Lecture notes on

Mathematical Methods for Engineering

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“*I knew it would take some time to get to that point.*
And I worked hard to get there.”
C. Schuldiner

Contents

Foreword

These notes contain the material covered by the 3rd/4th year course "*Metodi Matematici per l'Ingegneria*", which I gave in the second semester of the Academic Years from 2015/2016 to 2018/2019 at the University of Ferrara. The course lasted 48 *real* hours during a period of 12 weeks and was conceived for engineers. I used the following schedule:

- 3 weeks (i.e. 6 lectures) on Chapter 1
- 1 week (i.e. 2 lectures) on Chapter 2
- 1.5 week (i.e. 3 lectures) on Chapter 3
- 2 weeks (i.e. 4 lectures) on Chapter 4
- 2 weeks (i.e. 4 lectures) on Chapter 5
- 2.5 weeks (i.e. 5 lectures) on Chapter 6

Almost everything contained in these notes has been treated during the course, except for some advanced proofs or some exercises (also, I did not have time to treat the part on Volterra equations, which however is not part of the program). The contents of the course have been inherited from those treated by Prof. Daniela Mari, who previously held the course for many years. I only made some minor changes: for example, I enlarged the part on L^p spaces (Chapter 3), added a brief treatment of linear finite difference equations (in Chapter 2) and proved the *Sochocki-Plemelj formula* (in Chapter 6), which provides an elegant way to compute the Fourier transform of the Heaviside function. I also added a treatment of band-limited signals and the proof of the *Shannon-Whittaker sampling formula*, which is a beautiful result in the theory of Fourier transform.

The contents of the course aim at putting on a (reasonably) rigourous mathematical framework some standard tools used by engineers in signal processing. These are essentially the 3 kind of integral transforms presented in these notes:

- *$Z-$transform*
- *Laplace transform*

- *Fourier transform*

as well as the modern theory of *distributions*. Time permetting, usually I also briefly treat the bilateral Laplace transform, the Mellin transform and the Hilbert transform. As for the theory of distributions, I only treat the case of *tempered distributions*, essentially because this is the natural setting to define the Fourier transform in distributional sense.

Where possible, I tried to give a flavour of applications of these tools, mainly to *differential equations* and *finite difference equations*. The students were not supposed to be familiar with these two topics, but in the end this is not an issue. Indeed, by using the transforms one can offer a self-contained presentation (at least in the *constant coefficient linear case*).

It would be a good idea to include *Fourier series* among the contents of the course, but essentially there is no time to do it. For this reason, for completeness I added in Appendix C a brief summary of the main facts about Fourier series that the students should know. There are essentially two points where Fourier series enter in this course: in the proof of the Shannon-Whittaker formula and in the proof of the Poisson summation formula. I also singled out the connection between the singularities of the Laplace transform of a periodic function and its Fourier coefficients, see Remark 4.4.10.

I also added Appendix A and B about two standard facts in mathematical analysis, that usually are not very familiar to the students attending the course: the definitions and properties of $\liminf$ and $\limsup$ and a brief treatement of first order ordinary linear differential equations (possibly with varying coefficients).

Appendix D is essentially a *divertissement* for students that want to know a little bit more about harmonic functions in the plane. Even if they are not directly connected with the scopes of the course, they naturally arise in connection with holomorphic functions. I give some basic properties and construct some explicit examples.

Finally, in Appendix E one can find a summary of the main transforms computed throughout the lecture notes (Z, Laplace, bilateral Laplace, Mellin, Fourier, Hilbert).

Acknowledgments. I take the occasion to thank Daniela Mari for many helpful suggestions during the first preparation of the course in November 2015. I wish to express my gratitude to my friend and colleague Michele Miranda, who carefully read these notes, while teaching this course in the Academic Year 2019/2020. I also want to thank Mirko Ferracioli and Davide Zanellati, who spent some time in reading these notes and pointed out some typos and misprints.

List of symbols

We list below some basic notations used throughout these lecture notes:

Symbol	**Meaning**
i	imaginary unit
$\mathbb{C}$	field of complex numbers
$\mathrm{Re}(z)$	real part of $z \in \mathbb{C}$
$\mathrm{Im}(z)$	imaginary part of $z \in \mathbb{C}$
$\mathrm{Arg}(z)$	principal argument of $z \in \mathbb{C}$
$\mathbb{C}^*$	$\mathbb{C} \setminus \{0\}$
$\mathbb{C}^{**}$	$\mathbb{C} \setminus \{z \in \mathbb{C} : \mathrm{Im}(z) = 0 \text{ and } \mathrm{Re}(z) \leq 0\}$
H	*Heaviside step function*, defined by $H(t) = \begin{cases} 1, & \text{if } t \geq 0, \\ 0, & \text{if } t < 0 \end{cases}$
SW	*sawtooth wave function*, defined by $SW(t) = \sum_{k=0}^{\infty} (t-k) \Big[H(t-k) - H(t-k-1)\Big]$
R	*unitary ramp function*, defined by $R(t) = t\,H(t) = \begin{cases} t, & \text{if } t \geq 0, \\ 0, & \text{if } t < 0 \end{cases}$

Symbol	Meaning
rect	*rectangular function*, defined by $\mathrm{rect}(t) = \begin{cases} 1, & \text{if } -\dfrac{1}{2} \leq t \leq \dfrac{1}{2}, \\ 0, & \text{otherwise} \end{cases}$
tri	*triangular function*, defined by $\mathrm{tri}(t) = \begin{cases} 1 - \lvert t\rvert, & \text{if } -1 \leq t \leq 1, \\ 0, & \text{otherwise} \end{cases}$
sinc	*cardinal sine function*, defined by $\mathrm{sinc}(t) = \begin{cases} \dfrac{\sin(\pi\, t)}{\pi\, t}, & \text{if } t \neq 0, \\ 1, & \text{if } t = 0 \end{cases}$
$\square$	*square wave signal*, defined by $\square(t) = \sum_{n\in\mathbb{Z}} \left[\mathrm{rect}\left(t - \dfrac{1}{2} + 2\,n\right) - \mathrm{rect}\left(t + \dfrac{1}{2} + 2\,n\right)\right]$
$\mathcal{Z}[\{x_n\}]$	*Z−transform* of the sequence $\{x_n\}_{n\in\mathbb{N}}$, defined by $\mathcal{Z}[\{x_n\}](z) = \sum_{n=0}^{\infty} \dfrac{x_n}{z^n}$
$\mathcal{L}[f]$	*Laplace transform* of the causal signal f, defined by $\mathcal{L}[f](z) = \displaystyle\int_0^{+\infty} e^{-z\,t}\, f(t)\, dt$
σ_f	abscissa of convergence of $\mathcal{L}[f]$
$\mathcal{B}[f]$	*bilateral Laplace transform* of f, defined by $\mathcal{B}[f] = \displaystyle\int_{-\infty}^{+\infty} e^{-z\,t}\, f(t)\, dt$
Σ_f	upper abscissa of convergence of $\mathcal{B}[f]$
$\mathcal{M}[f]$	*Mellin transform* of the causal signal f, defined by $\mathcal{M}[f](z) = \displaystyle\int_0^{+\infty} t^{z-1}\, f(t)\, dt$
$\mathcal{F}[f]$	*Fourier transform* of f, defined by $\mathcal{F}[f](\omega) = \displaystyle\int_{\mathbb{R}} e^{-i\,t\,\omega}\, f(t)\, dt$
ω_f	band limit of the band-limited signal f

Symbol	**Meaning**
$\mathcal{S}$	Schwartz class
$[\varphi]_{m,k}$	$\sup\limits_{t\in\mathbb{R}} \lvert t^m\,\varphi^{(k)}(t)\rvert, \qquad m,k\in\mathbb{N}$
$\varphi \xrightarrow{\mathcal{S}} \varphi$	convergence in the Schwartz class $\mathcal{S}$
$\mathcal{S}'$	the space of tempered distributions
F_f	regular tempered distribution generated by f
δ_{t_0}	Dirac delta centered at $t_0\in\mathbb{R}$
$\mathrm{P.V.}\dfrac{1}{t}$	tempered distribution "principal value of $1/t$"
$F_n \xrightarrow{\mathcal{S}'} F$	convergence in $\mathcal{S}'$
$\mathcal{O}_M$	multipliers of the class $\mathcal{S}$
$\mathcal{O}_C$	convolvers of the class $\mathcal{S}$
P_τ	Dirac comb (with time step $\tau>0$)
$\mathcal{H}[f]$	*Hilbert transform* of f, defined by $\mathcal{H}[f]=f*\mathrm{P.V.}\dfrac{1}{t}$
$\widehat{f}(n)$	$n-$th *Fourier coefficient* of f, defined by $\widehat{f}(n)=\dfrac{1}{T}\displaystyle\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i\,n\,\frac{2\pi}{T}\,t}\,f(t)\,dt$
$\mathcal{J}[f]$	Fourier series of f
$\mathcal{J}_k[f]$	$k-$th partial Fourier sum

Chapter 1

Functions of one complex variable

1. Notation

We denote by $\mathbb{C}$ the field of complex numbers. If $z = x + i\,y \in \mathbb{C}$ is a complex number, we denote by

$$\mathrm{Re}(z) = x \qquad \text{and} \qquad \mathrm{Im}(z) = y,$$

its real and imaginary parts. Observe that if $z, w \in \mathbb{C}$, then

$$\mathrm{Re}(z + w) = \mathrm{Re}(z) + \mathrm{Re}(w) \qquad \text{and} \qquad \mathrm{Im}(z + w) = \mathrm{Im}(z) + \mathrm{Im}(w).$$

If $z \in \mathbb{C}$, we indicate by z^* its *conjugate*, which is defined by

$$z^* = x - i\,y.$$

We recall that $z\,z^* = |z|^2$, where

$$|z| = \sqrt{x^2 + y^2},$$

is the *modulus* of z. We recall that there holds

$$|z + w| \le |z| + |w|, \qquad \text{for every } z, w \in \mathbb{C}. \tag{1.1.1}$$

We set

$$\mathbb{C}^* = \mathbb{C} \setminus \{0\}, \qquad \mathbb{C}^{**} = \mathbb{C} \setminus \{z \in \mathbb{C} : \mathrm{Im}(z) = 0 \text{ and } \mathrm{Re}(z) \le 0\}.$$

Every $z \in \mathbb{C}^*$ can be written in polar coordinates as

$$z = |z|\,(\cos\vartheta + i\,\sin\vartheta),$$

where $\vartheta \in \mathbb{R}$ is called *an argument* of z. Of course, the argument is not unique, since any other argument of the form $\vartheta + 2\,k\,\pi$ with $k \in \mathbb{Z}$ would correspond to the same complex number z, thanks to the fact that

$$\cos(\vartheta + 2\,k\,\pi) = \cos\vartheta, \qquad \sin(\vartheta + 2\,k\,\pi) = \sin\vartheta.$$

We call *principal argument* of z the unique argument belonging to the interval $(-\pi, \pi]$. We will use the symbol $\mathrm{Arg}(z)$ to denote the principal argument of z.

