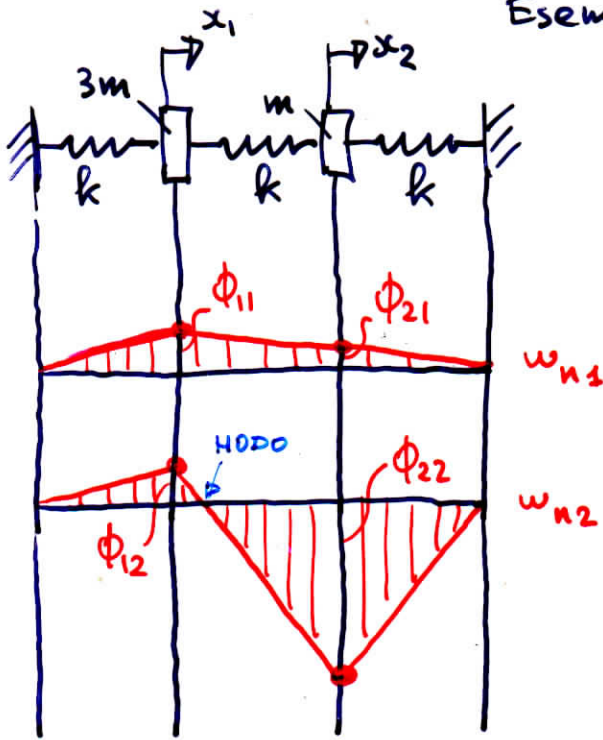


Esempio: Analisi modale e FRF  
di un sistema a 2 gdl

# MODELLO M-G-DL - VIBRAZIONI LIBERE

Esempio 2 G.D.L.



EQUAZIONI DEL MOTTO

$$\begin{cases} (3m)\ddot{x}_1 + 2kx_1 - kx_2 = 0 \\ m\ddot{x}_2 - kx_1 + 2kx_2 = 0 \end{cases}$$

$$\underline{M}\ddot{\underline{x}} + \underline{K}\underline{x} = \underline{0}$$

$$\underline{M} = \begin{bmatrix} 3m & 0 \\ 0 & m \end{bmatrix} ; \quad \underline{K} = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}$$

PROBLEMA AGLI AUTOVALORI

$$\omega_{n1} = 0,67 \sqrt{\frac{k}{m}} ; \quad \underline{\phi}_1 = \begin{Bmatrix} \phi_{11} \\ \phi_{21} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0,65 \end{Bmatrix} \cdot a_1$$

$R_1$

$$\omega_{n2} = 1,49 \sqrt{\frac{k}{m}} ; \quad \underline{\phi}_2 = \begin{Bmatrix} \phi_{12} \\ \phi_{22} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -4,65 \end{Bmatrix} \cdot a_2$$

$R_2$

SOLOZIONE GENERALE

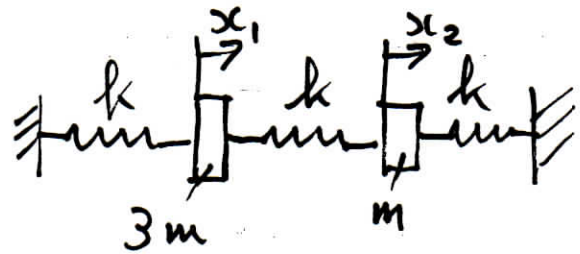
$$x_1(t) = a_1 \cos(\omega_{n1}t + d_1) + a_2 \cos(\omega_{n2}t + d_2)$$

