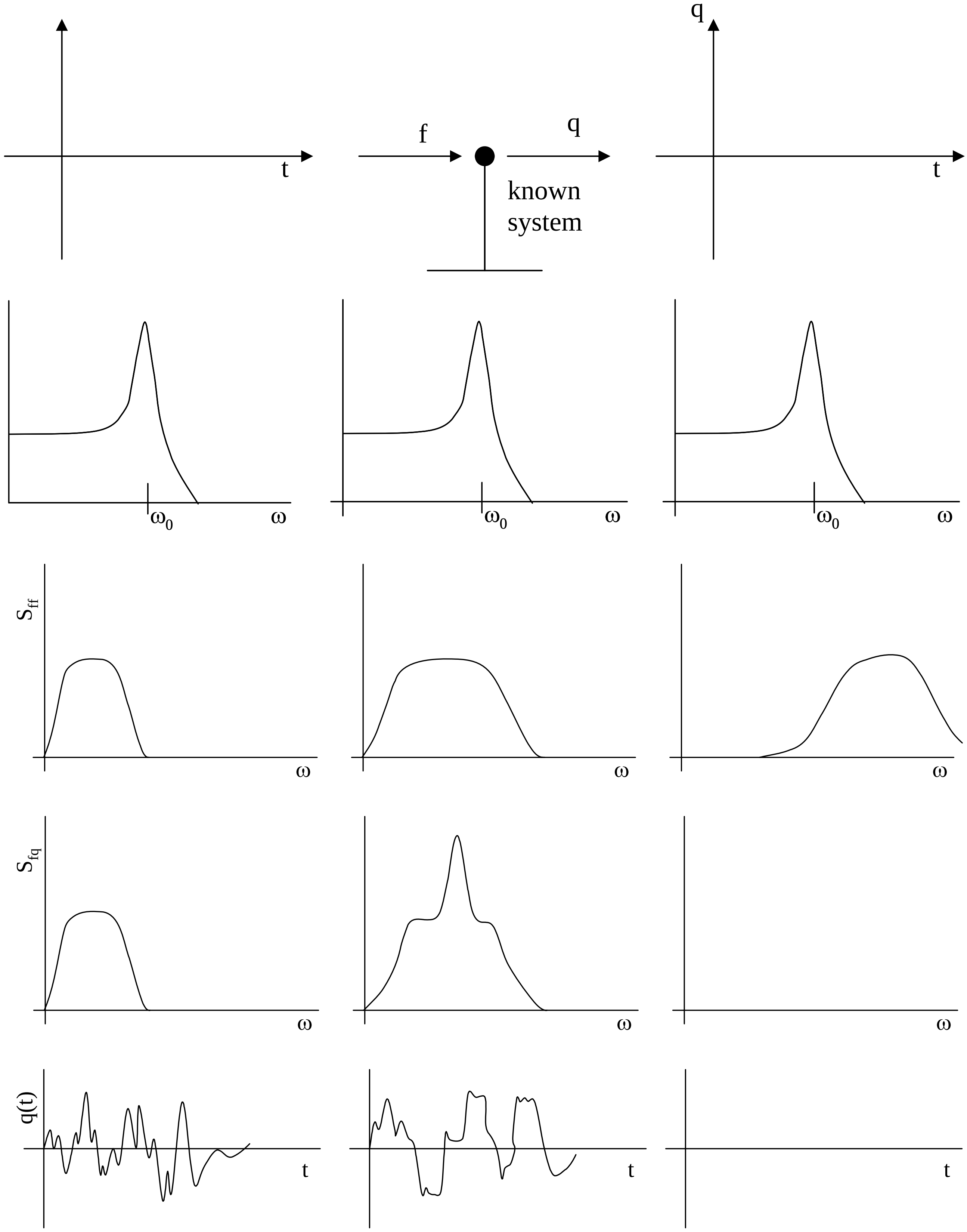
$S_{ff}(\omega) = 2|F(\omega)|$ = Fourier spectrum of the force =
= amplitude of the harmonic components of $f(t)$

$S_{fq}(\omega) = 2|Q(\omega)|$ = Fourier spectrum of the response =
= amplitude of the harmonic components of $q(t)$