Finally, we recall that if

$$z = |z|\,(\cos\vartheta + i\,\sin\vartheta) \qquad \text{and} \qquad w = |w|\,(\cos\varphi + i\,\sin\varphi),$$

then

$$z\,w = |z|\,|w|\,\Big(\cos(\vartheta+\varphi) + i\,\sin(\vartheta+\varphi)\Big). \tag{1.1.2}$$

2. A bit of topology

Let $z_0 \in \mathbb{C}$ and $r > 0$, we denote by $B_r(z_0)$ the disk centered at z_0 with radius $r > 0$, i.e.

$$B_r(z_0) = \{z \in \mathbb{C} \,:\, |z - z_0| < r\}.$$

We also introduce the notation $\dot{B}_r(z_0)$ for the *punctured disk* centered at z_0 with radius $r > 0$, i.e.

$$\dot{B}_r(z_0) = B_r(z_0) \setminus \{z_0\}.$$

We say that a subset $A \subset \mathbb{C}$ is *open* if for every $z_0 \in A$, there exists $r > 0$ such that $B_r(z_0) \subset A$. We say that A is *closed* if $\mathbb{C} \setminus A$ is open. A point $z_0 \in \mathbb{C}$ is said to be an *accumulation point* of a set $A \subset \mathbb{C}$ if for every $r > 0$ we have

$$\dot{B}_r(z_0) \cap A \neq \emptyset.$$

We say that z_0 is a *boundary point* of A if for every $r > 0$ we have

$$B_r(z_0) \cap A \neq \emptyset \qquad \text{and} \qquad B_r(z_0) \cap (\mathbb{C} \setminus A) \neq \emptyset.$$

Finally, for a subset $A \subset \mathbb{C}$ we denote by ∂A the collection of all boundary points of A. This set is called *boundary of* A.

For a set $A \subset \mathbb{C}$ we denote by $\overline{A}$ its *closure*. By definition, this is the smallest closed set containing A. For example, it is not difficult to see that

$$\overline{B_r(z_0)} = \{z \in \mathbb{C} \,:\, |z - z_0| \leq r\}.$$

We say that an open set $A \subset \mathbb{C}$ is *connected* if for every $z, w \in A$ there exists a continuous polygonal line $\gamma \subset A$ connecting z and w.

Example 1.2.1. Let $A = \{z \in \mathbb{C} \,:\, \mathrm{Re}\, z \geq 0\} \cup \{-1 + i\}$, this is a closed set. It is easy to see that

$$\partial A = \{z \in \mathbb{C} \,:\, \mathrm{Re}\, z = 0\} \cup \{-1 + i\},$$

but $\{-1 + i\}$ *is not* an accumulation point of A. Indeed, we have

$$\dot{B}_{1/2}(1 + i) \cap A = \emptyset,$$

since the only intersection point between $B_{1/2}(1 + i)$ and A is $1 + i$. Finally, A is not connected, since the point $1 + i$ and any point $z \in \mathbb{C}$ such that $\mathrm{Re}\, z \geq 0$ can not be connected by a polygonal line entirely contained in A.

3. Functions of one complex variable

We recall a couple of definitions that will be useful. Let $A, B \subset \mathbb{C}$ two non-empty sets and $f : A \to B$ a function. We say that

- f is *injective* if

  "for every $w \in B$, the equation $f(z) = w$ has **at most** a solution $z \in A$".

- f is *surjective* if

  "for every $w \in B$, the equation $f(z) = w$ has **at least** a solution $z \in A$".

- f is *bijective* if it is injective and surjective. This means that

  "for every $w \in B$, the equation $f(z) = w$ has a **unique** solution $z \in A$".

When $f : A \to B$ is bijective, it is well-defined its *inverse function* $f^{-1} : B \to A$. This is the function given by

$$\begin{array}{rcll} f^{-1} & : & B & \to & A \\ & & w & \mapsto & \begin{array}{c}\text{"the unique solution } z \in A \\ \text{of the equation } f(z) = w\text{"}\end{array} \end{array}$$

By construction, we have

$$f^{-1}(f(z)) = z \qquad \text{and} \qquad f(f^{-1}(w)) = w, \qquad \text{for every } z \in A,\ w \in B.$$

Definition 1.3.1 (Limits)**.** Let $A \subset \mathbb{C}$ be an open set and $f : A \to \mathbb{C}$ a function of one complex variable. Let $z_0 \in \mathbb{C}$ be an accumulation point of A, we say that f *admits limit* $L \in \mathbb{C}$ *at* z_0 if

$$\forall \varepsilon > 0,\ \exists \delta > 0 \ \text{ such that if } \ z \in \dot{B}_\delta(z_0), \ \text{ then } \ |f(z) - L| < \varepsilon.$$

In this case, we use the notation

$$\lim_{z \to z_0} f(z) = L.$$

Definition 1.3.2 (Continuity)**.** Let $A \subset \mathbb{C}$ be an open set and $f : A \to \mathbb{C}$ a function of one complex variable. We say that f is *continuous at* $z_0 \in A$ if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

We say that f is *continuous on* A if it is continuous at every $z \in A$.

Proposition 1.3.3. *A function f of a complex variable is continuous at $z_0 \in \mathbb{C}$ if and only if the two functions $\operatorname{Re} f$ and $\operatorname{Im} f$ are continuous at z_0.*

Proof. We observe that

$$|f(z) - f(z_0)| = \sqrt{|\operatorname{Re} f(z) - \operatorname{Re} f(z_0)|^2 + |\operatorname{Im} f(z) - \operatorname{Im} f(z_0)|^2}.$$

By using that (see Exercise 1.12.1 below)

$$\begin{aligned} \frac{|\operatorname{Re} f(z) - \operatorname{Re} f(z_0)| + |\operatorname{Im} f(z) - \operatorname{Im} f(z_0)|}{\sqrt{2}} &\le |f(z) - f(z_0)| \\ &\le |\operatorname{Re} f(z) - \operatorname{Re} f(z_0)| + |\operatorname{Im} f(z) - \operatorname{Im} f(z_0)|, \end{aligned}$$

we obtain that

$$\lim_{z \to z_0} |f(z) - f(z_0)| = 0 \quad \Longleftrightarrow \quad \left\{ \begin{array}{rcl} \displaystyle\lim_{z \to z_0} |\operatorname{Re} f(z) - \operatorname{Re} f(z_0)| &=& 0 \\ \\ \displaystyle\lim_{z \to z_0} |\operatorname{Im} f(z) - \operatorname{Im} f(z_0)| &=& 0, \end{array} \right.$$

which proves the claim. □

Example 1.3.4. The function *principal argument* $\operatorname{Arg} : \mathbb{C}^* \to (-\pi, \pi]$ is continuous on $\mathbb{C}^{**}$, but it has a discontinuity across the semiaxis of the negative real numbers. Indeed, for $x_0 < 0$ we have

$$\lim_{\vartheta \to \pi^-} \operatorname{Arg}\Big(|x_0|\,(\cos\vartheta + i\,\sin\vartheta)\Big) = \pi$$
$$\neq -\pi = \lim_{\vartheta \to -\pi^+} \operatorname{Arg}\Big(|x_0|\,(\cos\vartheta + i\,\sin\vartheta)\Big).$$

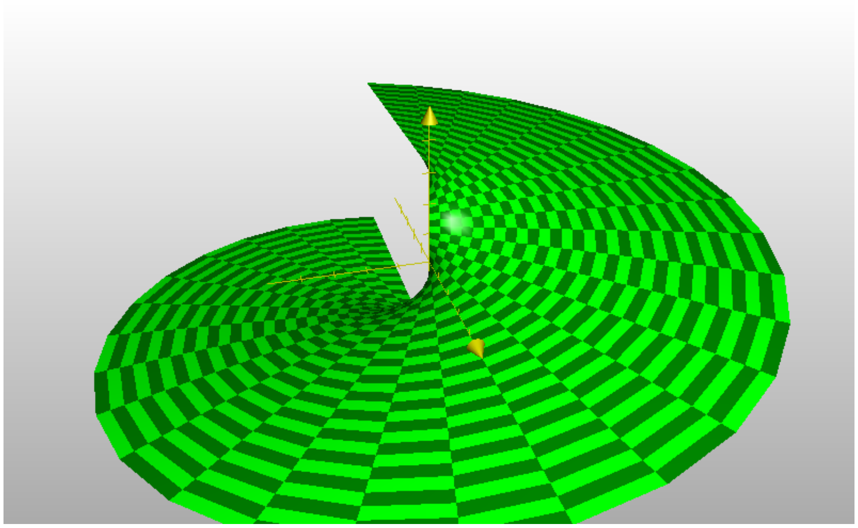

Figure 1. The graph of the function $(x, y) \mapsto \operatorname{Arg}(x + i\,y)$