$$x_2(t) = a_1 R_1 \cos(\omega_{n1}t + d_1) + a_2 R_2 \cos(\omega_{n2}t + d_2)$$

1° MODO

2° MODO

CONDITIONS INITIAL



If, at the instant  $t=0$ , one takes the initial conditions to be

$$x_1(0) = x_0$$

$$x_2(0) = 0$$

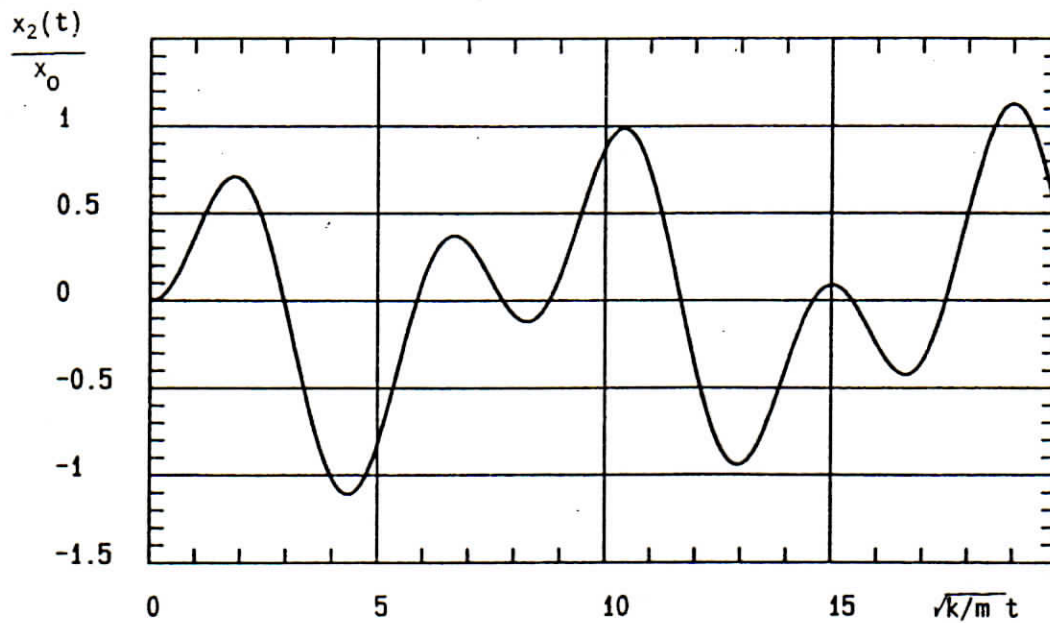
$$\dot{x}_1(0) = 0$$

$$\dot{x}_2(0) = 0$$

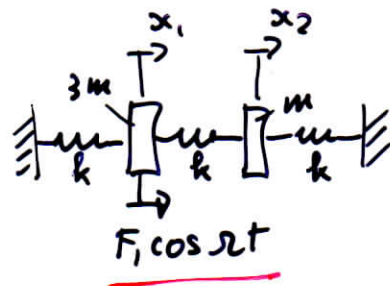
After completing this solution,

$$\begin{aligned} x_1(t) &= x_0(0.8780 \cos \omega_1 t + 0.1220 \cos \omega_2 t) \\ x_2(t) &= x_0(0.5669 \cos \omega_1 t - 0.5669 \cos \omega_2 t) \end{aligned} \quad (27)$$

Figure 2 shows  $x_2(t)/x_0$  as a function of  $t\sqrt{k/m}$ . Note that the solution is not sinusoidal.



# MODELLO MGD - VIBRAZIONI FORZATE METODO MODALE



$$\underline{M} \ddot{\underline{x}} + \underline{K} \underline{x} = \underline{F}$$

← TRASFORMAZIONE  
IN COORDINATE MODALI :  $\underline{x} = \underline{\Phi} \underline{q}$

MATRICE MODALE  $\underline{\Phi} = [\underline{\Phi}_1 \quad \underline{\Phi}_2] = \begin{bmatrix} 1 & 1 \\ R_1 & R_2 \end{bmatrix}$

$$\underbrace{\underline{\Phi}^T \underline{M} \underline{\Phi}}_{\underline{M}_p} \ddot{\underline{q}} + \underbrace{\underline{\Phi}^T \underline{K} \underline{\Phi}}_{\underline{K}_p} \underline{q} = \underline{\Phi}^T \underline{F}$$

← PROPRIETÀ DI ORTOGONALITÀ  
DEGLI AUTOVETTORI

$$\underline{M}_p \ddot{\underline{q}} + \underline{K}_p \underline{q} = \underline{\Phi}^T \underline{F}$$

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{bmatrix} 1 & R_1 \\ 1 & R_2 \end{bmatrix} \begin{Bmatrix} F_1 \cos \omega t \\ 0 \end{Bmatrix} = \begin{Bmatrix} F_1 \cos \omega t \\ F_1 \cos \omega t \end{Bmatrix}$$

$$\begin{cases} \ddot{q}_1 + \omega_{n1}^2 q_1 = F_1/M_1 \cos \omega t \\ \ddot{q}_2 + \omega_{n2}^2 q_2 = F_1/M_2 \cos \omega t \end{cases}$$

$$\omega_{n1}^2 = \frac{K_1}{M_1} ; \quad \omega_{n2}^2 = \frac{K_2}{M_2}$$

SOLUZIONE :  $\begin{cases} q_1 = Q_1 \cos \omega t \\ q_2 = Q_2 \cos \omega t \end{cases}$

con  $Q_1 = \frac{F_1/M_1}{\omega_{n1}^2 - \omega^2}$

$$Q_2 = \frac{F_1/M_2}{\omega_{n2}^2 - \omega^2}$$

RICORDANDO LA TRASFORMAZIONE DI COORDINATE :  $\underline{x} = \underline{\Phi} \underline{q}$

$$x_1 = q_1 + q_2 = \left( \frac{1/M_1}{\omega_{n1}^2 - \omega^2} + \frac{1/M_2}{\omega_{n2}^2 - \omega^2} \right) F_1 \cos \omega t = a_1 \cos \omega t$$

$$x_2 = q_1 \cdot R_1 + q_2 \cdot R_2 = \left( \frac{R_1/M_1}{\omega_{n1}^2 - \omega^2} + \frac{R_2/M_2}{\omega_{n2}^2 - \omega^2} \right) F_1 \cos \omega t = a_2 \cos \omega t$$

