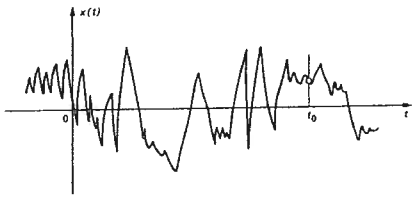
4

Classificazione dei segnali ed analisi in frequenza

1. Introduzione

- I dati che rappresentano un fenomeno fisico e che da questo sono ottenuti mediante l'impiego di opportuni sistemi di misura vengono spesso denominati segnali.
- Esempi di segnali sono:
  - fluttuazioni di temperatura in una stanza in funzione del tempo
  - variazioni di tensione in uscita da un trasduttore
  - modifiche di pressione in un punto di un campo acustico
- Il fenomeno fisico in esame è spesso tradotto da un trasduttore in un segnale elettrico e, se visualizzato mediante un oscilloscopio, può apparire come nell'esempio di figura:



1.1 Classificazione dei dati

- I segnali possono essere classificati come:
  - ***Deterministici***
  - ***Non-deterministici***
- Sono **deterministici** quei dati che possono essere descritti da esplicite relazioni matematiche.
- Si consideri, ad esempio, una massa rigida m sospesa ad un telaio mediante una molla priva di massa e avente costante elastica k
- Sappiamo che, spostata dalla sua posizione di equilibrio di una quantità X e abbandonata all'istante $t=0$ con velocità iniziale nulla, la massa oscillerà in modo tale che la sua distanza $x(t)$ dalla posizione di equilibrio sarà espressa dalla relazione:

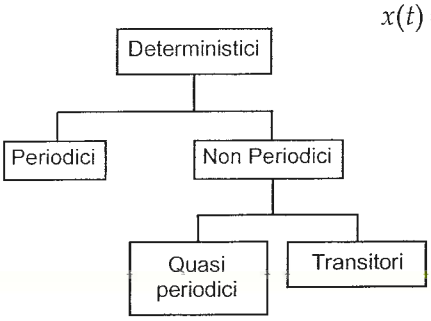
$$x(t) = X\cos\sqrt{\frac{k}{m}}t \qquad t \geq 0$$

Classificazione dei dati

- Esistono, tuttavia, molti altri fenomeni fisici che producono dati di tipo **non deterministico**.
- Ad esempio:
  - l'altezza delle onde in un mare agitato
  - la pressione acustica dell'aria che fluisce in un tubo
  - l'uscita elettrica di un generatore di rumore, ecc.
  - la risposta di una struttura ad un sisma
  - il comportamento di veicoli viaggianti su una strada irregolare
  - ...
- Non è possibile, in questi casi, prevedere il valore dei dati in un qualunque istante futuro.
- Non è possibile utilizzare esplicite relazioni matematiche, ma si deve ricorrere a strumenti di tipo probabilistico.
- Varie terminologie sono utilizzate in letteratura per descrivere questi segnali, ad esempio si parla di:
  - processi casuali
  - processi stocastici
  - serie temporali (time series)

2. Segnali deterministici

- I dati di tipo deterministico possono essere classificati secondo lo schema di figura:



- I dati ***periodici*** possono essere rappresentati analiticamente da una funzione che si ripete esattamente ad intervalli regolari di tempo.

$$x(t) = x(t \pm nT_p) \qquad n = 1,2,3,\cdots$$

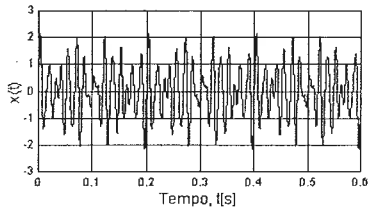
- L'intervallo di tempo necessario per un intero ciclo è detto *periodo Tp* .
- Il numero di cicli per unità di tempo è chiamato *frequenza fondamentale f1*.

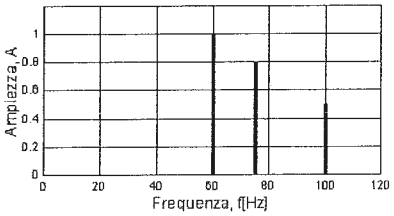
2.1 Segnali periodici

- I dati periodici possono essere sviluppati in serie di Fourier.
- In altre parole, consistono di una componente statica (valore medio) e di un numero infinito di componenti sinusoidali dette armoniche, che hanno ampiezza e fase.
- Le frequenze delle componenti armoniche sono tutte multipli interi della frequenza fondamentale.
- Le frequenze 60, 75 e 100 hanno come massimo comune divisore 5.
- La frequenza fondamentale è *f1* = 5 Hz.
- Sono nulle tutte le ampiezze corrispondenti alle armoniche superiori, tranne quelle corrispondenti alle armoniche numero 12, 15 e 20.
- Una funzione periodica può quindi essere espressa come una somma di termini sinusoidali aventi frequenze multiple di una frequenza fondamentale. Reciprocamente, la somma di più termini sinusoidali, le cui frequenze siano tra loro commensurabili, ossia i cui rapporti siano numeri razionali, dà luogo ad una funzione periodica.
- Così, nell'esempio sopra riportato, i rapporti 60/75, 60/100 e 75/100 sono tutti numeri razionali.

$$x(t) = 1.0\sin(2\pi 60t) + 0.8\sin(2\pi 75t) + 0.5\sin(2\pi 100t)$$

Segnali periodici





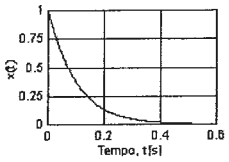
$$x(t) = 1.0\sin(2\pi 60t) + 0.8\sin(2\pi 75t) + 0.5\sin(2\pi 100t)$$

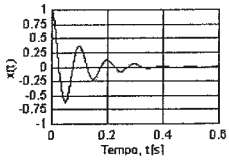
Segnali quasi periodici e transitori

- Si consideri, ora, la seguente funzione:

$$x(t) = X_1 \sin(2t + \vartheta_1) + X_2 \sin(\sqrt{2}\,t + \vartheta_2)$$

- Il rapporto $2/\sqrt{2}$ è irrazionale e pertanto il periodo della fondamentale risulta infinitamente lungo.
- Questi dati sono definiti come ***quasi periodici*** in quanto le frequenze delle componenti non sono multipli interi di una frequenza fondamentale.
- I dati non periodici diversi da quelli definiti come quasi periodici sono indicati come ***transitori***.
- Esempi di dati transitori sono mostrati in figura





2.2 Serie di Fourier

- Come è noto, una funzione $x(t)$ periodica di periodo T si può rappresentare mediante la serie di Fourier:

$$x(t) = X_0 + X_1 \cos(2\pi f_1 t + \varphi_1) + X_2 \cos(2\pi 2 f_1 t + \varphi_2) + \ldots + X_1 \cos(2\pi n f_1 t + \varphi_n)$$

$$x(t) = X_0 + \sum_{n=1}^{\infty} X_n \cos(2\pi n f_1 t + \varphi_n)$$

- f_1 è la frequenza fondamentale (frequenza dell'armonica fondamentale, che ha ampiezza X_1)
- X_0 è il valore medio di $x(t)$
- X_n è l'ampiezza della n–esima armonica, di frequenza nf_1
- φ_n è la fase della n–esima armonica

- La funzione deve essere assolutamente integrabile: $\int_0^T |x(t)|dt < \infty$

- Il termine corrispondente ad n=1 è detto fondamentale.
- Quelli di ordine superiore (n > 1) sono detti armoniche.