Lemma 1.3.5. *Let $A \subset \mathbb{C}$ be an open set and let $g : A \to \mathbb{C}$ be a function. Suppose that g is continuous at $z_0 \in A$ and that*

$$g(z_0) \neq 0.$$

Then there exists $r > 0$ such that $B_r(z_0) \subset A$ and

$$g(z) \neq 0, \qquad \textit{for } z \in B_r(z_0).$$

Proof. By continuity, for every $\varepsilon > 0$ there exists $r > 0$ such that

$$|g(z) - g(z_0)| < \varepsilon, \qquad \text{for } z \in B_r(z_0).$$

By using the triangle inequality (1.1.1) with the choices

$$z = g(z_0) - g(z) \qquad \text{and} \qquad w = g(z)$$

we get

$$|g(z) - g(z_0)| + |g(z)| \geq |g(z_0)|,$$

thus in particular

$$\varepsilon + |g(z)| \geq |g(z_0)|, \qquad \text{for } z \in B_r(z_0). \tag{1.3.1}$$

We now observe that

$$|g(z_0)| \neq 0,$$

by hypothesis, thus we can choose

$$\varepsilon = \frac{1}{2}\,|g(z_0)| > 0.$$

From (1.3.1), we get

$$|g(z)| > |g(z_0)| - \varepsilon = \frac{1}{2}\,|g(z_0)|, \qquad \text{for } z \in B_r(z_0).$$

This in turn implies that g can not vanish in $B_r(z_0)$. □

4. Holomorphic functions

Definition 1.4.1. Let $A \subset \mathbb{C}$ be an open set and let $f : A \to \mathbb{C}$ be a function. We say that f is *derivable at* $z_0 \in A$ if the limit

$$\lim_{\mathbb{C}\ni h\to 0} \frac{f(z_0+h) - f(z_0)}{h},$$

exists in $\mathbb{C}$. This means that there exists $\lambda \in \mathbb{C}$ such that

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that if } 0 < |h| < \delta, \text{ then } \left|\frac{f(z_0+h) - f(z_0)}{h} - \lambda\right| < \varepsilon.$$

In this case λ is called *derivative of* f *at* z_0 and we use one of the notations

$$f'(z_0), \qquad \frac{df}{dz}(z_0).$$

Remark 1.4.2. As in the case of functions of one real variable, we have that *if f is derivable at z_0, then it is continuous as well at this point.* Indeed, by definition of derivative we have

$$\frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) + o(1), \qquad \text{for } |z - z_0| \to 0,$$

that is

$$f(z) - f(z_0) = f'(z_0)\,(z - z_0) + o(z - z_0), \qquad \text{for } |z - z_0| \to 0.$$

This implies that

$$\lim_{z\to z_0} \Big[f(z) - f(z_0)\Big] = \lim_{z\to z_0} \Big[f'(z_0)\,(z - z_0)\Big] = 0.$$

The usual properties of derivatives hold true. We state the next three propositions without proofs, which are left to the reader.

Proposition 1.4.3 (Sums & products)**.** *Let $A \subset \mathbb{C}$ be an open set. Let $f : A \to \mathbb{C}$ and $g : A \to \mathbb{C}$ two functions. If f and g are derivable at $z_0 \in A$, then we have:*

- *for every $\alpha, \beta \in \mathbb{C}$, the function $z \mapsto \alpha\, f(z) + \beta\, g(z)$ is derivable at z_0 and we have*

$$\frac{d}{dz}\Big(\alpha\, f(z) + \beta\, g(z)\Big)_{|z=z_0} = \alpha\, f'(z_0) + \beta\, g'(z_0);$$

- *the product function $f\,g$ is derivable at z_0 and we have*

$$\frac{d}{dz}\Big(f(z)\, g(z)\Big)_{|z=z_0} = f'(z_0)\, g(z_0) + f(z_0)\, g'(z_0);$$

Proposition 1.4.4 (Compositions)**.** *Let $A, B \subset \mathbb{C}$ be two open sets and let $f : A \to \mathbb{C}$ and $g : B \to A$ be two functions of a complex variable. If g is derivable at $z_0 \in B$ and f is derivable at $g(z_0) \in A$, then the composition $f \circ g$ is derivable at z_0 and we have*

$$\frac{d}{dz}\Big(f \circ g(z)\Big)_{|z=z_0} = f'(g(z_0))\, g'(z_0);$$

Proposition 1.4.5 (Inverse function). *Let $A, B \subset \mathbb{C}$ be two open sets and let us suppose that $f : A \to B$ is bijective. Let us assume that*

- $f'(f^{-1}(z_0)) \neq 0$;
- *the inverse function $f^{-1} : B \to A$ is continuous at $z_0 \in B$.*

Then f^{-1} is derivable at z_0 and we have

$$\frac{d}{dz} f^{-1}(z)_{|z=z_0} = \frac{1}{f'(f^{-1}(z_0))}.$$

Lemma 1.4.6. *Let $A \subset \mathbb{C}$ be a connected open set and let $f : A \to \mathbb{C}$ be such that*

$$f'(z) = 0, \qquad \text{for every } z \in A.$$

Then f is constant.

Proof. Let us take $z, w \in A$, since A is connected we know that there exists a polygonal line γ contained in A and joining z to w. If we prove that

$$f(z) = f(w),$$

we get the conclusion, by arbitrariness of z and w. In order to prove this, it is sufficient to prove that f is constant on every segment of the polygonal line γ. Such a segment can be parametrized by

$$\eta(t) = (1-t)\, p_i + t\, p_{i+1}, \qquad t \in [0,1],$$

for a suitable choice of distinct points $p_1, \dots, p_n \in \mathbb{C}$. Then the function of one real variable $g(t) = f(\eta(t))$ defined on $[0,1]$ is such that

$$g'(t) = f'(\eta(t))\, \eta'(t) = 0, \qquad \text{for every } t \in [0,1].$$

This implies that g is constant, as desired. □

The previous properties are similar to those for differentiable functions of one real variable. On the contrary, the next important property is characteristic of functions of one complex variable.

Theorem 1.4.7. *Let $A \subset \mathbb{C}$ be an open set and let $f : A \to \mathbb{C}$ be a function which is differentiable as a function of the two real variables x and y. Then f is derivable as a function of the complex variable z if and only if we have*

$$\frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y}. \tag{1.4.1}$$

In this case, we have

$$f'(z) = \frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y}. \tag{1.4.2}$$

Proof. Let us assume that f is derivable as a function of z. By definition of complex derivative, we know that

$$f'(z) = \lim_{\mathbb{R} \ni h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h} = f_x,$$

and also

$$f'(z) = \lim_{\mathbb{R} \ni h \to 0} \frac{f(z + i\, h) - f(z)}{i\, h} = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{i\, h} = \frac{1}{i}\, f_y.$$

Thus we immediately obtain (1.4.1).

Let us now assume that (1.4.1) is verified. By using the fact that f is differentiable as a function of x and y, for $h = h_1 + i\,h_2 \in \mathbb{C}$ we get

$$\begin{aligned} f(z+h) - f(z) &= f(x+h_1, y+h_2) - f(x,y) \\ &= f_x(x,y)\,h_1 + f_y(x,y)\,h_2 + o(|h|) \\ &= f_x(x,y)\,h_1 + i\,f_x(x,y)\,h_2 + o(|h|) \\ &= f_x(x,y)\,h + o(|h|). \end{aligned}$$

This implies that

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = f_x(x,y) + \lim_{h\to 0}\frac{o(|h|)}{h} = f_x(x,y),$$

so that f is derivable as a function of z and (1.4.2) holds true. □

Remark 1.4.8. It is useful to recall that a sufficient condition for a function of two real variables $(x,y) \mapsto f(x,y)$ to be differentiable is that the partial derivatives f_x and f_y exist and are continuous.

Corollary 1.4.9 (Cauchy-Riemann equations)**.** *Under the previous hypotheses, the function $f(x+i\,y) = u(x,y) + i\,v(x,y)$ is derivable as a function of $z = x + i\,y$ if and only if*

$$\begin{cases} u_x &= \quad v_y, \\ v_x &= -u_y. \end{cases} \tag{1.4.3}$$

Proof. It is sufficient to observe that

$$f_x = u_x + i\,v_x \qquad \text{and} \qquad \frac{1}{i}\,f_y = \frac{1}{i}\,u_y + v_y = -i\,u_y + v_y,$$

then (1.4.1) becomes (1.4.3). □

Remark 1.4.10. The equation (1.4.1) will be called *Cauchy-Riemann equations in complex form*, while (1.4.3) will be called *Cauchy-Riemann equations in real form.*

Example 1.4.11. We can now give an example of function which is not derivable in z, but it is differentiable as a function of x and y. Let us take

$$f(z) = z^*,$$

as a (complex-valued) function of the variables x and y this is

$$f(x,y) = x - i\,y.$$

This is of course differentiable as a function of x and y, since the partial derivatives f_x and f_y exist and are continuous (they are actually constant functions!). On the other hand

$$f_x = 1 \neq -1 = \frac{1}{i}\,f_y,$$

thus (1.4.1) is not satisfied and f is not derivable as a function of the complex variable z.