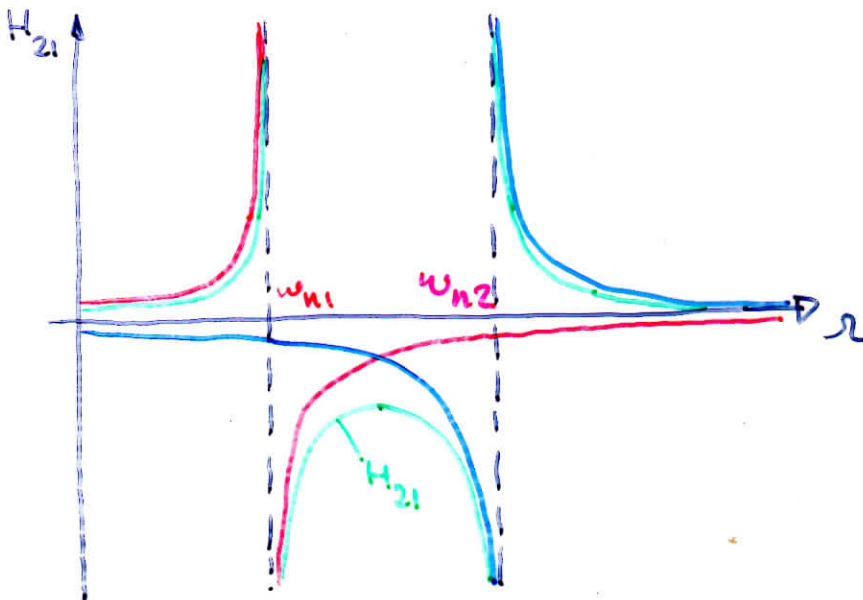
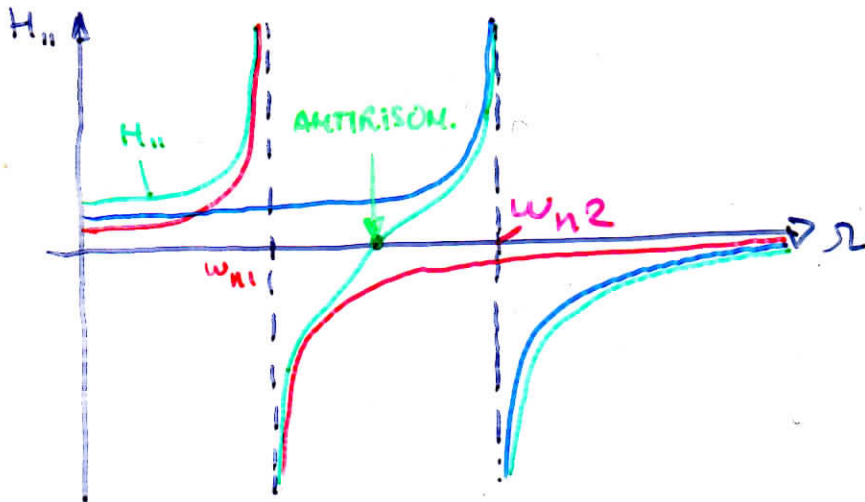
FRF

1<sup>st</sup> MODE

2<sup>nd</sup> MODE

$$H_{11} = \frac{a_1}{F_1} = \frac{1/M_1}{\omega_{n1}^2 - \omega^2} + \frac{1/M_2}{\omega_{n2}^2 - \omega^2}$$

$$H_{21} = \frac{a_2}{F_1} = \frac{R_1/M_1}{\omega_{n1}^2 - \omega^2} + \frac{R_2/M_2}{\omega_{n2}^2 - \omega^2}$$



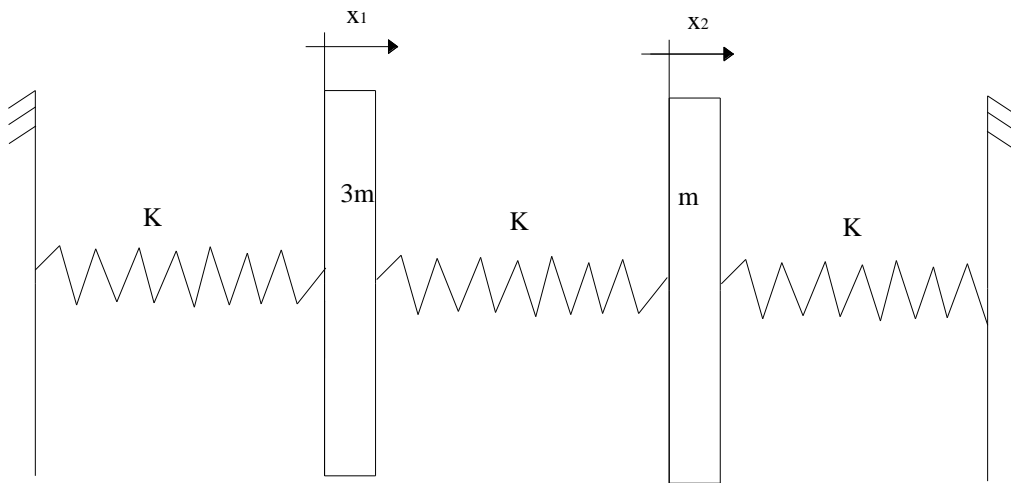
## Esempio: FRF di un sistema a 2 gdl

Per il sistema a due gradi di libertà in Figura 1, le equazioni del moto sono:

$$\begin{cases} 3m\ddot{x}_1 + 2kx_1 - kx_2 = 0 \\ m\ddot{x}_2 - kx_1 + 2kx_2 = 0 \end{cases}$$