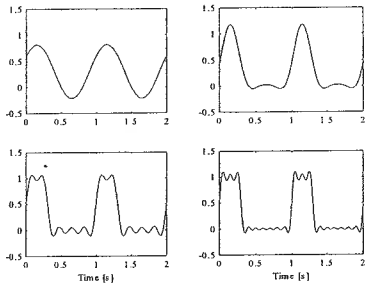
- Si è riportata la notazione più usata, cioè quella solo in coseno ma, naturalmente, si può trovarla anche solo in seno, in seno e coseno, o nella forma esponenziale.

Serie di Fourier

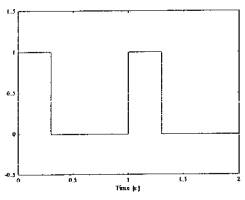
- Se si ha una funzione periodica, effettuarne l'analisi di Fourier significa ricavare le ampiezze X_n e le fasi φ_n.

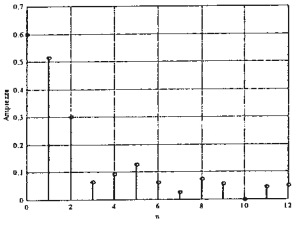
- Si può pensare di compiere l'analisi di Fourier con un filtro che abbia la caratteristica di lasciar passare solo le componenti comprese tra una certa frequenza f^* e la f^* più un certo incremento.

- La serie dei valori dei coefficienti X_n e φ_n costituiscono lo ***spettro*** della funzione $x(t)$ e ne forniscono una rappresentazione nel dominio delle frequenze.



Fenomeno di Gibbs





2.3 Trasformata di Fourier

- Per una funzione $x(t)$ non periodica, con la condizione che l'integrale da $-\infty$ a $+\infty$ del valore assoluto di $x(t)$ sia una quantità finita, al posto della serie si definisce la Trasformata di Fourier:

$$X(f) = \mathbb{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-i2\pi f t} dt$$

- La trasformata di Fourier è una funzione complessa, per cui si rappresenta con la parte reale e la parte immaginaria:

$$X(f) = \Re[X(f)] + i\Im[X(f)]$$

- Oppure mediante modulo e fase:

$$X(f) = |X(f)| e^{i\Phi(f)}$$

$$|X(f)| = \sqrt{\Re[X(f)]^2 + \Im[X(f)]^2} \qquad tg[\Phi(f)] = \frac{\Im[X(f)]}{\Re[X(f)]}$$

Trasformata inversa di Fourier

- Si definisce inoltre *trasformata* ***inversa*** *di Fourier* la seguente funzione:

$$x(t) = \mathbb{F}^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f)\, e^{i2\pi f t} df$$

- La trasformata inversa di Fourier esprime il fatto che ogni funzione $x(t)$ può essere descritta da un integrale che rappresenta il contributo di componenti armoniche aventi uno spettro di frequenza continuo da $-\infty$ a $+\infty$. Si può anche dire che la quantità $X(f)df$ rappresenta il contributo a $x(t)$ delle armoniche comprese nell'intervallo da f a $f+df$.

2. 4 Proprietà della Trasformata di Fourier

- Trasformata di Fourier della convoluzione di due funzioni

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

$$Y(f) = H(f)X(f)$$

$$Y(f) = \int_{-\infty}^{+\infty} y(t)e^{-j2\pi ft}dt = \int_{-\infty}^{+\infty}(\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau)e^{-j2\pi ft}dt =$$

$$= \int_{-\infty}^{+\infty}(\int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau)e^{-j2\pi f(t+\tau-\tau)}dt =$$

$$= \int_{-\infty}^{+\infty} h(\tau)e^{-j2\pi f\tau}d\tau \int_{-\infty}^{+\infty} x(t-\tau)e^{-j2\pi f(t-\tau)}d(t-\tau) =$$

$$= H(f)X(f)$$

Proprietà della Trasformata di Fourier

- Trasformata di Fourier del prodotto di due funzioni

$$y(t) = s(t)w(t)$$

$$\mathbf{F}\{s(t)w(t)\} = \int_{-\infty}^{\infty} S(g)W(f-g)dg$$

- Infatti, essendo:

$$\mathbf{F}\{y(t)\} = \mathbf{F}\{s(t)w(t)\} = \int_{-\infty}^{\infty} s(t)w(t)e^{-j2\pi ft}dt$$

- si ha:

$$\mathbf{F}\{y(t)\} = \iint_{-\infty}^{\infty} S(f_1)W(f_2)e^{-j2\pi(f-f_1-f_2)t}dt\,df_1df_2 =$$

$$= \int\int_{-\infty}^{\infty} S(f_1)W(f_2)\delta(f-f_1-f_2)df_1df_2 =$$

$$= \int_{-\infty}^{\infty} S(f_1)W(f-f_1)df_1$$

CHARACTERISTICS OF VIBRATION AND SHOCK

2.1. PERIODIC VIBRATION

Periodic vibration may be looked upon as an oscillating motion of a particle, or body, about a reference position, the motion repeating itself exactly after certain periods of time. The simplest form of periodic vibration is the so-called harmonic motion which when plotted as a function of time, is represented by a sinusoidal curve, Fig.2.1. Here T is the period of vibration, i.e. the time elapsed between two successive, exactly equal conditions of motion.

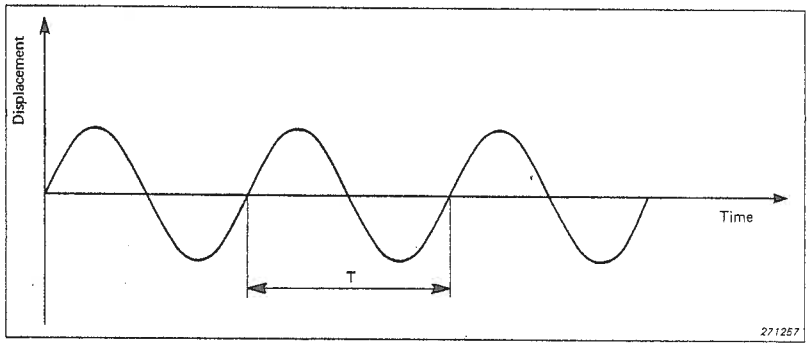


Fig.2.1. Example of a pure harmonic (sinusoidal) vibration signal

The *frequency* of the vibration is given by:

$$f = \frac{1}{T} \qquad (2.1)$$

Turning to the magnitude of the vibration this may be characterized by different quantities, all of which have definite mathematical relationships to each other *as long as harmonic motion is considered.*