Example 1.4.12. Another function of a complex variable which is not derivable in z is given by $f(z) = |z|$. Indeed, observe that for $x^2 + y^2 \neq 0$ we have

$$f_x = \frac{x}{\sqrt{x^2+y^2}} \qquad \text{and} \qquad f_y = \frac{y}{\sqrt{x^2+y^2}}.$$

Thus (1.4.1) is not satisfied.

Remark 1.4.13. More generally, every function f of one complex variable z which only takes real values can not be derivable in z, unless it is constant.

Remark 1.4.14 (Conjugate harmonic functions)**.** Let us suppose that $f : A \to \mathbb{C}$ is derivable on some open set $A \subset \mathbb{C}$. Then by writing

$$f(z) = u(x, y) + i\, v(x, y), \qquad z = x + i\, y \in A,$$

We have seen that the real part u and the imaginary part v are linked to the system of Cauchy-Riemann equations (1.4.3). If we suppose that $u, v \in C^2(A)$, then we can differentiate the equations in (1.4.3), i.e.

$$u_x = v_y \qquad \Longrightarrow \qquad u_{xy} = v_{yy},$$

and

$$u_y = -v_x \qquad \Longrightarrow \qquad u_{yx} = -v_{xx},$$

By using that $u \in C^2(A)$, we obtain from *Schwarz's Theorem* that

$$v_{yy} = u_{xy} = u_{yx} = -v_{xx}.$$

In other words, the imaginary part v satisfies the partial differential equation

$$v_{xx} + v_{yy} = 0, \qquad \text{in } A.$$

A function satisfying such an equation is called a *harmonic function*. By proceeding in a similar way, we can also prove that

$$u_{xx} + u_{yy} = 0, \qquad \text{in } A.$$

Then the functions u and v are said to be *conjugate harmonic functions*. We refer to Appendix D for more details on harmonic functions.

Definition 1.4.15 (Holomorphic function)**.** Let $A \subset \mathbb{C}$ be open set, we say that f is *holomorphic in A* if f is derivable for every $z \in A$ and f' is a continuous function on A.

Definition 1.4.16 (Entire function)**.** A function $f : \mathbb{C} \to \mathbb{C}$ which is holomorphic on the whole complex plane $\mathbb{C}$ is called *entire*.

5. Some examples of holomorphic functions

We now present some remarkable holomorphic functions.

• **Power functions.** For $n \in \mathbb{N}$, this is defined in the usual way by

$$z^n = \underbrace{z \cdot z \cdot \dots \cdot z}_{n}, \qquad z^0 = 1.$$

This is derivable for every $z \in \mathbb{C}$, the proof is the same as in the real case (*use Newton's bynomial formula*). We have

$$\frac{d}{dz} z^n = n\, z^{n-1}, \qquad \text{for every } z \in \mathbb{C}.$$

Since the latter is continuous, the function $z \mapsto z^n$ is holomorphic. By writing a complex number $z \in \mathbb{C}^*$ in polar coordinates as

$$z = \varrho\,(\cos\vartheta + i\,\sin\vartheta), \qquad \varrho > 0,\ \vartheta \in (-\pi, \pi],$$

from (1.1.2) we have

$$z^n = \varrho^n\,(\cos(n\,\vartheta) + i\,\sin(n\,\vartheta)).$$

This is *not an injective function*, unless we are in the trivial case $n = 1$. Indeed, for every point

$$z = \varrho\,(\cos\vartheta + i\,\sin\vartheta),$$

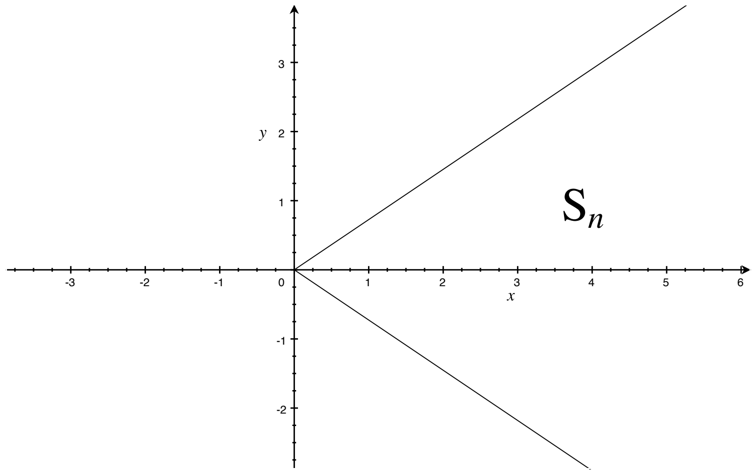

Figure 2. The restriction of the function $z \mapsto z^n$ on the region $\mathcal{S}_n$ is bijective (here $n = 5$).

such that

$$-\frac{\pi}{n} < \vartheta \leq \frac{\pi}{n}, \qquad z \in \mathbb{C},$$

we obtain that the points

$$z_k = \varrho \left(\cos \left(\vartheta + \frac{2\,k\,\pi}{n} \right) + i\, \sin \left(\vartheta + \frac{2\,k\,\pi}{n} \right) \right), \qquad k = 1 \ldots, n-1,$$

are distinct and such that

$$z_1^n = \cdots = z_{n-1}^n = z^n,$$

i.e. they have the same image. Indeed, we know that for every $w \in \mathbb{C}^*$, the equation

$$z^n = w, \tag{1.5.1}$$

admits n distinct solutions, given by the formula

$$z_k = \sqrt[n]{|w|} \left(\cos \left(\frac{\vartheta}{n} + \frac{2\,k\,\pi}{n} \right) + i\, \sin \left(\frac{\vartheta}{n} + \frac{2\,k\,\pi}{n} \right) \right), \qquad k = 0, 1 \ldots, n-1, \tag{1.5.2}$$

where ϑ is now an argument of w. On the other hand, if we take the restriction of the $n-th$ *power function* to the sector

$$\mathcal{S}_n = \left\{ z \in \mathbb{C}^* \,:\, -\frac{\pi}{n} < \mathrm{Arg}(z) \leq \frac{\pi}{n} \right\} \cup \{0\},$$

then this becomes injective. Moreover, since we showed that (1.5.1) always admits at least a solution $z \in \mathcal{S}_n$ for every $w \in \mathbb{C}$, this is surjective as well.

• **Principal $n-$th root.** We have seen that for every $n \in \mathbb{N} \setminus \{0, 1\}$, the function

$$\begin{array}{ccc} \mathcal{S}_n & \to & \mathbb{C} \\ z & \mapsto & z^n \end{array}$$

is bijective. Thus its inverse function is well-defined and called *principale value $n-th$ root*. This is the function

$$\begin{array}{ccc} \mathbb{C} & \to & \mathcal{S}_n \\ w & \mapsto & \text{“the unique solution } z \in \mathcal{S}_n \text{ to the equation } z^n = w\text{''}, \end{array}$$

which will be denoted by the usual symbol $\sqrt[n]{z}$. By construction is the function defined by

$$\sqrt[n]{z} = \sqrt[n]{|z|}\,\left[\cos\left(\frac{\mathrm{Arg}(z)}{n}\right) + i\sin\left(\frac{\mathrm{Arg}(z)}{n}\right)\right], \qquad \sqrt[n]{0} = 0.$$

We observe that for $n \geq 2$ this function *has a discontinuity* along the semiaxis of negative real numbers, due to the presence of the principal argument (recall Example 1.3.4). In other words, the function $z \mapsto \sqrt[n]{z}$ is continuous only on $\mathbb{C}^{**}$. Moreover, for every $z_0 \in \mathbb{C}^{**}$ we have

$$\frac{d}{dz}(z^n)_{|\sqrt[n]{z_0}} = n\,\left(\sqrt[n]{z_0}\right)^{n-1} \neq 0,$$

thus by using the formula for the derivative of the inverse function (see Proposition 1.4.5), we easily get

$$\frac{d}{dz}\sqrt[n]{z} = \frac{1}{n\,\left(\sqrt[n]{z}\right)^{n-1}} = \frac{1}{n}\,z^{\frac{1}{n}-1}, \qquad \text{for every } z \in \mathbb{C}^{**}.$$

Such a function is continuous on $\mathbb{C}^{**}$ and thus the principal $n-$th root is holomorphic on $\mathbb{C}^{**}$.