E' pertanto possibile ottenere le matrici massa e rigidezza  $M$  ,  $K$  :

$$M = \begin{pmatrix} 3m & 0 \\ 0 & m \end{pmatrix}; K = \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}$$



**Figura 1. Sistema a 2 g.d.l.**

Dato  $m=85 \text{ kg}$  e  $k=1e6 \text{ N/m}$ , di seguito alcuni risultati in termini di modi frequenze naturali e FRF.

	<b>Frequenza [Hz]</b>
<b>Primo modo</b>	11.6
<b>Secondo modo</b>	25.7

**Tabella 1. Frequenze naturali .modi di vibrare.**

<b>Primo modo</b>	<b>Secondo modo</b>
$\Phi_{11} = 1$	$\Phi_{12} = 1$
$\Phi_{21} = 0.65$	$\Phi_{22} = -4.65$

**Tabella 2. Modi di vibrare normalizzati**

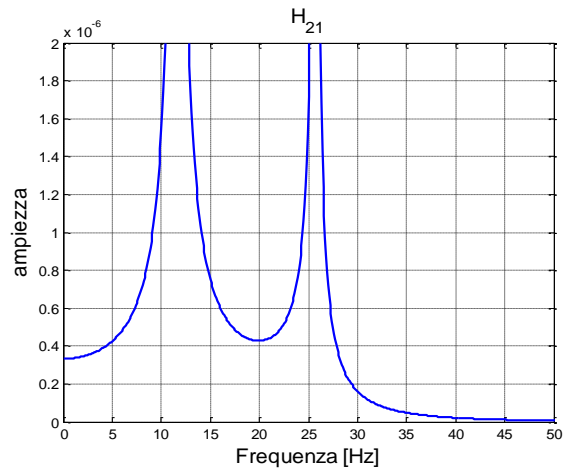
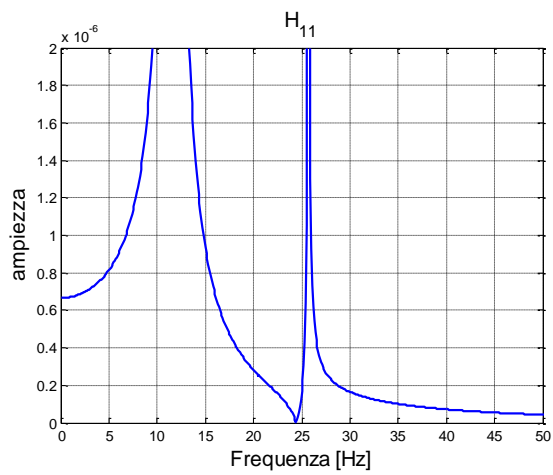


Figura 2. Ampiezza [m/N] della  $H_{11}$  e  $H_{21}$  per il sistema non smorzato.