If the vibration has the form of a pure translational oscillation along one axis (x) only, the instantaneous *displacement* of the particle (or body) from the reference position can be mathematically described by means of the equation:

$$x = X_{peak}\, sin\left(2\pi\frac{t}{T}\right) = X_{peak}\, sin\,(2\pi ft) = X_{peak}\, sin\,(\omega t) \qquad (2.2)$$

where

$\omega = 2\pi f$ = angular frequency

X_{peak} = Maximum displacement from the reference position

t = time

As the *velocity* of a moving particle (or body) is the time rate of change of the displacement, the motion can also be described in terms of velocity (v):

$$v = \frac{dx}{dt} = \omega\, X_{peak}\, cos\,(\omega t) = V_{peak}\, cos\,(\omega t) = V_{peak}\, sin\,(\omega t + \pi/2) \qquad (2.3)$$

Finally, the *acceleration (a)* of the motion is the time rate of change of the velocity:

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -\omega^2\, X_{peak}\, sin\,(\omega t) = -A_{peak}\, sin\,(\omega t) = A_{peak}\, sin\,(\omega t + \pi) \qquad (2.4)$$

From the above equations it can be seen that the form and period of vibration remain the same whether it is the displacement, the velocity or the acceleration that is being studied. However, the velocity leads the displacement by a phase angle of 90° ($\pi/2$) and the acceleration again leads the velocity by a phase angle of 90° ($\pi/2$). As characterizing values for the magnitude the peak values have been used, i.e. X_{peak} , V_{peak} and A_{peak} . The magnitude description in terms of peak values is quite useful as long as pure harmonic vibration is considered because it applies directly in the equations given above. If, on the other hand, more complex vibrations are being studied other descriptive quantities may be preferred. One of the reasons for this is that the peak value describes the vibration in terms of a quantity which depends only upon an instantaneous vibration magnitude regardless of the time history producing it.

A further descriptive quantity, which does take the time history into account, is the *average absolute* value, defined as (see also Fig.2.2):

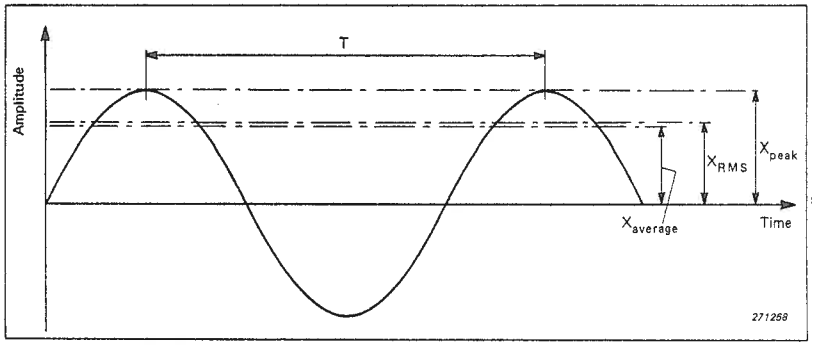


Fig.2.2. Example of a harmonic vibration signal with indication of the peak, the RMS and the average absolute value

$$X_{Average} = \frac{1}{T}\int_0^T |x|\,dt$$

Even though this quantity takes into account the time history of the vibration over one period (T) it has been found to be of limited practical interest. A much more useful descriptive quantity which also takes the time history into account, is the *RMS (root mean square)* value (Fig.2.2):

$$X_{RMS} = \sqrt{\frac{1}{T}\int_0^T x^2(t)\,dt} \qquad (2.5)$$

The major reason for the importance of the RMS-value as a descriptive quantity is its simple relationship to the power content of the vibrations.

For a *pure harmonic motion* the relationship between the various values is:

$$X_{RMS} = \frac{\pi}{2\sqrt{2}} X_{Average} = \frac{1}{\sqrt{2}} X_{peak}$$

A more general form of these relationships may be given by:

$$X_{RMS} = F_f\, X_{Average} = \frac{1}{F_c} X_{peak} \qquad (2.6)$$

or

$$F_f = \frac{X_{RMS}}{X_{Average}}; \qquad F_c = \frac{X_{peak}}{X_{RMS}}$$

The factors F_f and F_c are called "form-factor" and "crest-factor", respectively, and give some indication of the waveshape of the vibrations being studied.

For pure harmonic motion:

$$F_f = \frac{\pi}{2\sqrt{2}} = 1.11\ (\approx 1\ \text{dB})$$

and

$$F_c = \sqrt{2} = 1.414\ (= 3\ \text{dB})$$

Most of the vibrations encountered in daily life are not pure harmonic motions even though many of them may be characterized as periodic. A typical non-harmonic periodic motion is shown in Fig.2.3 (piston acceleration of a combustion engine). By determining the Peak, Average Absolute and RMS-value of this vibration as well as the form-factor and crest-factor a lot of useful information is obtained, and it can be clearly concluded that the motion is not harmonic. However, it will be practically impossible, on the basis of this information, to predict all the various effects that the vibration might produce in connected structural elements. Other methods of description must therefore be used.

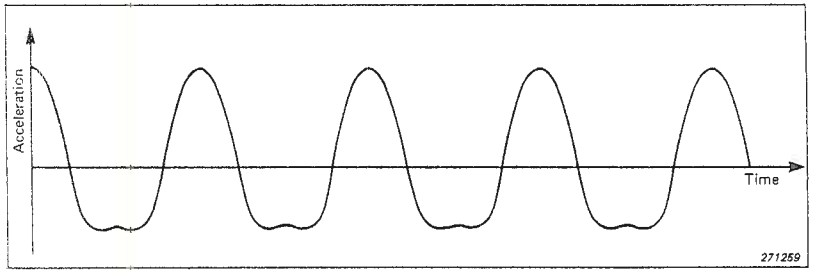


Fig.2.3. Example of a non-harmonic periodic motion (piston acceleration of a combustion engine)

One of the most powerful descriptive methods is the method of frequency analysis. This is based on a mathematical theorem, first formulated by FOURIER, which states that any periodic curve, no matter how complex, may be looked upon as a combination of a number of pure sinusoidal curves with harmonically related frequencies.

$$f(t) = X_0 + X_1 \sin(\omega t + \varphi_1) + X_2 \sin(2\omega t + \varphi_2) + X_3 \sin(3\omega t + \varphi_3) + \ldots + X_n \sin(n\omega t + \varphi_n) \qquad (2.7)$$