• **Complex exponential.** This is defined by

$$e^z = e^{x+i\,y} := e^x\,\left(\cos y + i\,\sin y\right), \qquad z = x + i\,y \in \mathbb{C}.$$

By its definition, we immediately get

(1.5.3) $$|e^z| = e^x\,|\cos y + i\,\sin y| = e^x,$$

thanks to the well-known trigonometric relation

$$\cos^2 y + \sin^2 y = 1, \qquad \text{for every } y \in \mathbb{R}.$$

Observe that this is derivable for every $z \in \mathbb{C}$, since

$$\frac{\partial}{\partial y}e^z = e^x\,\left(-\sin y + i\,\cos y\right) = i\,e^z = i\,\frac{\partial}{\partial x}e^z,$$

thus by Proposition 1.4.7 we get

$$\frac{d}{dz}e^z = e^z,$$

as for the usual exponential function. Moreover, since the derivative is continuous, we get that the complex exponential is an entire function. Observe that from the previous formula for the derivative, we get in particulat

$$1 = e^0 = \frac{d}{dz}e^z_{|z=0} = \lim_{z\to 0}\frac{e^z - e^0}{z},$$

that is

(1.5.4) $$\lim_{z\to 0}\frac{e^z - 1}{z} = 1.$$

Observe that we have

$$e^z \neq 0, \qquad \text{for every } z \in \mathbb{C}.$$

By definition, we have

$$e^{z+2\,\pi\,i} = e^z, \qquad \text{for every } z \in \mathbb{C},$$

thus the complex exponential is periodic, with (complex) period $2\,\pi\,i$. In particular, the complex exponential *is not injective*. We also observe that

(1.5.5) "*the complex exponential sends the vertical line* $\{z\,:\,\mathrm{Re}(z) = x\}$ *into the circle of radius* e^x *and center* 0"

On the other hand, the complex exponential is surjective as a function from $\mathbb{C}$ to $\mathbb{C}^*$: indeed, for every $w \in \mathbb{C}^*$, we have

$$
\begin{aligned}
e^z = w \quad &\Longleftrightarrow \quad e^x\,(\cos y + i\,\sin y) = |w|\,\Big(\cos(\mathrm{Arg})(w) + i\,\sin(\mathrm{Arg})(w)\Big)\\
&\Longleftrightarrow \quad \left\{\begin{array}{lcl} e^x &=& |w|,\\ y &=& \mathrm{Arg}(w) + 2\,k\,\pi,\ k\in\mathbb{Z}\end{array}\right.\\
&\Longleftrightarrow \quad \left\{\begin{array}{lcl} x &=& \log|w|,\\ y &=& \mathrm{Arg}(w) + 2\,k\,\pi,\ k\in\mathbb{Z}\end{array}\right.
\end{aligned}
\tag{1.5.6}
$$

The complex exponential becomes bijective when restricted to

$$
\mathcal{S} = \{z\in\mathbb{C}\,:\,-\pi < \mathrm{Im}(z) \le \pi\},
\tag{1.5.7}
$$

since for every $w \in \mathbb{C}^*$ the set $\mathcal{S}$ contains one and one only of the solutions found in (1.5.6) (i.e. the one corresponding to $k = 0$).

• **Principal logarithm.** As for the usual logarithm, roughly speaking this is defined as the inverse function of the (complex) exponential. Once again, we should be careful, since the complex exponential is not a bijective function and thus the concept of inverse function is not well-defined. From the discussion above, we know that

$$
\begin{array}{ccc} \mathcal{S} & \to & \mathbb{C}^*\\ z & \mapsto & e^z \end{array}
$$

is a bijective function. Thus we can define the inverse function

$$
\begin{array}{ccc} \mathbb{C}^* & \to & \mathcal{S}\\ w & \mapsto & \text{“ the unique solution } z\in\mathcal{S} \text{ to the equation } e^z = w'' \end{array}
$$

We use the notation $\mathrm{Log}\,w$ for this function and call it *principal logarithm*. From (1.5.6), we know that this function has an explicit expression, given by

$$
\mathrm{Log}\,w = \log|w| + i\,\mathrm{Arg}(w), \qquad \text{for every } w\in\mathbb{C}^*.
$$

Observe that we can now give a sense to expressions like $\mathrm{Log}\,(-7)$, since by definition of principal logarithm we have

$$
\mathrm{Log}\,(-7) = \log 7 + i\,\pi.
$$

We observe that the principal logarithm is discontinuous across the semiaxis of negative real numbers, exactly like it happens for the principal value $n-$th root. Again, this is due to the presence of the principal argument.

On the set $\mathbb{C}^{**}$, the principal logarithm is a holomorphic function, with derivative (again, it is sufficient to use the formula for the inverse function)

$$
\frac{d}{dz}\mathrm{Log}\,z = \frac{1}{e^{\mathrm{Log}\,z}} = \frac{1}{z}, \qquad z\in\mathbb{C}^{**},
$$

which is analogous to the case of the usual real logarithm.

• **Complex trigonometric functions.** We observe that by definition of complex exponential, we have the identities for x real number

$$
\cos x = \frac{e^{i\,x}+e^{-i\,x}}{2}, \qquad \sin x = \frac{e^{i\,x}-e^{-i\,x}}{2\,i}, \qquad x\in\mathbb{R}.
$$

It is then natural to extend the cosinus and sinus functions to the complex variable, by defining them as

$$\cos z = \frac{e^{i\,z} + e^{-i\,z}}{2}, \qquad \sin z = \frac{e^{i\,z} - e^{-i\,z}}{2\,i}, \qquad z \in \mathbb{C}.$$

By the definition, we immediately get that these are entire functions, as sums of entire functions. Moreover, we have

$$\frac{d}{dz}\cos z = \frac{d}{dz}\frac{e^{i\,z} + e^{-i\,z}}{2} = i\,\frac{e^{i\,z} - e^{-i\,z}}{2} = -\frac{e^{i\,z} - e^{-i\,z}}{2\,i} = -\sin z,$$

and

$$\frac{d}{dz}\sin z = \frac{d}{dz}\frac{e^{i\,z} - e^{-i\,z}}{2\,i} = \frac{e^{i\,z} + e^{-i\,z}}{2} = \cos z.$$

We also have

$$\cos(z + 2\,\pi) = \cos z \qquad \text{and} \qquad \sin(z + 2\,\pi) = \sin z,$$

thus these are still periodic functions, with (real) period $2\,\pi$. By recalling that

$$\cosh x = \frac{e^{x} + e^{-x}}{2}, \qquad \sinh x = \frac{e^{x} - e^{-x}}{2}, \qquad \text{for } x \in \mathbb{R},$$

with some elementary manipulations we obtain

$$\cos(x + i\,y) = \cos x\,\cosh y - i\,\sin x\,\sinh y,$$

$$\sin(x + i\,y) = \sin x\,\cosh y + i\,\cos x\,\sinh y. \tag{1.5.8}$$

In particular, we obtain that[1]

$$\cos z = 0 \quad \Longleftrightarrow \quad \begin{cases} \cos x\,\cosh y &= 0 \\ \sin x\,\sinh y &= 0 \end{cases}$$

$$\Longleftrightarrow \quad \begin{cases} \cos x = 0 \\ \sin x = 0 \quad \text{or} \quad \sinh y = 0 \end{cases}$$

By observing that if $\sinh y \neq 0$ the previous system does not admit solution and that $\sinh y$ vanishes at $y = 0$ only, we finally get

$$\cos z = 0 \qquad \Longleftrightarrow \qquad z = \frac{\pi}{2}\,(2\,k + 1),\ \ k \in \mathbb{Z}.$$

In other words, *the zeros of the complex cosinus coincide with the zeros of its restriction to the real axis.* In a similar fashion, we can prove the same property for the sinus.

By appealing to the definition, it is not difficult to see that we still have the usual addition formulas for every $z, w \in \mathbb{C}$

$$\cos(z + w) = \cos z\,\cos w - \sin z\,\sin w,$$

$$\sin(z + w) = \sin z\,\cos w + \cos z\,\sin w.$$

We also have the fundamental relation

$$\cos^2 z + \sin^2 z = 1, \qquad z \in \mathbb{C}.$$

This may be proved by observing that $\cos^2 z + \sin^2 z$ is an entire function and

$$\frac{d}{dz}\left(\cos^2 z + \sin^2 z\right) = -2\,\cos z\,\sin z + 2\,\sin z\,\cos z = 0.$$

[1] Recall that $\cosh y \geq 1$, for every $y \in \mathbb{R}$. In particular $\cosh y$ never vanishes.

By Lemma 1.4.6 we get that $\cos^2 z + \sin^2 z$ has to be constant. In particular

$$\cos^2 z + \sin^2 z = \cos^2 0 + \sin^2 0 = 1, \qquad z \in \mathbb{C}.$$

As in the case of one real variable, we have

$$\lim_{z\to 0} \frac{\sin z}{z} = 1 \qquad \text{and} \qquad \lim_{z\to 0} \frac{1-\cos z}{z^2} = \frac{1}{2}. \tag{1.5.9}$$

In order to prove the first one, it is sufficient to observe that by definition this coincides with the derivative of sinus at $z = 0$. Thus we get

$$\lim_{z\to 0} \frac{\sin z}{z} = \frac{d \sin z}{dz}_{|z=0} = \cos(0) = 1.$$

For the second limit we proceed as follows

$$\begin{aligned} \lim_{z\to 0} \frac{1-\cos z}{z^2} = \lim_{z\to 0} \frac{1-\cos z}{z^2} \frac{1+\cos z}{1+\cos z} &= \lim_{z\to 0} \frac{1}{1+\cos z} \frac{1-\cos^2 z}{z^2} \\ &= \frac{1}{2} \lim_{z\to 0} \frac{\sin^2 z}{z^2} = \frac{1}{2} \lim_{z\to 0} \left(\frac{\sin z}{z}\right)^2, \end{aligned}$$

which gives the desired conclusion.

6. Integrals in the complex plane

Definition 1.6.1. Let $a < b$ be two real numbers, a *curve in the complex plane* is a function $\gamma : [a, b] \to \mathbb{C}$. We will denote by

$$\Gamma_\gamma := \{z \in \mathbb{C} \,:\, \exists t \in [a, b] \text{ such that } z = \gamma(t)\},$$

the image of γ. We say that γ is *regular* if is C^1 and

$$|\gamma'(t)| \neq 0, \qquad t \in [a, b].$$

We say that a continuous curve $\gamma : [a, b] \to \mathbb{C}$ is

- *closed* if $\gamma(a) = \gamma(b)$;
- *simple* if γ is injective on $[a, b)$;
- a *loop* if it is a closed simple curve (see figure below).

Definition 1.6.2 (Reparametrization)**.** Let $\gamma : [a, b] \to \mathbb{C}$ be a regular curve. Let $\phi : [c, d] \to [a, b]$ be a C^1 strictly monotone surjective function, with $\phi'(t) \neq 0$ for every $t \in [c, d]$. Then the new curve $\widetilde{\gamma} := \gamma \circ \phi : [c, d] \to \mathbb{C}$ is said to be a *reparametrization* of γ. We say that the reparametrization is:

- *orientation preserving* if $\phi'(t) > 0$ for every $t \in [c, d]$;
- *orientation reversing* if $\phi'(t) < 0$ for every $t \in [c, d]$.