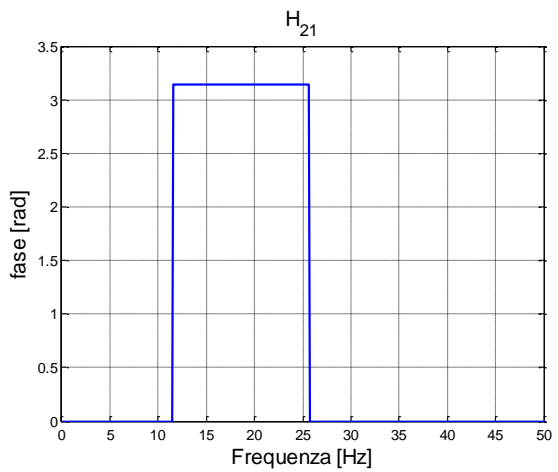
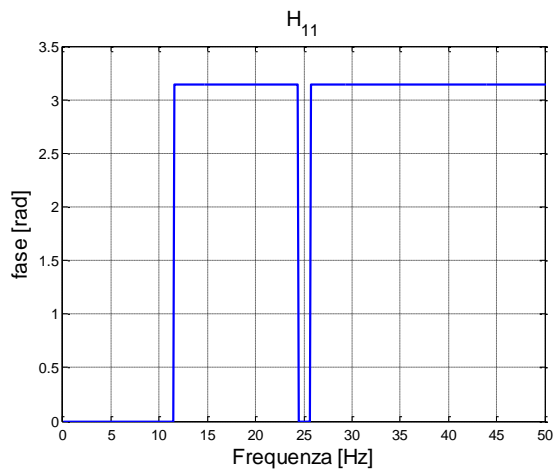
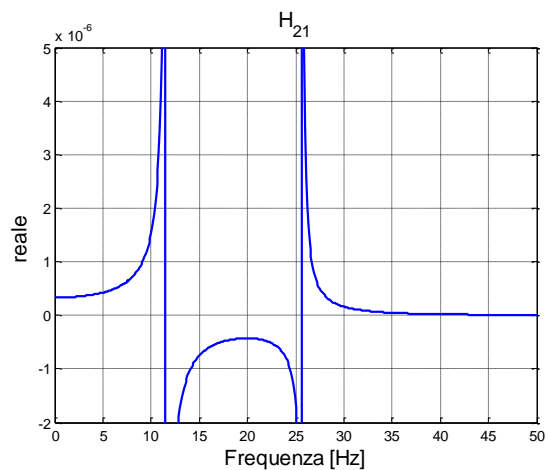
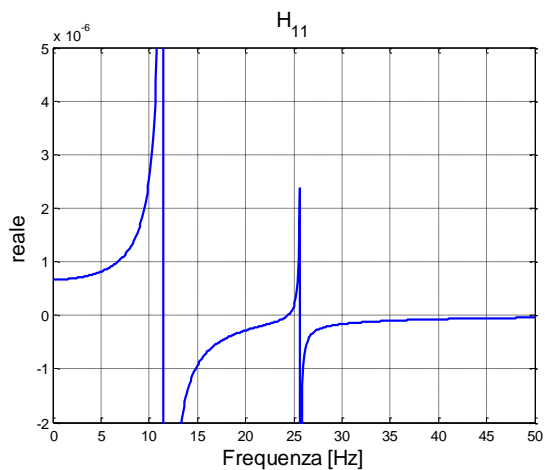
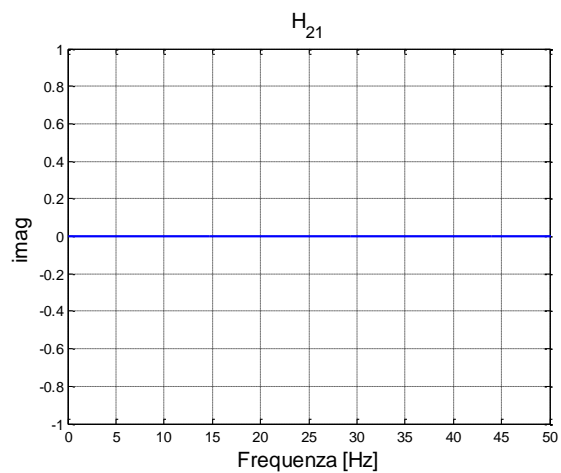
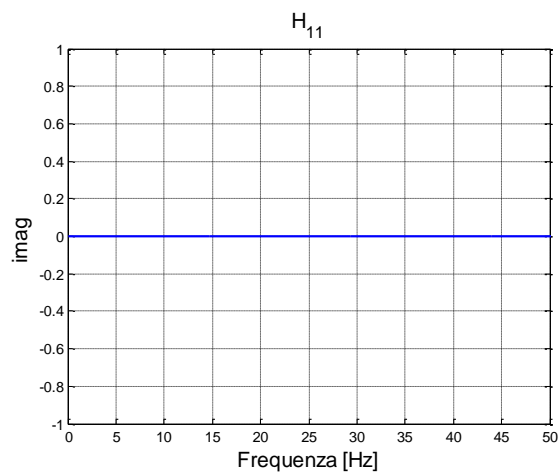


Figura 3. Fase [rad] della  $H_{11}$  e  $H_{21}$  per il sistema non smorzato.