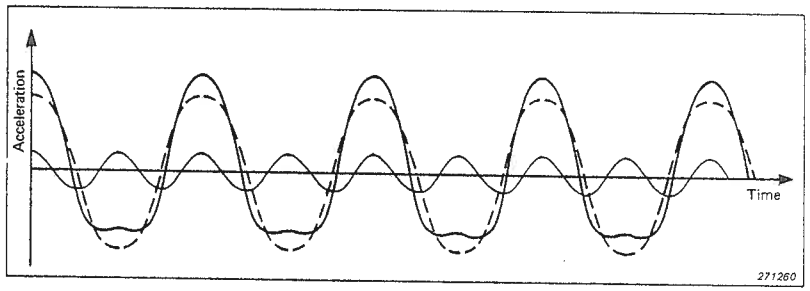


Fig.2.4. Illustration of how the waveform shown in Fig.2.3 can be "broken up" into a sum of harmonically related sinewaves

The number of terms required may be infinite, but in that case as the number of elements in the series is increased it becomes an increasingly better approximation to the original curve. The various elements constitute the *vibration frequency spectrum.* In Fig.2.4 the nonharmonic periodic motion of Fig.2.3 is redrawn together with the two most important harmonic curves representing its frequency spectrum. A somewhat more convenient method of representing this spectrum is shown in Fig.2.5 b, while Fig.2.6 shows some further examples of periodic time functions and their frequency spectra. A specific feature of periodic vibrations, which becomes clear by looking at Fig.2.5 and 2.6 is that their spectra consist of *discrete lines* when presented

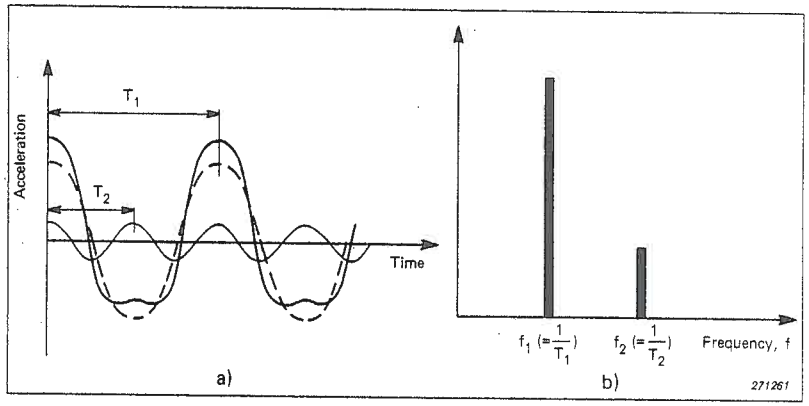


Fig.2.5. Illustration of how the signal, Fig.2.3 can be described in terms of a frequency spectrum
a) Description in the time domain
b) Description in the frequency domain

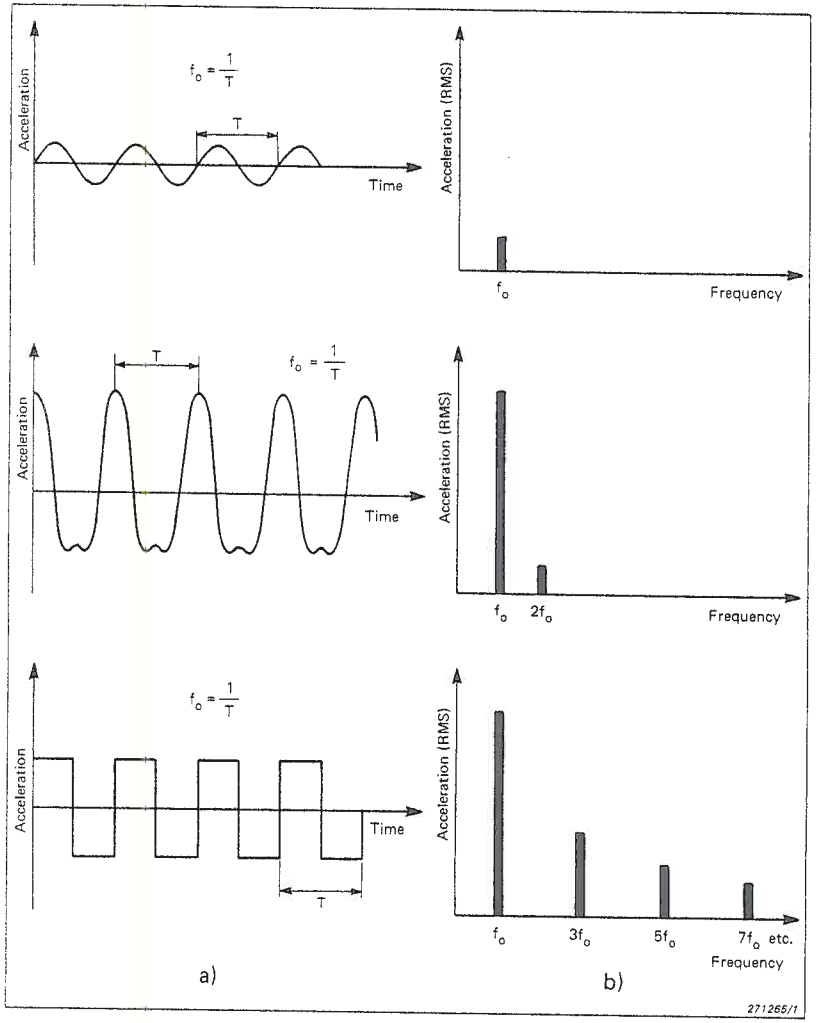


Fig.2.6. Examples of periodic signals and their frequency spectra
a) Descriptions in the time domain
b) Descriptions in the frequency domain

in the so-called frequency domain (Figs.2.5 b and 2.6 b). This is in contrast to random vibrations which show continuous frequency spectra (section 2.2, Fig.2.12).

2.2. STATIONARY RANDOM VIBRATION

Random vibrations are met rather frequently in nature and may be characterized as vibratory processes in which the vibrating particles undergo irregular motion cycles that never repeat themselves exactly, see Fig.2.7. To obtain a complete description of the vibrations, an infinitely long time record is thus theoretically necessary. This is of course an impossible requirement, and finite time records would have to be used in practice. Even so, if the time record becomes too long it will also become a very inconvenient means of description and other methods have therefore been devised and are commonly used. These methods have their origin in statistical mechanics and communication theory and involve concepts such as amplitude probability distributions and probability densities, and continuous vibration frequency spectra in terms of mean square spectral densities*.

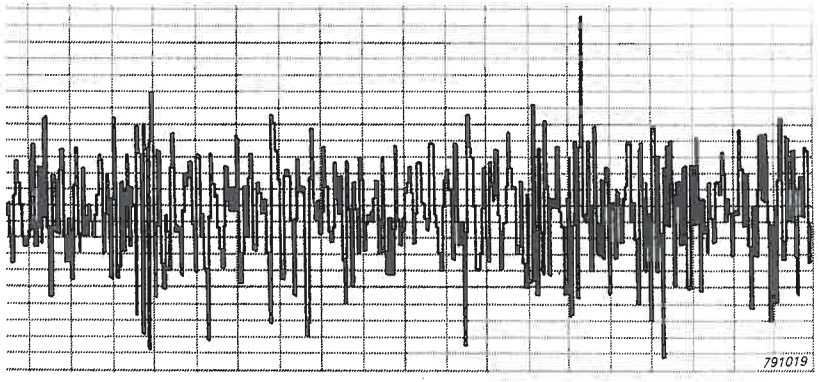


Fig.2.7. Example of a random vibration signal

Without going into too much mathematical detail the meaning of the above concepts should be briefly reviewed because of their importance in relation to practical vibration measurements.