We use the notation γ^- for the particular orientation reversing reparametrization $\gamma^- : [a, b] :\to \mathbb{C}$ defined by

$$\gamma^-(t) = \gamma(b - t + a). \tag{1.6.1}$$

Roughly speaking, this is the curve γ "run in the opposite sense".

Figure 3. From left to right: the image of a closed curve which is not simple; the image of a simple curve which is not closed; the image of a loop.

Example 1.6.3 (Circle)**.** Let $z_0 \in \mathbb{C}$ and $r > 0$. The curve $\gamma : [0,1] \to \mathbb{C}$ defined by

$$\gamma(t) = z_0 + r\, e^{2\,\pi\, i\, t}, \qquad t \in [0,1],$$

is a regular loop. Its image is the circle of radius r and center z_0. The curve $\widetilde{\gamma} : [0, 2\,\pi] \to \mathbb{C}$ defined by

$$\widetilde{\gamma}(t) = z_0 + r\, e^{i\, t}, \qquad t \in [0, 2\,\pi],$$

is an orientation preserving reparametrization of γ. Indeed, we have $\widetilde{\gamma} = \gamma \circ \phi$, with $\phi(t) = t/(2\,\pi)$. We also observe that the orientation reversing reparametrization γ^- is given by

$$\gamma^-(t) = z_0 + r\, e^{2\,\pi\, i\,(1-t)} = z_0 + r\, e^{-2\,\pi\, i\, t}, \qquad t \in [0, 2\,\pi].$$

Definition 1.6.4 (Glueing of curves)**.** Let $\gamma_1 : [a,b] \to \mathbb{C}$ and $\gamma_2 : [b,c] \to \mathbb{C}$ be two continuous curves such that

$$\gamma_1(b) = \gamma_2(b).$$

Then we can *glue* the two curves together, by defining the new curve $\widehat{\gamma_1\, \gamma_2} : [a,c] \to \mathbb{C}$ through

$$\widehat{\gamma_1\, \gamma_2}(t) = \begin{cases} \gamma_1(t), & \text{if } t \in [a,b], \\ \gamma_2(t), & \text{if } t \in [b,c]. \end{cases}$$

We say that a curve $\gamma : [a,b] \to \mathbb{C}$ is *piecewise regular* if it is obtained by gluing together a finite number of regular curves.

Definition 1.6.5. Let $f : A \to \mathbb{C}$ be a continuous function on the open set $A \subset \mathbb{C}$. Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise regular curve such that $\Gamma_\gamma \subset A$. Then we set

$$\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\, \gamma'(t)\, dt.$$

Remark 1.6.6. The value of the integral does not change if we replace γ by an orientation preserving reparametrization. Indeed, let $\widetilde{\gamma} = \gamma \circ \phi : [c,d] \to \mathbb{C}$ be such a reparametrization (by hypothesis

$\phi' > 0$), then we have

$$\int_{\widetilde{\gamma}} f(z)\,dz = \int_c^d f(\widetilde{\gamma}(t))\,\widetilde{\gamma}'(t)\,dt = \int_c^d f(\gamma(\phi(t)))\,\gamma'(\phi(t))\,\phi'(t)\,dt$$
$$= \int_a^b f(\gamma(\tau))\,\gamma'(\tau)\,d\tau = \int_\gamma f(z)\,dz,$$

where we used the usual change of variable formula for integrals of one real variable. With similar manipulations we obtain that if $\widetilde{\gamma} = \gamma \circ \phi$ is orientation reversing (so that $\phi' < 0$), then

$$\int_{\widetilde{\gamma}} f(z)\,dz = -\int_\gamma f(z)\,dz.$$

In particular, by recalling the definition (1.6.1), we have

$$\int_\gamma f(z)\,dz + \int_{\gamma^-} f(z)\,dz = 0, \tag{1.6.2}$$

for every piecewise regular curve γ.

The following simple result can be regarded as the *fundamental Theorem of Calculus*, in the complex plane.

Lemma 1.6.7. *Let $A \subset \mathbb{C}$ be a connected open set and let $f : A \to \mathbb{C}$ be an holomorphic function. For every $z_0, z_1 \in A$ we have*

$$f(z_1) = f(z_0) + \int_\gamma f'(z)\,dz,$$

where $\gamma : [a, b] \to \mathbb{C}$ is any piecewise regular curve such that $\Gamma_\gamma \subset A$ and

$$\gamma(b) = z_1 \qquad \text{and} \qquad \gamma(a) = z_0.$$

Proof. It is sufficient to use the definition of integral, i.e.

$$\int_\gamma f'(z)\,dz = \int_a^b f'(\gamma(t))\,\gamma'(t)\,dt,$$

and observe that

$$\frac{d}{dt} f(\gamma(t)) = f'(\gamma(t))\,\gamma'(t).$$

Thus, by the fundamental Theorem of Calculus for functions of one real variable, we get

$$\int_\gamma f'(z)\,dz = \int_a^b f'(\gamma(t))\,\gamma'(t)\,dt = f(\gamma(b)) - f(\gamma(a)) = f(z_1) - f(z_0),$$

as desired. □

Definition 1.6.8. Let $\gamma : [a, b] \to \mathbb{C}$ be a regular loop. We define its *tangent versor* by

$$\mathbf{T}_\gamma(t) = \frac{\gamma'(t)}{|\gamma'(t)|}, \qquad t \in [a, b].$$

Its *normal versor* is defined by[2]

$$\mathbf{N}_\gamma(t) = -i\,\mathbf{T}_\gamma(t), \qquad t \in [a, b].$$

[2] By recalling formula (1.1.2), the multiplication by $-i$ geometrically corresponds to rotate the versor $\mathbf{T}_\gamma(t)$ clockwise of an angle $\pi/2$.

Definition 1.6.9. Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise regular loop. We denote by D the set entoured by γ. We say that γ is *positively oriented* if for every $t \in [a, b]$ the normal versor $\mathbf{N}_\gamma(t)$ is exiting from D.

Example 1.6.10. Let $z_0 \in \mathbb{C}$ and $r > 0$, then the regular loop

$$\gamma(t) = z_0 + r\, e^{2\,\pi\, i\, t}, \qquad \text{for } t \in [0, 1],$$

is positively oriented, while γ^- is negatively oriented.

The following simple result will be useful.

Lemma 1.6.11. *Let $\{g_k\}_{k\in\mathbb{N}}$ be a sequence of continuous functions on the open set $A \subset \mathbb{C}$. Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise regular curve, whose image in contained in A. Assume that $\{g_k\}_{k\in\mathbb{N}}$ converges uniformly on the image of γ to some continuous function g, i.e.*

$$\lim_{k\to\infty} \left(\max_{z\in\Gamma_\gamma} |g_k(z) - g(z)| \right) = 0.$$

Then we have

$$\lim_{k\to\infty} \int_\gamma g_k(z)\, dz = \int_\gamma g(z)\, dz.$$

Proof. Let $\varepsilon > 0$, by definition of uniform convergence there exists $k_0 \in \mathbb{N}$ such that

$$|g_k(\gamma(t)) - g(\gamma(t))| < \varepsilon, \qquad \text{for every } k \geq k_0 \;\text{ and }\; t \in [a, b].$$

We thus obtain for every $k \geq k_0$

$$\begin{aligned}\left| \int_\gamma g_k(z)\, dz - \int_\gamma g(z)\, dz \right| &= \left| \int_a^b \Big[g_k(\gamma(t)) - g(\gamma(t)) \Big]\, \gamma'(t)\, dt \right| \\ &\leq \int_a^b |g_k(\gamma(t)) - g(\gamma(t))|\, |\gamma'(t)|\, dt \\ &\leq \varepsilon \int_a^b |\gamma'(t)|\, dt.\end{aligned}$$

By the arbitrariness of $\varepsilon > 0$, we get the conclusion. □

Theorem 1.6.12 (Cauchy's Theorem)**.** *Let $A \subset \mathbb{C}$ be a connected open set and let $f : A \to \mathbb{C}$ be an holomorphic function. For every positively oriented piecewise regular loop γ such that $\Gamma_\gamma \subset A$ and such that the region entoured by Γ_γ is contained in A, we have*

$$\int_\gamma f(z)\, dz = 0.$$

Proof. We write

$$f(z) = u(x, y) + i\, v(x, y), \qquad z = x + i\, y,$$

and

$$\gamma(t) = \gamma_1(t) + i\, \gamma_2(t), \qquad t \in [a, b],$$

with $\gamma_1, \gamma_2 : [a, b] \to \mathbb{R}$ piecewise C^1 functions. We can write the integral of f on γ as

$$\begin{aligned}\int_\gamma f(z)\, dz &= \int_a^b \Big[u(\gamma_1(t), \gamma_2(t)) + i\, v(\gamma_1(t), \gamma_2(t))\Big]\, (\gamma_1'(t) + i\, \gamma_2'(t))\, dt \\ &= \int_a^b \Big[u(\gamma_1(t), \gamma_2(t))\, \gamma_1'(t) - v(\gamma_1(t), \gamma_2(t))\, \gamma_2'(t)\Big]\, dt \\ &+ i \int_a^b \Big[u(\gamma_1(t), \gamma_2(t))\, \gamma_2'(t) + v(\gamma_1(t), \gamma_2(t))\, \gamma_1'(t)\Big]\, dt.\end{aligned}$$