**Figura 4. Parte reale [m/N] della  $H_{11}$  e  $H_{21}$  per il sistema non smorzato.**



**Figura 5. Parte immaginaria [m/N] della  $H_{11}$  e  $H_{21}$  per il sistema non smorzato.**



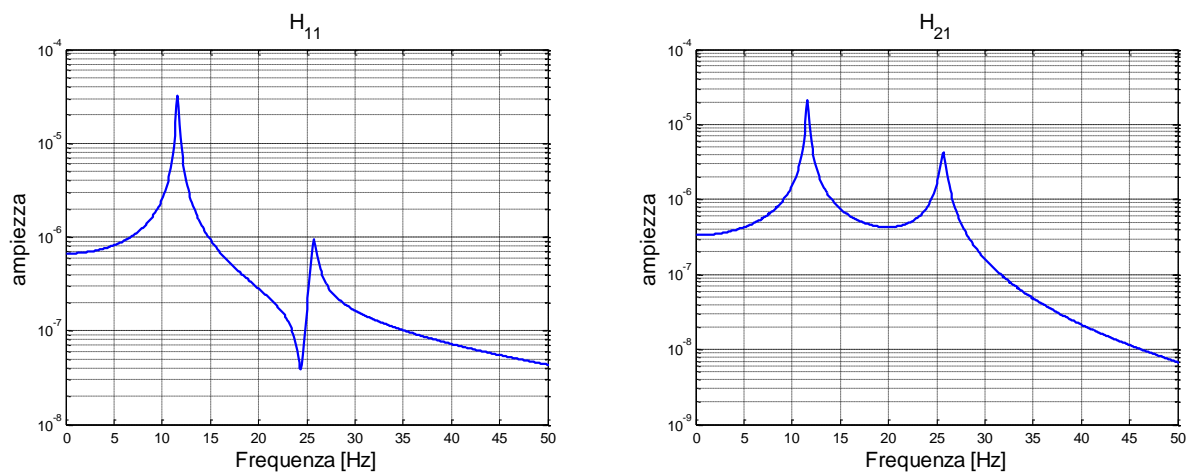


Figura 6. Ampiezza [m/N] della  $H_{11}$  e  $H_{21}$  per il sistema con smorzamento  $\zeta_1 = \zeta_2 = 0.01$ .

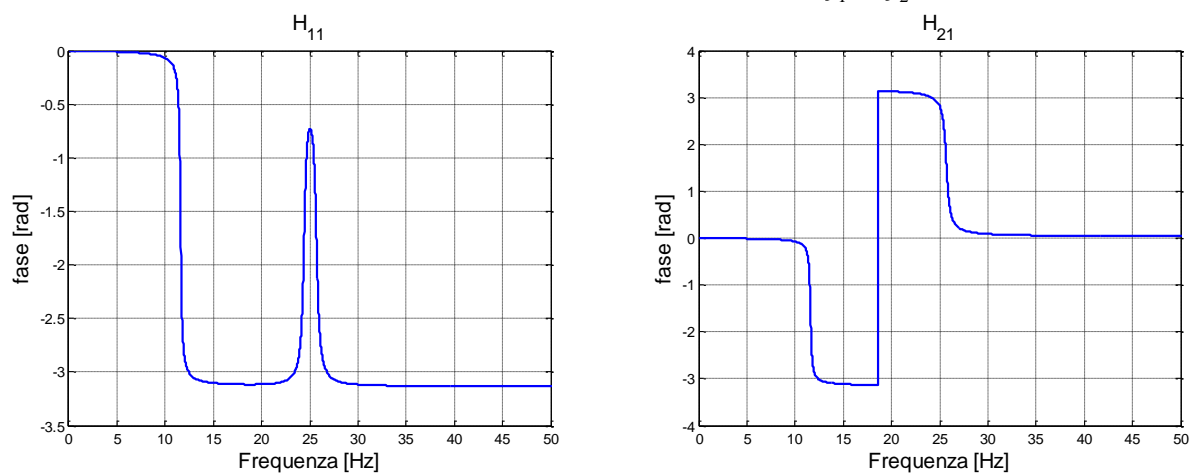


Figura 7. Fase della  $H_{11}$  e  $H_{21}$  per il sistema con smorzamento  $\zeta_1 = \zeta_2 = 0.01$ .

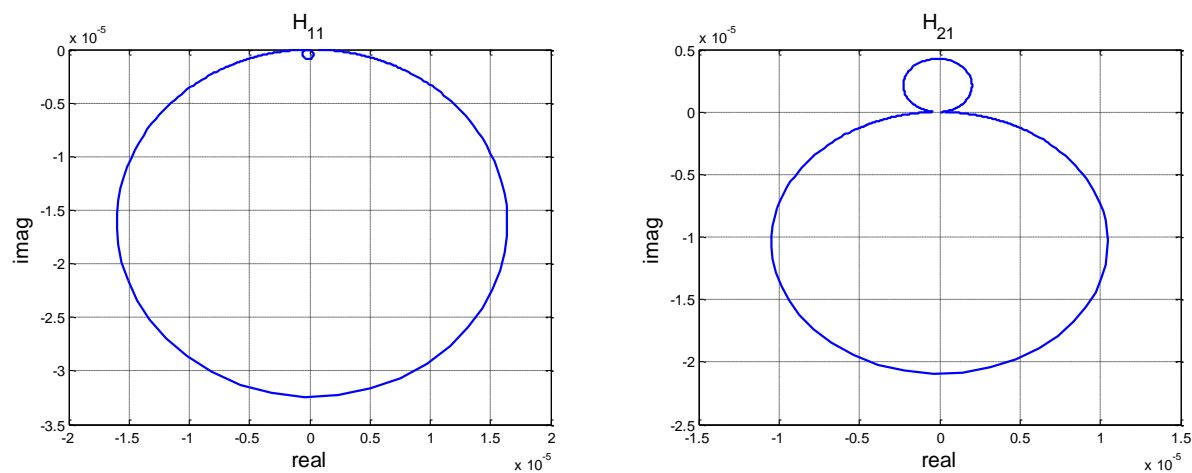


Figura 8. diagramma di Nyquist [m/N] della  $H_{11}$  e  $H_{21}$  per il sistema con smorzamento  $\zeta_1 = \zeta_2 = 0.01$ .

Figura 5. Parte immaginaria  $[m/N]$  della  $H_{11}$  e  $H_{21}$  per il sistema non smorzato.