The concept of probability is of a mathematical origin and denotes the chance of a particular event happening. If the event in question is absolutely certain to happen the probability of occurrence of the event is said to be 1. On the other hand, if the event in question is certain *not* to happen the probability of occurrence is said to be 0. Thus probabilities are, in the sense used here, positive real numbers between 1 and 0.

* Mean square spectral density is also often termed "Power Spectral Density" (P.S.D.) because the mean square is a quantity proportional to power.

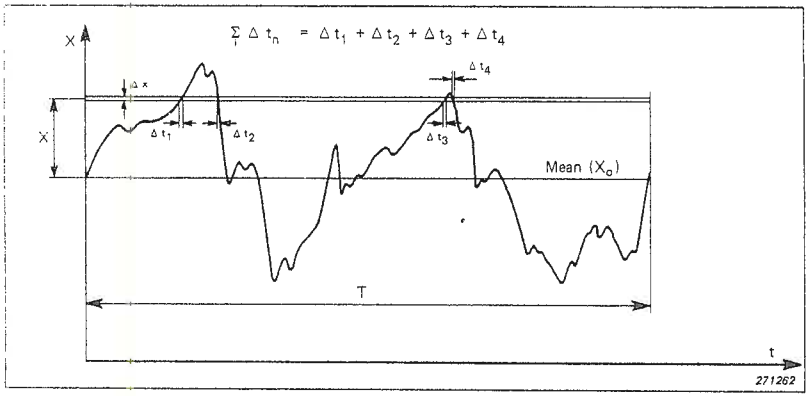


Fig.2.8. Sketch illustrating the concepts of probability and probability density

In the study of continuous processes such as stationary* random vibrations it is often convenient to use the concept of probability *density* instead of probability. Physically the probability density can be defined as the probability of finding instantaneous amplitude values within a certain amplitude interval, Δx, divided by the size of that interval (thus: density), see Fig.2.8. This means that while probabilities are dimensionless quantities the probability density is a quantity having a certain dimension.

Mathematically formulated the probability density at some specified amplitude level, x, is:

$$p(x) = \lim_{\Delta x \to o} \frac{P(x) - P(x + \Delta x)}{\Delta x} \qquad (2.8)$$

Here $p(x)$ designates the probability density while $P(x)$ is the probability that any instantaneous amplitude value exceeds the level x and $P(x + \Delta x)$ is the probability of occurrence of instantaneous amplitude values exceeding the level $x+\Delta x$. By plotting the value of $p(x)$ for all values of x a probability density curve is obtained which has the feature that an integration of the curve from a value x_1 to a value x_2 immediately tells the probability of occurrence of instantaneous amplitude values within the interval $(x_2 - x_1)$, independent of the actual magnitude of x_1 and x_2. The presentation of experimental probability data in terms of probability density curves bears some advantages because it allows for a direct comparison of data between experiments (and between experimenters) independent of the width of the amplitude interval, Δx, used in the experiment. Finally, theoretical probability data are commonly presented

* Stationary random vibrations are defined as random vibrations whose *statistical* characteristics do not change with time.

in the form of probability density curves and this method of presentation must therefore be considered the most generally acceptable one.

From the definition of probability density it follows that by integrating the probability density curve over all possible amplitude values the magnitude of the integral will be 1 (because the probability of finding a certain amplitude value within all possible amplitude values is 1). The practical procedure involved in converting experimental and/or theoretical data into probability density data ensuring that the area under the probability density curve is 1, is called normalization. The most commonly known normalized probability density curve, the normal (Gaussian) curve, is shown in Fig.2.9.

Even though probability density data are very useful signal descriptions and give excellent information on how, on the average, the instantaneous amplitudes in a vibratory signal are distributed, they give little or no information as to the time history or frequency content of the process being studied. To try and remedy this, and to obtain further descriptive data, statistical physicists introduced a function called the *autocorrelation function*, $\psi(\tau)$. This function describes (on the average) how a particular instantaneous amplitude value depends upon previously occurring instantaneous amplitude values in that $\psi(\tau)$ is defined as:

$$\psi(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) f(t+\tau)\, dt \qquad (2.9)$$

where $f(t)$ is the magnitude of the vibratory process at an arbitrary instant of

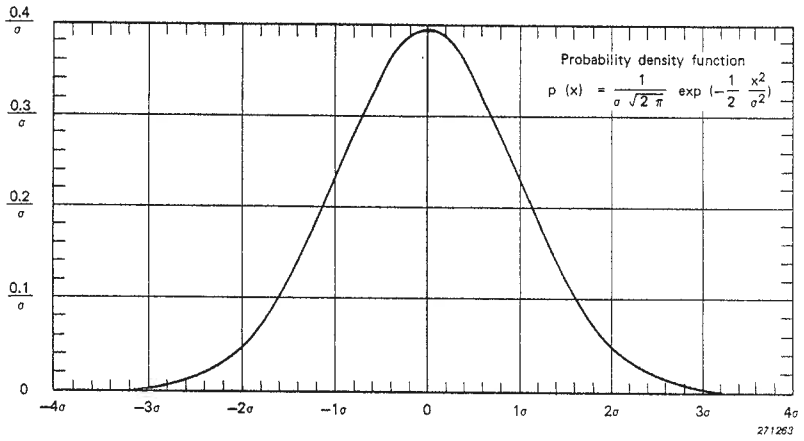


Fig.2.9. The normalized Gaussian probability density curve

time, t, and $f(t+\tau)$ designates the magnitude of the same process observed at a time, τ, later, see Fig.2.10.

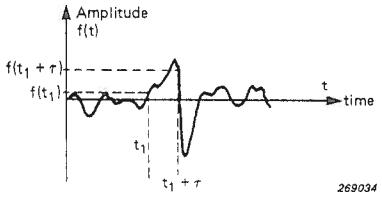


Fig.2.10. Basic concepts involved in deriving the autocorrelation function

In the case of an "ideal" stationary random process (white noise) the autocorrelation function would consist of an infinitely narrow impulse-function around zero ($\tau = 0$), see Fig.2.11 a), as in such a process each instantaneous amplitude value should be completely independent of all other instantaneous amplitude values.

However, in practice the autocorrelation functions associated with stationary random vibrations cluster around $\tau = 0$, but are never "infinitely narrow" impulse-functions, Fig.2.11 b) and c). The reason for this spreading out of the curve around zero is that all practical random processes are frequency limited, and the narrower the frequency limits the more spread-out are the corresponding autocorrelation functions (because the rate at which a signal can change from its current value is much more limited).