If we introduce the two vector fields

$$\mathbf{V}(x, y) = (u(x, y), -v(x, y)) \qquad \text{and} \qquad \mathbf{W}(x, y) = (v(x, y), u(x, y)),$$

we can rewrite the previous formula

$$\int_\gamma f(z)\, dz = \int_\gamma \langle \mathbf{V}, \mathbf{T}_\gamma \rangle\, d\ell + i \int_\gamma \langle \mathbf{W}, \mathbf{T}_\gamma \rangle\, d\ell,$$

and the last two integrals represent the work of the two vector fields along the curve γ. Let us call $D \subset A$ the region entoured by γ, so that γ is a positively oriented parametrization of ∂D. Then by using the *Gauss-Green formula* we know that

$$\int_\gamma \langle \mathbf{V}, \mathbf{T}_\gamma \rangle\, d\ell = \iint_D \left[-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]\, dx\, dy,$$

and

$$\int_\gamma \langle \mathbf{W}, \mathbf{T}_\gamma \rangle\, d\ell = \iint_D \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right]\, dx\, dy.$$

We thus obtained

$$\int_\gamma f(z)\, dz = \iint_D \left[-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]\, dx\, dy + i \iint_D \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right]\, dx\, dy.$$

We now get the conclusion by recalling that

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},$$

which are the Cauchy-Riemann equations (1.4.3) in real form. □

Corollary 1.6.13 (Deformation of contour)**.** *Let $A \subset \mathbb{C}$ be a connected open set and let $f : A \to \mathbb{C}$ be a holomorphic function. Let γ_1 and γ_2 be two piecewise regular loops contained in A, both positively oriented. We indicate with D_1 and D_2 the regions entoured by Γ_{γ_1} and Γ_{γ_2} respectively. We suppose that $D_2 \subset D_1$ and that $D_1 \setminus D_2 \subset A$. Then we have*

$$\int_{\gamma_1} f(z)\, dz = \int_{\gamma_2} f(z)\, dz.$$

Proof. We give an idea of the proof. We connect the images of γ_1 and γ_2 through two simple curves ξ_1 and ξ_2, as in Figure 4. We thus obtain two new pairs of simple curves, that we call γ_1^N, γ_1^S and γ_2^S, γ_2^N. By construction, we have

$$\widehat{\gamma_1^N\, \gamma_1^S} = \gamma_1 \qquad \text{and} \qquad \widehat{\gamma_2^N\, \gamma_2^S} = \gamma_2. \tag{1.6.3}$$

We now define two piecewise regular positively oriented loops as follows

$$\widehat{\gamma_1^N\, \xi_2\, (\gamma_2^N)^-\, \xi_1} \qquad \text{and} \qquad \widehat{\gamma_1^S\, (\xi_1)^-\, (\gamma_2^S)^-\, (\xi_2)^-},$$

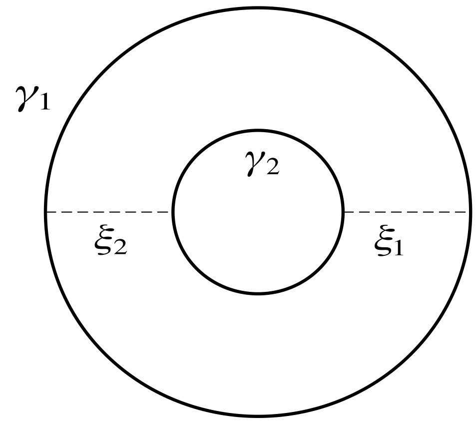

Figure 4. An illustration of the hypotheses and the construction of Corollary 1.6.13. The integral on γ_1 of a function f is the same as that on γ_2, provided that f is holomorphic in the annular region in between the two curves.

i.e. they are both obtained by glueing 4 simple curves. By construction and thanks to the hypotheses, we have that both loops entour a region on which f is holomorphic. Thus by Theorem 1.6.12, we have

$$\int_{\widehat{\gamma_1^N\,\xi_2\,(\gamma_2^N)^-\,\xi_1}} f(z)\,dz = 0,$$

and

$$\int_{\widehat{\gamma_1^S\,(\xi_1)^-\,(\gamma_2^S)^-\,(\xi_2)^-}} f(z)\,dz = 0.$$

By using the definition of glueing of curves and property (1.6.2), these equations become

$$\int_{\gamma_1^N} f(z)\,dz + \int_{\xi_2} f(z)\,dz - \int_{\gamma_2^N} f(z)\,dz + \int_{\xi_1} f(z)\,dz = 0,$$

and

$$\int_{\gamma_1^S} f(z)\,dz - \int_{\xi_1} f(z)\,dz - \int_{\gamma_2^S} f(z)\,dz - \int_{\xi_2} f(z)\,dz = 0.$$

By summing these two equation and erasing the integrals over ξ_1 and ξ_2, we obtain

$$\int_{\gamma_1^N} f(z)\,dz + \int_{\gamma_1^S} f(z)\,dz - \int_{\gamma_2^N} f(z)\,dz - \int_{\gamma_2^S} f(z)\,dz = 0.$$

By recallig (1.6.3), we get the desired conclusion. □

In turn, the *deformation of contour* implies the following remarkable result.

Theorem 1.6.14 (Cauchy's integral formula)**.** *Let $A \subset \mathbb{C}$ be a connected open set and let $f : A \to \mathbb{C}$ be a holomorphic function. Let γ be a positively oriented piecewise regular loop contained in A, together with the region D entoured by γ. For every $z \in D$ we have*

$$f(z) = \frac{1}{2\,\pi\,i} \int_\gamma \frac{f(s)}{s-z}\,ds. \qquad (1.6.4)$$

Proof. Let $z \in D$ and let $r > 0$ be small enough, so that the disk $B_r(z)$ is contained in D. Let

$$\gamma_r(t) = r\, e^{2\pi\, i\, t} + z, \qquad t \in [0,1],$$

and observe that this is positively oriented. Then by applying Corollary 1.6.13 to the function $s \mapsto f(s)/(s-z)$ (which is holomorphic in the open set $A \setminus \{z\}$), we obtain

$$\begin{aligned}\int_\gamma \frac{f(s)}{s-z}\, ds = \int_{\gamma_r} \frac{f(s)}{s-z}\, ds &= 2\,\pi\, i \int_0^1 \frac{f(z + r\, e^{2\,\pi\, i\, t})}{r\, e^{2\,\pi\, i\, t}}\, r\, e^{2\,\pi\, i\, t}\, dt \\ &= 2\,\pi\, i \int_0^1 f(z + r\, e^{2\,\pi\, i\, t})\, dt.\end{aligned}$$

This identity implies in particular that the last integral is independent of $r > 0$. Thus we get

$$\int_\gamma \frac{f(s)}{s-z}\, ds = \lim_{r\to 0} \int_0^1 f(z + r\, e^{2\,\pi\, i\, t})\, dt \tag{1.6.5}$$

By using that the function of one real variable

$$t \mapsto f(z + r\, e^{2\,\pi\, i\, t}), \qquad t \in [0,1],$$

converges uniformly to $f(z)$, as r goes to 0, we get

$$\lim_{r\to 0} \int_0^1 f(z + r\, e^{2\,\pi\, i\, t})\, dt = \int_0^1 f(z)\, dt = f(z),$$

i.e. we can pass the limit under the integral sign. By using this in (1.6.5), we get the conclusion. □

7. *Intermezzo:* complex power series

Let $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ be a sequence of complex numbers. For a fixed $z_0 \in \mathbb{C}$, we can consider the power series centered at z_0

$$\sum_{n=0}^{\infty} a_n\, (z - z_0)^n.$$

This is well-defined for every $z \in \mathbb{C}$ such that the sequence

$$s_k(z) := \sum_{n=0}^{k} a_n\, (z - z_0)^n, \tag{1.7.1}$$

converges to a complex number $\lambda \in \mathbb{C}$. This means that

$$\forall \varepsilon > 0,\ \exists k_0 \in \mathbb{N} \ \text{ such that } \ |s_k(z) - \lambda| < \varepsilon \ \text{ for every } k \geq k_0.$$

We observe that a power series is a particular case of the larger class of *series of functions*.

Definition 1.7.1. We say that the power series

$$\sum_{n=0}^{\infty} a_n\, (z - z_0)^n,$$

converges:

- *absolutely* if
$$\sum_{n=0}^{\infty} |a_n|\, |z - z_0|^n < +\infty;$$
- *uniformly on* $A \subset \mathbb{C}$ if the sequence of functions $\{s_k\}_{k\in\mathbb{N}}$ defined by (1.7.1) converge uniformly on A;

- *totally on* $A \subset \mathbb{C}$ if

$$\sum_{n=0}^{\infty} \sup_{z\in A} \left(|a_n|\, |z-z_0|^n\right) < +\infty.$$

Remark 1.7.2. We observe that if for some $z_1 \neq z_0$ a power series is absolutely convergent, then it is automatically totally convergent on the closed disk $\overline{B_\varrho(z_0)}$, where $\varrho = |z_1 - z_0|$. Indeed, for every $n \in \mathbb{N}$ we have

$$\sup_{z\in\overline{B_\varrho(z_0)}} \left(|a_n|\, |z-z_0|^n\right) = |a_n|\, \varrho^n = |a_n|\, |z_1 - z_0|^n,$$

and thus

$$\sum_{n=0}^{\infty} \sup_{z\in\overline{B_\varrho(z_0)}} \left(|a_n|\, |z-z_0|^n\right) = \sum_{n=0} |a_n|\, |z_1 - z_0|^n < +\infty.$$

This property is of course a peculiarity of power series.