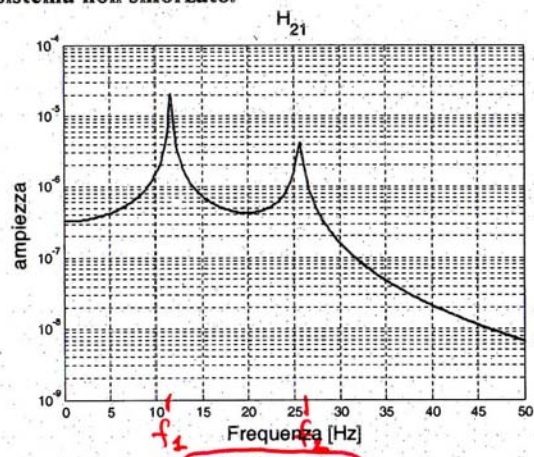
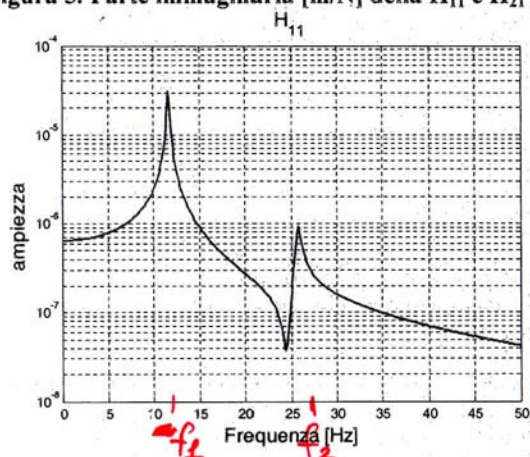


Figura 6. Ampiezza  $[m/N]$  della  $H_{11}$  e  $H_{21}$  per il sistema con smorzamento  $\zeta_1 = \zeta_2 = 0.01$ .

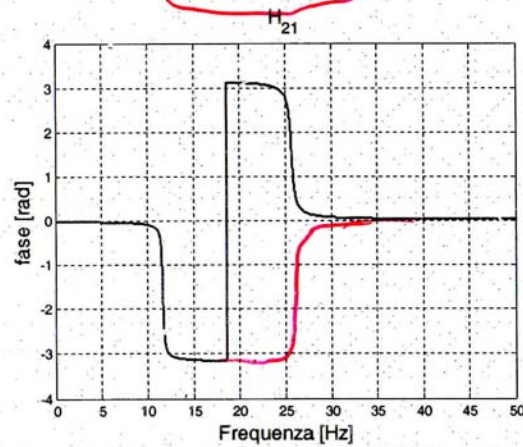
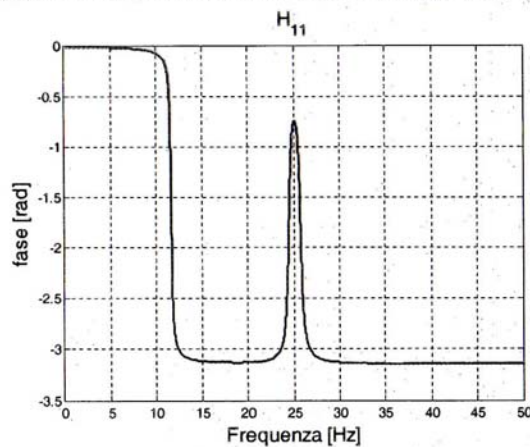


Figura 7. Fase della  $H_{11}$  e  $H_{21}$  per il sistema con smorzamento  $\zeta_1 = \zeta_2 = 0.01$ .