$|H(\omega)| = N(\omega)/m\omega_o^2$ = Ratio between the amplitudes of the harmonic components of the response and of the force

Thus: $S_{fq}(\omega) = |H(\omega)| \cdot S_{ff}(\omega)$

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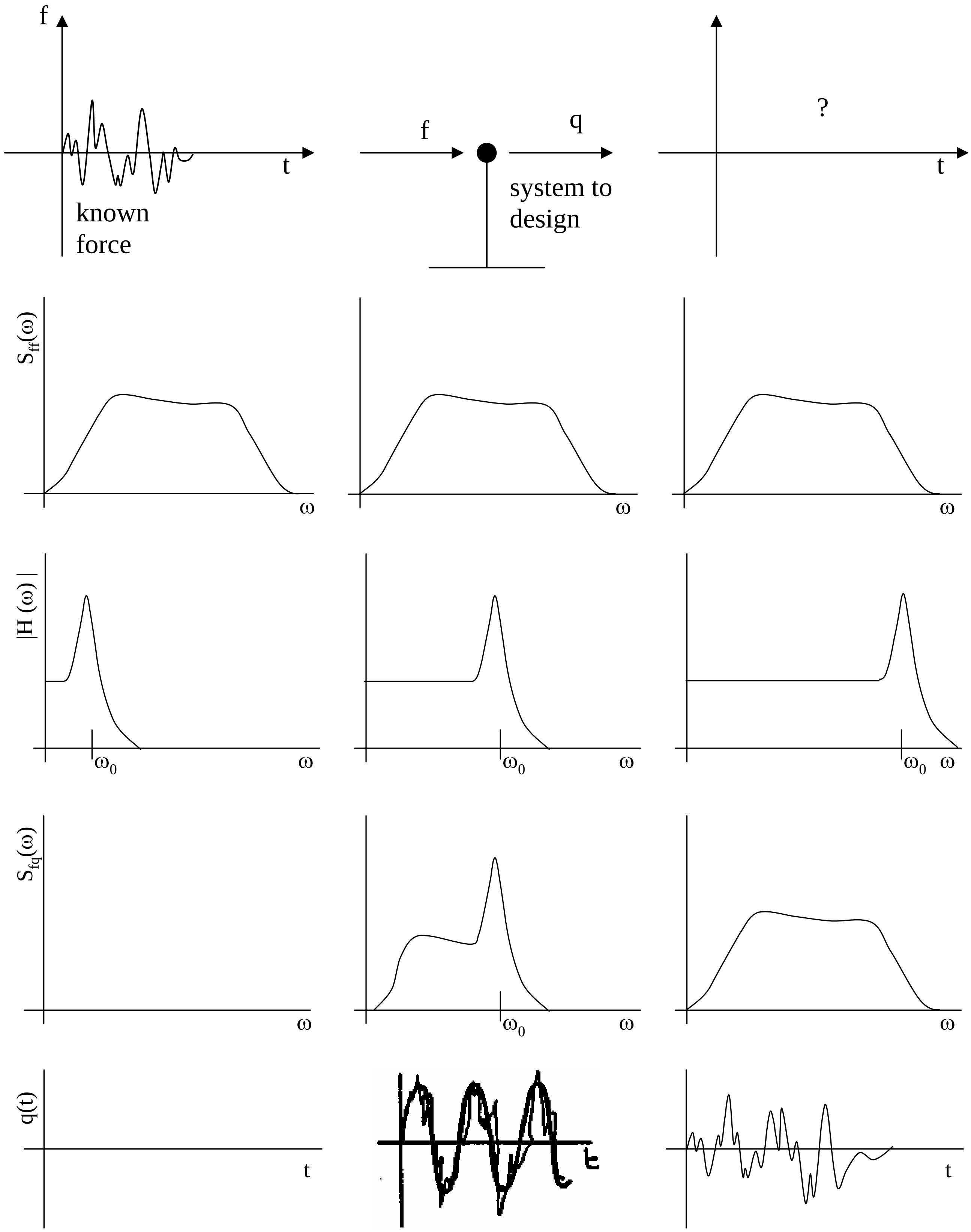
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