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C.1 Definitions

Matrix. A matrix is a rectangular array of numbers. An array having m rows and n columns enclosed in brackets is called an m-by-n matrix. If $[A]$ is an $m \times n$ matrix, it is denoted as

$$[A] = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{C.1}$$

where the numbers a_{ij} are called the *elements* of the matrix. The first subscript i denotes the row and the second subscript j specifies the column in which the element a_{ij} appears.

Square Matrix. When the number of rows (m) is equal to the number of columns (n), the matrix is called a *square matrix of order n*.

Column Matrix. A matrix consisting of only one column—that is, an $m \times 1$ matrix—is called a *column matrix* or more commonly a *column vector.* Thus if $\vec{a}$ is a column vector having m elements, it can be represented as

$$\vec{a} = \begin{Bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_m \end{Bmatrix} \tag{C.2}$$

Row Matrix. A matrix consisting of only one row—that is a $1 \times n$ matrix—is called a *row matrix* or a *row vector.* If $\lfloor b \rfloor$ is a row vector, it can be denoted as

$$\lfloor b \rfloor = [b_1 \; b_2 \cdots b_n] \tag{C.3}$$

Diagonal Matrix. A square matrix in which all the elements are zero except those on the principal diagonal is called a *diagonal matrix.* For example, if $[A]$ is a diagonal matrix of order n, it is given by

$$[A] = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} \tag{C.4}$$

Identity Matrix. If all the elements of a diagonal matrix have a value 1, then the matrix is called an *identity matrix* or *unit matrix* and is usually denoted as $[I]$.

Zero Matrix. If all the elements of a matrix are zero, it is called a *zero* or *null matrix* and is denoted as $[0]$. If $[0]$ is of order 2×4, it is given by

$$[0] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{C.5}$$

Symmetric Matrix. If the element in ith row and jth column is the same as the one in jth row and ith column in a square matrix, it is called a *symmetric matrix.* This means that if $[A]$ is a symmetric matrix, we have $a_{ji} = a_{ij}$. For example,

$$[A] = \begin{bmatrix} 4 & -1 & -3 \\ -1 & 0 & 7 \\ -3 & 7 & 5 \end{bmatrix} \tag{C.6}$$

is a symmetric matrix of order 3.

Transpose of a Matrix. The transpose of an $m \times n$ matrix $[A]$ is the $n \times m$ matrix obtained by interchanging the rows and columns of $[A]$ and is denoted as $[A]^T$. Thus if

$$[A] = \begin{bmatrix} 2 & 4 & 5 \\ 3 & 1 & 8 \end{bmatrix} \tag{C.7}$$

then $[A]^T$ is given by

$$[A]^T = \begin{bmatrix} 2 & 3 \\ 4 & 1 \\ 5 & 8 \end{bmatrix} \tag{C.8}$$

Note that the transpose of a column matrix (vector) is a row matrix (vector), and vice versa.

Trace. The sum of the main diagonal elements of a square matrix $[A] = [a_{ij}]$ is called the *trace* of $[A]$ and is given by

$$\text{Trace}[A] = a_{11} + a_{22} + \cdots + a_{nn} \tag{C.9}$$

Determinant. If $[A]$ denotes a square matrix of order n, then the determinant of $[A]$ is denoted as $|[A]|$. Thus

$$|[A]| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \tag{C.10}$$

The value of a determinant can be found by obtaining the minors and cofactors of the determinant.

The *minor* of the element a_{ij} of the determinant $|[A]|$ of order n is a determinant of order $(n - 1)$ obtained by deleting the row i and the column j of the original determinant. The minor of a_{ij} is denoted as M_{ij}.

The *cofactor* of the element a_{ij} of the determinant $|[A]|$ of order n is the minor of the element a_{ij}, with either a plus or a minus sign attached; it is defined as

$$\text{Cofactor of } a_{ij} = \beta_{ij} = (-1)^{i+j} M_{ij} \tag{C.11}$$

where M_{ij} is the minor of a_{ij}. For example, the cofactor of the element a_{32} of

$$\det[A] = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \tag{C.12}$$

is given by

$$\beta_{32} = (-1)^5 M_{32} = -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \tag{C.13}$$

The value of a second order determinant $|[A]|$ is defined as

$$\det[A] = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \tag{C.14}$$

The value of an nth order determinant $|[A]|$ is defined as

$$\det[A] = \sum_{j=1}^{n} a_{ij}\beta_{ij} \text{ for any specific row } i$$

or

$$\det[A] = \sum_{i=1}^{n} a_{ij}\beta_{ij} \text{ for any specific column } j \tag{C.15}$$

For example, if

$$\det[A] = |[A]| = \begin{vmatrix} 2 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \tag{C.16}$$

then, by selecting the first column for expansion, we obtain

$$\det[A] = 2\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 4\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} + 7\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}$$

$$= 2(45 - 48) - 4(18 - 24) + 7(12 - 15) = -3 \tag{C.17}$$

Properties of Determinants

1. The value of a determinant is not affected if rows (or columns) are written as columns (or rows) in the same order.
2. If all the elements of a row (or a column) are zero, the value of the determinant is zero.
3. If any two rows (or two columns) are interchanged, the value of the determinant is multiplied by -1.
4. If all the elements of one row (or one column) are multiplied by the same constant a, the value of the new determinant is a times the value of the original determinant.