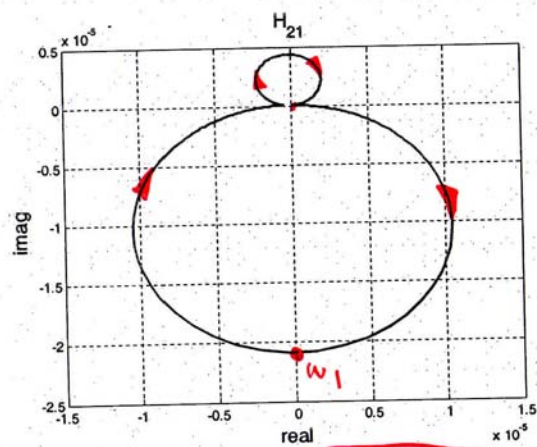
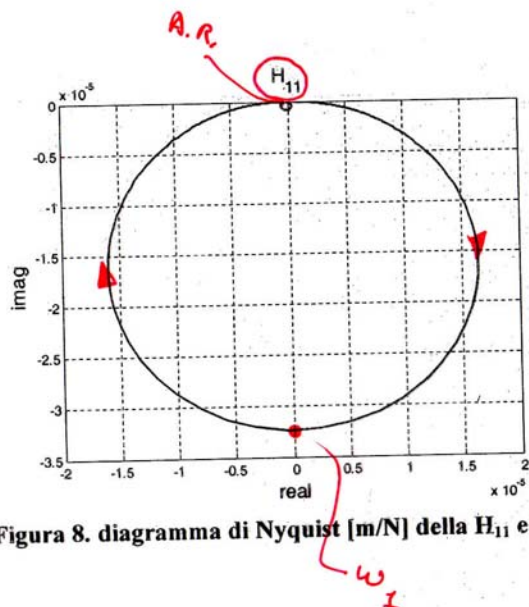
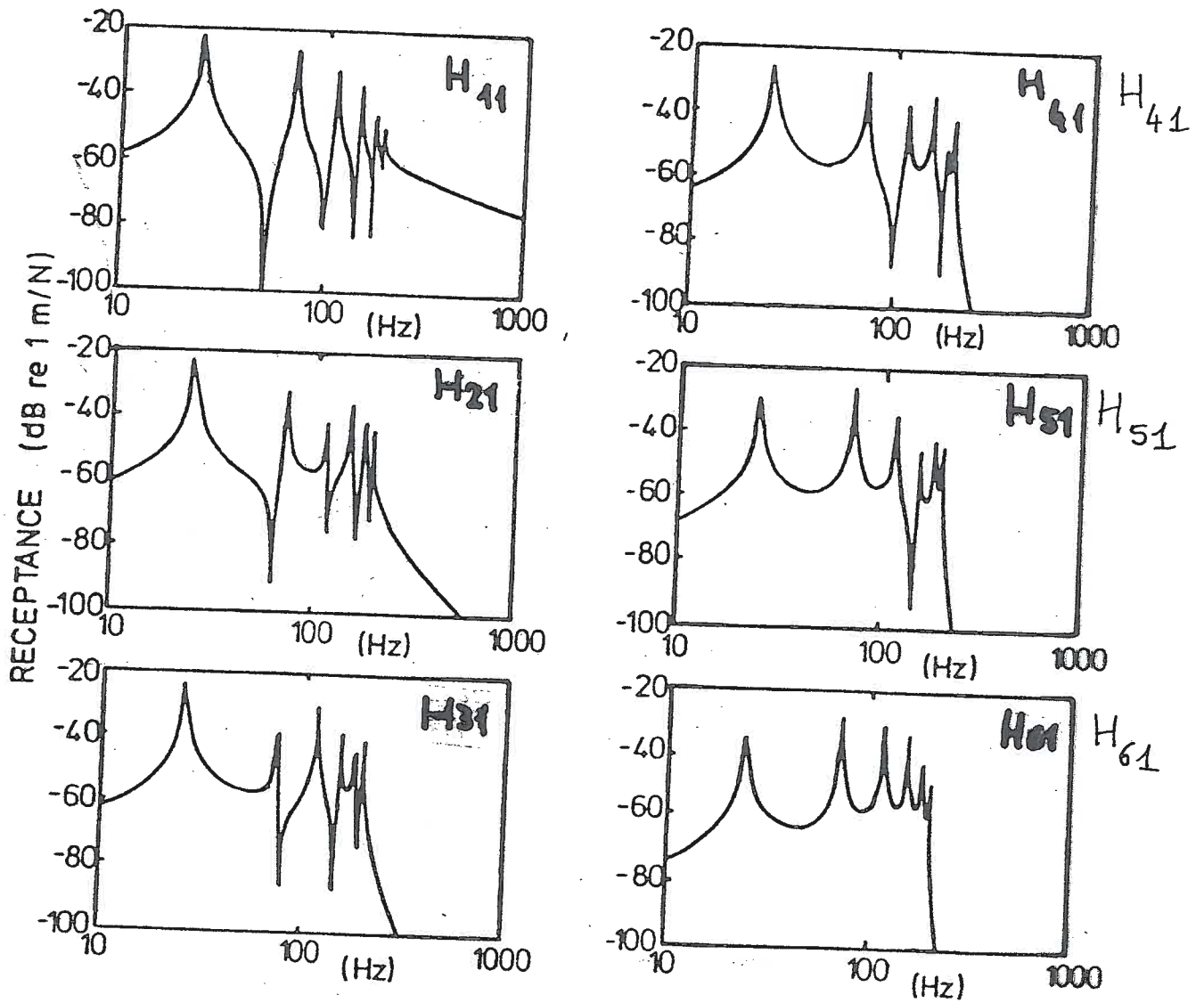
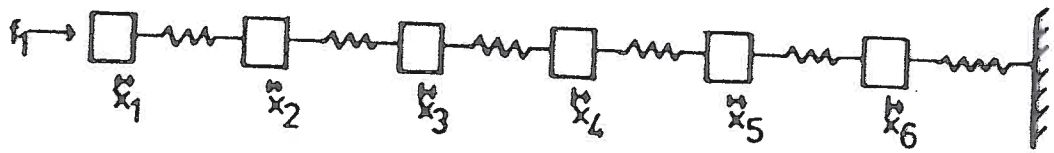


Figura 8. diagramma di Nyquist  $[m/N]$  della  $H_{11}$  e  $H_{21}$  per il sistema con smorzamento  $\zeta_1 = \zeta_2 = 0.01$ .



4 bus