From the autocorrelation function another, very important function in practice, can be deduced, which has a certain resemblance to the Fourier frequency spectra described in section 2.1 for periodic vibrations. This function has been termed the *mean square spectral density function* (power spectral density function) and can be derived from the autocorrelation function as follows: Assuming that the integral of $\psi(\tau)$ from $-\infty$ to $+\infty$ is finite (see Fig.2.11) one can write:

$$S(f) = \int_{-\infty}^{\infty} \psi(\tau)\, e^{-j2\pi f\tau}\, d\tau \qquad (2.10)$$

where f is frequency.

From the theory of Fourier integrals it is furthermore known that $\psi(\tau)$ can also be found from the above integral by inversion:

$$\psi(\tau) = \int_{-\infty}^{\infty} S(f)\, e^{j2\pi f\tau}\, df \qquad (2.11)$$

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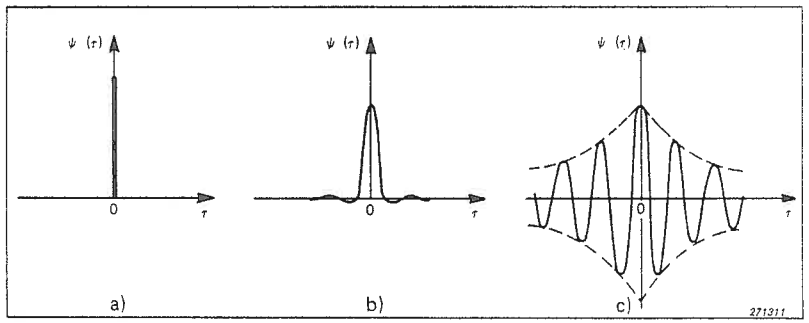

Fig.2.11. Examples of autocorrelation functions
a) Autocorrelation function for an ideal stationary random process containing frequencies from 0 to ∞ (constant spectral density)
b) Autocorrelation function for a "practical" wide band stationary random process
c) Autocorrelation function for a narrow band stationary random process

The Fourier integral relations between $\psi(\tau)$ and $S(f)$ are often called the Wiener-Khinchin relations and play a very important role in the theory of random processes.

In physically realizable stationary processes one operates with positive frequencies only* and $\psi(\tau) = \psi(-\tau)$ whereby the integral for $\psi(\tau)$ becomes:

$$\psi(\tau) = 2\int_0^\infty S(f)\cos(2\pi f\tau)\,df$$

or, if a function $G(f)$ is defined so that

$$G(f) = 2S(f) \qquad \text{for } f>0$$

then

$$\psi(\tau) = \int_0^\infty G(f)\cos(2\pi f\tau)\,df \tag{2.12}$$

* Note that frequency can be interpreted as rate of change of phase, in which case the concept of positive and negative frequencies is meaningful. A 2-sided frequency domain is useful analytically because of symmetry with the time domain, but in practical measurements it is most common to combine positive and negative frequency contributions to obtain a one-sided power spectrum. For a more detailed discussion see the B & K book "Frequency Analysis".

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To interpret the function $G(f)$ consider the case where $\tau = 0$:

$$\psi(o) = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)f(t+o)\,dt = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f^2(t)\,dt$$

and

$$\psi(o) = \int_0^\infty G(f)\,df$$

thus

$$\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f^2(t)\,dt = \int_0^\infty G(f)\,df \tag{2.13}$$

Both of these integrals are measures of the power involved in the process, one in terms of the process time function, $f(t)$, and the other in terms of a frequency function, $G(f)$. Because of the squaring involved in the above time function description, $G(f)$ has been designated as the *mean square spectral density function* (or power spectral density function).

Traditionally, power spectra have been measured using analog frequency analyzers whose mode of operation may be understood as follows:

An ideal analog frequency analyzer will allow only that part of the signal to be measured which has frequency components within a narrow frequency band, B, see Fig.2.12. Assuming that no attenuation or amplification of these frequency components takes place in the analyzer the signal which is passed on to its indicating arrangement is:

$$\int_0^\infty G(f)\,df = \int_f^{f+B} G(f)\,df = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f_B^2(t)\,dt$$

Here $f_B(t)$ is the above-mentioned part of the complete signal, $f(t)$, which has frequency components within B. If now B is made so small that $G(f)$ can be considered constant within this frequency range then

$$\int_f^{f+B} G(f)\,df = G(f)B$$

thus, in the limiting case when $B \to 0$, one obtains:

$$G(f) = \lim_{B\to 0}\lim_{T\to\infty}\frac{1}{BT}\int_{-\frac{T}{2}}^{\frac{T}{2}} f_B^2(t)\,dt \tag{2.14}$$

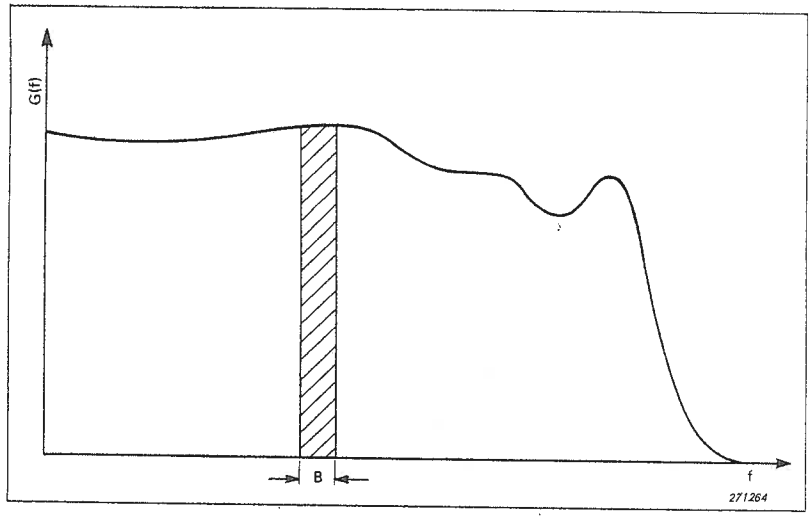

Fig.2.12. Determination of the mean square spectral density by means of ideal filters

This equation forms the basis of most analog experimental techniques used in the mean square spectral density analysis of random signals, although the actual bandwidth B used must of course be finite, in order that the measurement time T does not need to be infinite. The results (as a power spectral density) will only be valid, however, if B is sufficiently small that the above assumption is valid, i.e. that $G(f)$ is approximately constant within B. This will be the case for practical purposes if B is, say, less than 1/3 of the width of any peaks in the spectrum being measured.