5. If the corresponding elements of two rows (or two columns) of a determinant are proportional, the value of the determinant is zero. For example,

$$\det[A] = \begin{vmatrix} 4 & 7 & -8 \\ 2 & 5 & -4 \\ -1 & 3 & 2 \end{vmatrix} = 0 \tag{C.18}$$

Adjoint Matrix. The adjoint matrix of a square matrix $[A] = [a_{ij}]$ is defined as the matrix obtained by replacing each element a_{ij} by its cofactor β_{ij} and then transposing. Thus

$$\text{Adjoint}[A] = \begin{bmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \beta_{n1} & \beta_{n2} & \cdots & \beta_{nn} \end{bmatrix}^T = \begin{bmatrix} \beta_{11} & \beta_{21} & \cdots & \beta_{n1} \\ \beta_{12} & \beta_{22} & \cdots & \beta_{n2} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \beta_{1n} & \beta_{2n} & \cdots & \beta_{nn} \end{bmatrix} \tag{C.19}$$

Inverse Matrix. The inverse of a square matrix $[A]$ is written as $[A]^{-1}$ and is defined by the following relationship:

$$[A]^{-1}[A] = [A][A]^{-1} = [I] \tag{C.20}$$

where $[A]^{-1}[A]$, for example, denotes the product of the matrix $[A]^{-1}$ and $[A]$. The inverse matrix of $[A]$ can be determined (see Ref. [A.1]):

$$[A]^{-1} = \frac{\text{adjoint}[A]}{\det[A]} \tag{C.21}$$

when $\det[A]$ is not equal to zero. For example, if

$$[A] = \begin{bmatrix} 2 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \tag{C.22}$$

its determinant has a value $\det[A] = -3$. The cofactor of a_{11} is

$$\beta_{11} = (-1)^2 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = -3 \tag{C.23}$$

In a similar manner, we can find the other cofactors and determine

$$[A]^{-1} = \frac{\text{adjoint}[A]}{\det[A]} = \frac{1}{-3}\begin{bmatrix} -3 & 6 & -3 \\ 6 & -3 & 0 \\ -3 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & 0 \\ 1 & 2/3 & -2/3 \end{bmatrix} \tag{C.24}$$

Singular Matrix. A square matrix is said to be singular if its determinant is zero.

Basic Matrix Operations

Equality of Matrices. Two matrices $[A]$ and $[B]$, having the same order, are equal if and only if $a_{ij} = b_{ij}$ for every i and j.

Addition and Subtraction of Matrices. The sum of the two matrices $[A]$ and $[B]$, having the same order, is given by the sum of the corresponding elements. Thus if $[C] = [A] + [B] = [B] + [A]$, we have $c_{ij} = a_{ij} + b_{ij}$ for every i and j. Similarly, the difference between two matrices $[A]$ and $[B]$ of the same order, $[D]$, is given by $[D] = [A] - [B]$ with $d_{ij} = a_{ij} - b_{ij}$ for every i and j.

Multiplication of Matrices. The product of two matrices $[A]$ and $[B]$ is defined only if they are conformable—that is, if the number of columns of $[A]$ is equal to the number of rows of $[B]$. If $[A]$ is of order $m \times n$ and $[B]$ is of order $n \times p$, then the product $[C] = [A][B]$ is of order $m \times p$ and is defined by $[C] = [c_{ij}]$, with

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} \tag{C.25}$$

This means that c_{ij} is the quantity obtained by multiplying the ith row of $[A]$ and the jth column of $[B]$ and summing these products. For example, if

$$[A] = \begin{bmatrix} 2 & 3 & 4 \\ 1 & -5 & 6 \end{bmatrix} \quad \text{and} \quad [B] = \begin{bmatrix} 8 & 0 \\ 2 & 7 \\ -1 & 4 \end{bmatrix} \tag{C.26}$$

then

$$[C] = [A][B] = \begin{bmatrix} 2 & 3 & 4 \\ 1 & -5 & 6 \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 2 & 7 \\ -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 8 + 3 \times 2 + 4 \times (-1) & 2 \times 0 + 3 \times 7 + 4 \times 4 \\ 1 \times 8 + (-5) \times 2 + 6 \times (-1) & 1 \times 0 + (-5) \times 7 + 6 \times 4 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 37 \\ -8 & -11 \end{bmatrix} \tag{C.27}$$

If the matrices are conformable, the matrix multiplication process is associative:

$$([A][B])[C] = [A]([B][C]) \tag{C.28}$$

and is distributive:

$$([A] + [B])[C] = [A][C] + [B][C] \tag{C.29}$$

The product $[A][B]$ denotes the premultiplication of $[B]$ by $[A]$ or the postmultiplication of $[A]$ by $[B]$. It is to be noted that the product $[A][B]$ is not necessarily equal to $[B][A]$.

The transpose of a matrix product can be found to be the product of the transposes of the separate matrices in reverse order. Thus, if $[C] = [A][B]$,

$$[C]^T = ([A][B])^T = [B]^T[A]^T \tag{C.30}$$

The inverse of a matrix product can be determined from the product of the inverse of the separate matrices in reverse order. Thus if $[C] = [A][B]$,

$$[C]^{-1} = ([A][B])^{-1} = [B]^{-1}[A]^{-1} \tag{C.31}$$

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C.1 Barnett, *Matrix Methods for Engineers and Scientists,* McGraw-Hill, New York, 1982.