At one time, a digital alternative to analog analysis was based on the already-mentioned Wiener-Khinchin relationship. The autocorrelation function was first calculated digitally, and this then Fourier transformed by digital evaluation of the Fourier integral. It is only in the last few years, however, that digital alternatives to analog analyzers have been competitive in the sense that they could be incorporated as hardware in a portable standalone unit. One of the major factors here has been the increasing speed and miniaturization of digital components in general, resulting in continually reducing costs for a given calculation. Another major factor has been the development of the so-called Fast Fourier Transform (FFT) algorithm which has typically allowed savings of 100:1 in digital evaluation of the Fourier integral. This has in fact meant that it is now quicker to calculate the autocorrelation function by inverse transformation of the power spectrum, the latter being obtained by Fourier transforming the time signal.

The FFT procedure produces a constant bandwidth spectrum, but for constant percentage bandwidth (where the filter bandwidth is a constant percentage of its centre frequency) another digital analysis technique known as recursive digital filtering is found to be better. The choice of appropriate analysis method is discussed in Chapter 7.

Before closing the discussion on methods used to describe and analyze random vibration phenomena some important "practical" facts should be pointed out:

Returning to the equation (2.13)

$$\lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f^2(t)\,dt = \int_0^\infty G(f)\,df$$

it can be seen that the expression on the left hand side of this equation has a close resemblance to the square of the expression previously used to define the RMS-value of a periodic vibration signal (Equation 2.5). This means that *the description of a complex signal in terms of its overall RMS-value is equally meaningful whether the signal has a periodic or a random character.*

When it comes to *spectral* description, however, *a periodic signal* may well be described in terms of *the RMS-values of its various components* (its frequency spectrum), while *random vibration signals* are best described in terms of *mean square spectral density functions*. This is due to the fact that random signals produce continuous frequency spectra and the RMS-value measured within a certain frequency band will therefore depend upon the width of the band. The detailed measurement evaluation techniques will, in view of the above, normally also differ, a fact which is more specially discussed in Chapter 7 of this book and in the B & K book "Frequency Analysis".

2.3. TRANSIENT PHENOMENA AND SHOCKS

Transient phenomena and mechanical shocks are, like random vibrations encountered relatively often in daily life. They may originate from such widely different releases of energy as rough handling of equipment, explosions and supersonic motion. However, common for this type of energy release is its short duration and sudden occurrence.

A simple shock may be defined as a transmission of kinetic energy to a system which takes place in a relatively short time compared with the natural period of oscillation of the system, while transient phenomena (also termed complex shocks) may last for several periods of vibration of the system.

Shocks and transient vibrations may be described in terms of force, acceleration, velocity or displacement and for a complete description it is necessary to obtain an exact time history record of the quantity in question.

In many cases the ultimate goal is not the waveform itself, but rather a means to estimate the effect that the corresponding shock or transient vibration would have on a certain mechanical system. A more useful method of description might then again be found in the form of Fourier analysis. If the time function for a shock is $f(t)$ then its Fourier transform is given by:

$$F(f) = \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi ft} dt \qquad (2.15)$$

The analogy between this expression and the mean square spectral density function of stationary random vibrations (Equation (2.10)) is readily seen. There is, however, a very distinct difference in that the mean square spectral density function for stationary random vibrations is the Fourier transform of an already time-averaged, even function, the autocorrelation function, with the dimensions of amplitude squared. In the above Fourier integral for transient or shock functions the function $f(t)$ itself must be time-limited and has the dimensions of amplitude only. Because it in general is not an even function, its Fourier transform will be complex, but it is found that the square of the amplitude of the Fourier transform at each frequency gives a measure of the *energy* distribution of the transient.

It may be useful to see how this difference in dimensions influences the units in a particular case. Assuming that signal amplitude is expressed in volts (V), then the autocorrelation function for a stationary random signal would have units of volts squared (V^2) or power. The Fourier transform of this has the units $V^2 s$, or V^2/Hz, i.e. power per unit frequency or *power spectral density*. A shock or transient function, however, has units of Volts and its Fourier transform (amplitude) units of Volt-seconds (Vs). The amplitude squared thus has units $V^2 s^2$ or $V^2 s/Hz$, i.e. energy per unit frequency or *energy spectral density*. A transient of course has finite energy when integrated over all time, while a stationary random signal would have infinite energy, though finite power.

Most analyzers assume a signal is continuous and give a result in terms of power per analysis bandwidth. The conversion of this to the correct units is discussed in Chapter 7 and in more detail in the B & K book "Frequency Analysis".

In Fig.2.13 various shock time functions and the amplitudes of their Fourier spectra are given. It is seen from the figure that in general a shock pulse contains energy spread over all frequencies from zero to infinity, and that the spectra are continuous with no discrete frequency components.

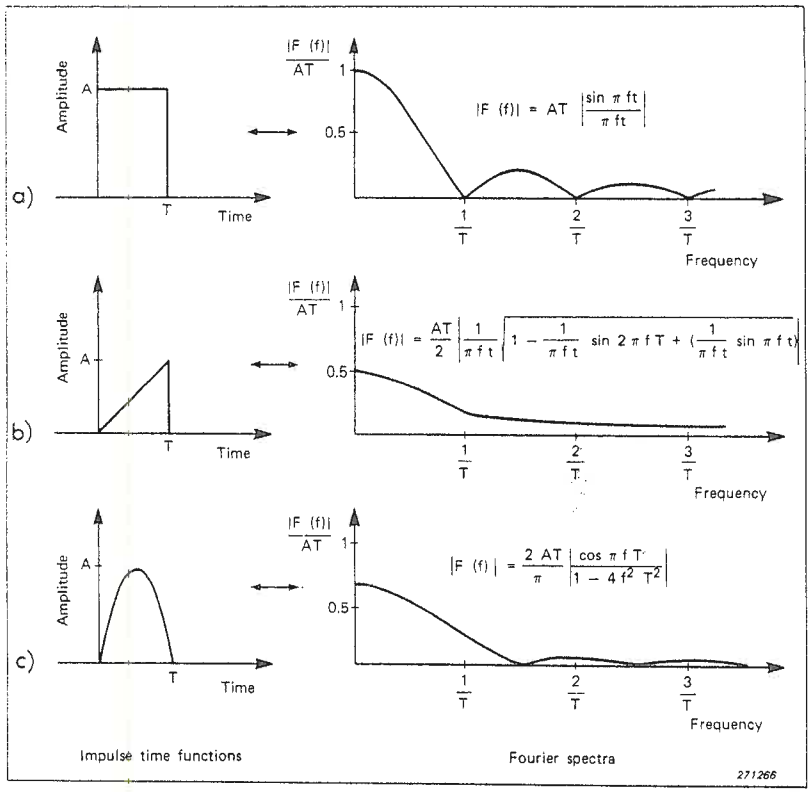

Fig.2.13. Example of shock time functions and their Fourier transforms (amplitude spectra)
a) A rectangular shock pulse
b) A final peak sawtooth shock pulse
c) A half-sine shock pulse

In the expressions for $F(f)$ given in the figure all the expressions within the parallel brackets approach unity as f goes to zero, so that at very low frequencies the magnitude of the spectrum component is equal to the area (amplitude-time integral) of the shock pulse, irrespective of the pulse shape. This fundamental relationship is of considerable practical importance, for example in shock testing. It means that so long as the shock pulse is short compared with the natural period of the mechanical system on which its acts, the severity of the shock is determined by the area of the shock pulse alone (see also Fig.3.13 b II).

In the case of transient phenomena the situation is somewhat different. Such phenomena, in the sense used in this book, may consist either of a single period "shock-wave", or of an oscillating transient. The Fourier spectrum function of a typical oscillating transient is shown in Fig.2.14 and it is seen that the magnitude of the spectrum components in this special case tends towards zero as the frequency f goes to zero. Also, a maximum magnitude of the spectrum is reached around f_o which corresponds roughly to the frequency of oscillation of the transient. This maximum is broader the quicker the transient phenomenon ceases.

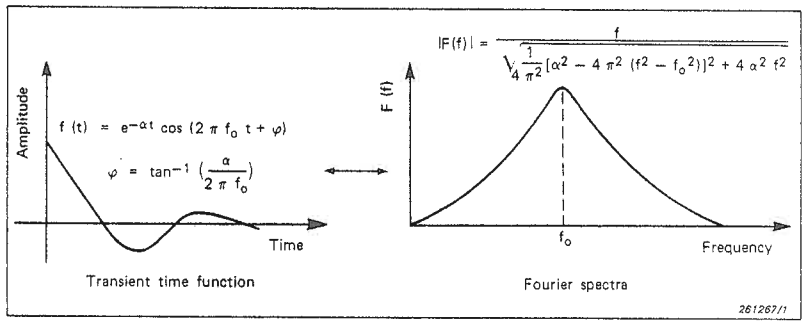

Fig.2.14. Example of an oscillating transient and its Fourier spectrum function

If the "transient" does not cease at all, i.e. when the "transient" is no longer a transient but a periodic phenomenon (in this case a harmonic vibration), the frequency spectrum degenerates into a discrete spectral line (infinitely narrow maximum at f_o).

2.4. NON-STATIONARY RANDOM VIBRATION

Theoretically all kinds of random vibrations encountered in practice are non-stationary because their statistical properties vary with time. However, from an engineering point of view this variation in statistical properties may be so slow, or of such a character, that many of the phenomena studied can be considered stationary in a practical sense.

Non-stationary random vibrations may therefore, in practice, be defined as random vibrations whose statistical properties vary with time within time intervals considered essential for their proper description. To analyze and describe such vibration data it is thus necessary to take their variation in statistical properties with time into account. A typical example of seriously non-stationary random vibrations is the vibrations induced in space vehicles during launch and re-entry.

To theoretically analyze non-stationary random vibrations properly it is necessary to introduce the concept of *ensemble averaging*. An ensemble average is an average taken over a large number (an ensemble) of repeated experiments, see Fig.2.15. As can be seen from the figure an ensemble average can be taken at any particular instant of time t_1, t_2, t_3 etc., and when the average values are plotted against time a more or less complete description of the vibration is obtained. There are, on the other hand, several reasons why this method of description is not very useful in practice. Firstly, it requires that the non-stationary process can be repeated a very large number of times. In the case of space vehicle launch and re-entry for instance this is not possible due to the cost of such experiments. Secondly, the amount of data necessary for a thorough description is so large that their proper measurement will pose serious problems.

It is therefore normally necessary to seek other methods of description, and in general some sort of time averaging is used. There are, however, certain limitations imposed upon this kind of time averaging in that the response and averaging time of the measurement equipment employed should preferably

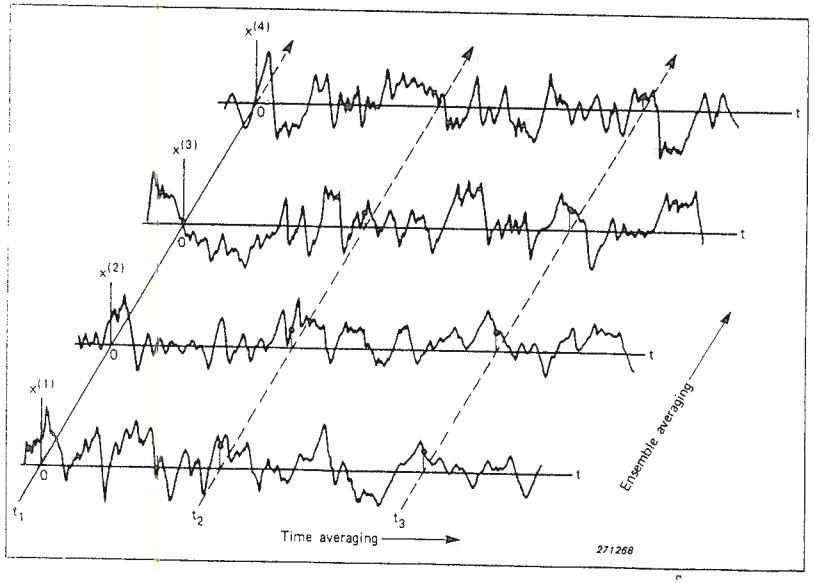

Fig.2.15. Illustration of an ensemble of random functions

be small relative to important time trends in the non-stationary data. This again may lead to considerable statistical uncertainties in the measurements.

Fig.2.16 illustrates some basic and important types of non-stationary random vibrations.

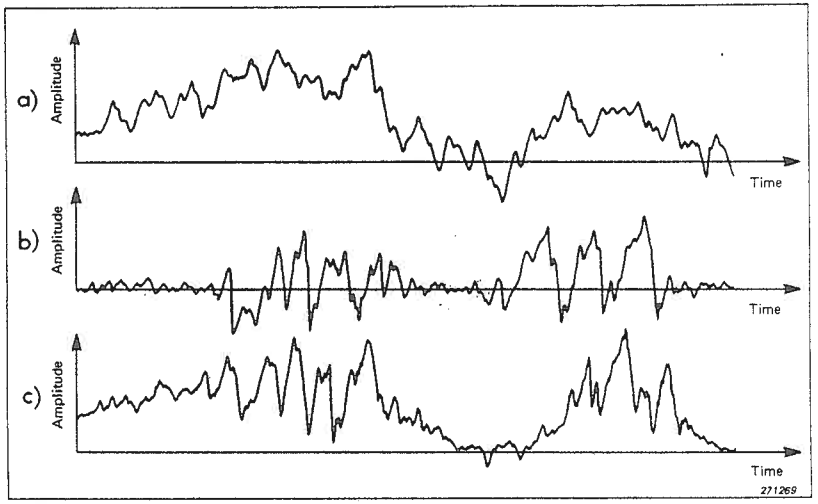

Fig.2.16. Examples of some basic types of nonstationary random vibrations
a) Time-varying mean value
b) Time-varying mean square value
c) Time-varying mean and mean square value

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