Theorem 1.7.3. *Let* $\sum_{n=0}^{\infty} a_n\,(z-z_0)^n$ *be a power series with*

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = L < +\infty. \tag{1.7.2}$$

i) The power series is totally convergent on every closed disk $\overline{B_\varrho(z_0)}$ *with radius* $\varrho < 1/L$ *(with the convention that if* $L = 0$*, then* $1/L = +\infty$*).*

ii) The power series does not converge for every z *such that* $|z - z_0| > 1/L$*.*

Proof. In order to prove *i)*, we first observe that by Remark 1.7.2 it is sufficient to prove that for every $\varrho < 1/L$, the power series

$$\sum_{n=0}^{\infty} a_n \left((\varrho + z_0) - z_0\right)^n = \sum_{n=0}^{\infty} a_n\, \varrho^n,$$

is absolutely convergent. From (1.7.2), we know that for every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\sqrt[n]{|a_n|} < L + \varepsilon, \qquad \text{for every } n \geq n_\varepsilon.$$

In particular, if we take

$$\varepsilon = \frac{1}{2}\left(\frac{1}{\varrho} - L\right) > 0,$$

then there exists $n_\varepsilon \in \mathbb{N}$ such that

$$|a_n| < (L+\varepsilon)^n = \left(\frac{L}{2} + \frac{1}{2\,\varrho}\right)^n, \qquad \text{for every } n \geq n_\varepsilon.$$

Thus we get

$$|a_n|\, \varrho^n < \left(\frac{L}{2} + \frac{1}{2\,\varrho}\right)^n \varrho^n = \left(\frac{L\,\varrho + 1}{2}\right)^n, \qquad \text{for every } n \geq n_\varepsilon.$$

Since by construction

$$\frac{L\,\varrho + 1}{2} < 1,$$

we get the desired result by comparison with the geometric series.

Let us now prove *ii)*. Still by (1.7.2), we know that for every $\varepsilon > 0$ there exists a subsequence $\{a_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\sqrt[n_k]{|a_{n_k}|} > L - \varepsilon, \qquad \text{for every } k.$$

We pick $z \in \mathbb{C}$ such that $|z - z_0| = \varrho > 1/L$ and choose

$$\varepsilon = \frac{1}{2}\left(L - \frac{1}{\varrho}\right) > 0,$$

thus there exists a subsequence $\{a_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\begin{aligned} |a_{n_k}|\,|z - z_0|^{n_k} > (L-\varepsilon)^{n_k}\ \varrho^{n_k} &= \left(\frac{1}{2\,\varrho} + \frac{L}{2}\right)^{n_k}\ \varrho^{n_k} \\ &= \left(\frac{1 + L\,\varrho}{2}\right)^{n_k} > 1, \qquad \text{for every } k. \end{aligned}$$

This implies that $a_n\,(z-z_0)^n$ doest not converge to zero and thus the power series can not converge.

□

Definition 1.7.4. Let $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ be a sequence such that

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = L.$$

Then $R = 1/L$ is called *radius of convergence* of the power series $\sum_{n=0}^{\infty} a_n\,(z - z_0)^n$. We use the following conventions:

$$R = \frac{1}{L} = +\infty, \qquad \text{if } L = 0,$$

and

$$R = \frac{1}{L} = 0, \qquad \text{if } L = +\infty.$$

Proposition 1.7.5. *Let*

$$s(z) = \sum_{n=0}^{\infty} a_n\,(z - z_0)^n,$$

be a power series with radius of convergence $R > 0$. Then the new series $\sum_{n=1}^{\infty} n\,a_n\,(z - z_0)^{n-1}$ has the same radius of convergence $R > 0$.

Moreover, s is a holomorphic function on $B_R(z_0)$, with

$$s'(z) = \sum_{n=1}^{\infty} n\,a_n\,(z - z_0)^{n-1}, \quad z \in B_R(z_0).$$

Proof. We first verify the first statement about the radius of convergence. We first rewrite

$$\sum_{n=1}^{\infty} n\,a_n\,(z - z_0)^{n-1} = \frac{1}{z - z_0}\sum_{n=1}^{\infty} n\,a_n\,(z - z_0)^n,$$

then the radius of convergence of this power series is given by

$$\frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{n\,|a_n|}}.$$

It is then sufficient to observe that[3]

$$\limsup_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} e^{\frac{\log n}{n}} = 1,$$

thus we obtain

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty} \sqrt[n]{n\,|a_n|},$$

[3] We use that $\log n = o(n)$ for $n \to \infty$.

as desired.

We now show that the function s can be differentiated in complex sense. We take $z \in B_R(z_0)$ and $h \in \mathbb{C}^*$ such that we still have $z + h \in B_R(z_0)$. For this, it is sufficient to take

$$0 < |h| < \frac{R - |z - z_0|}{2}. \tag{1.7.3}$$

We now write

$$\begin{aligned}\frac{s(z+h) - s(z)}{h} &= \sum_{n=0}^{\infty} a_n \left[\frac{(z+h-z_0)^n - (z-z_0)^n}{h}\right] \\ &= a_1 \frac{(z - z_0 + h) - (z - z_0)}{h} + \sum_{n=2}^{\infty} a_n \left[\frac{(z+h-z_0)^n - (z-z_0)^n}{h}\right] \\ &= a_1 + \sum_{n=2}^{\infty} a_n \left[\frac{(z+h-z_0)^n - (z-z_0)^n}{h}\right].\end{aligned}$$

We take $\gamma : [0, 1] \to B_R(z_0)$ to be the regular curve such that

$$\gamma(t) = z + t\,h, \qquad \text{for } t \in [0, 1].$$

We observe that each function

$$s \mapsto (s - z_0)^n,$$

is holomorphic, thus by Exercise 1.13.1 for every $n \geq 2$ we can infer

$$(z + h - z_0)^n = (z - z_0)^n + n\,(z - z_0)^{n-1}\,h + \int_\gamma n\,(n-1)\,(w - z_0)^{n-2}\,(z + h - w)\,dw.$$

In other words, we have

$$\begin{aligned}\left[\frac{(z+h-z_0)^n - (z-z_0)^n}{h}\right] &= n\,(z - z_0)^{n-1} \\ &\quad + n\,(n-1)\,\frac{1}{h} \int_\gamma (w - z_0)^{n-2}\,(z + h - w)\,dw \\ &= n\,(z - z_0)^{n-1} \\ &\quad + n\,(n-1)\,h \int_0^1 (z + t\,h - z_0)^{n-2}\,(1 - t)\,dt.\end{aligned} \tag{1.7.4}$$

We now observe that the last term is the $n-$th term of a converging series. More precisely, we have

$$\begin{aligned}\left| n\,(n-1)\,|h| \int_0^1 |z + t\,h - z_0|^{n-2}\,(1 - t)\,dt \right| &\leq n\,(n-1)\,|h| \int_0^1 (|z - z_0| + t\,|h|)^{n-2}\,dt \\ &\leq n\,(n-1)\,|h| \int_0^1 \left(|z - z_0| + t\,\frac{R - |z - z_0|}{2}\right)^{n-2}\,dt \\ &\leq n\,(n-1)\,|h| \left(\frac{R + |z - z_0|}{2}\right)^{n-2},\end{aligned}$$

where we used (1.7.3), to estimate the integral from above. The claimed convergence above now follows from the convergence of the power series

$$\sum_{n=2}^{\infty} n\,(n-1)\,a_n \left(\frac{R + |z - z_0|}{2}\right)^{n-2}.$$

Indeed, we have

$$\lim_{n\to\infty} \sqrt[n]{n\,(n-1)\,|a_n|} = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{R},$$

and

$$\frac{R + |z - z_0|}{2} < R.$$

If we now set for simplicity

$$g(z) = \sum_{n=2}^{\infty} n\,(n-1) \int_0^1 (z + t\,h - z_0)^{n-2}\,(1-t)\,dt,$$

this is a finite quantity, by the previous discussion. We have obtained from (1.7.4)

$$\begin{aligned}\frac{s(z+h) - s(z)}{h} &= a_1 + \sum_{n\geq 2}^{\infty} n\,a_n\,(z - z_0)^{n-1} + h\,g(z)\\ &= \sum_{n=1}^{\infty} n\,a_n\,(z - z_0)^{n-1} + h\,g(z).\end{aligned}$$

By taking the limit as h goes to 0, we finally obtain that the function s is derivable in every $z \in B_R(z_0)$ and

$$s'(z) = \sum_{n=1}^{\infty} n\,a_n\,(z - z_0)^{n-1}.$$

By Theorem 1.7.3, the convergence of the last series is uniform on every $\overline{B_\varrho(z_0)}$ with $\varrho < R$. As every function $z \mapsto (z - z_0)^{n-1}$ is continuous, we get that s' is continuous as well on $\overline{B_\varrho(z_0)}$, for every $\varrho < R$. This finally shows that s is holomorphic on $B_R(z_0)$. □

By iterating the previous result, we obtain the following.

Corollary 1.7.6. *Let*

$$s(z) = \sum_{n=0}^{\infty} a_n\,(z - z_0)^n,$$

be a power series with radius of convergence $R > 0$. Then s is derivable infinitely many times in $B_R(z_0)$ and we have

$$s^{(k)}(z) = \sum_{n=k}^{\infty} n\,(n-1)\dots(n-k+1)\,a_n\,(z - z_0)^{n-k}, \qquad \textit{for } z \in B_R(z_0). \tag{1.7.5}$$

Remark 1.7.7. We observe that by taking $z = z_0$ in (1.7.5), we get

$$s^{(k)}(z_0) = k\,(k-1)\dots 1\,a_k = k!\,a_k.$$

Thus s can be rewritten as

$$s(z) = \sum_{n=0}^{\infty} \frac{s^{(n)}(z_0)}{n!}\,(z - z_0)^n.$$

In other words, *a power series centered at z_0 with positive radius of convergence is a C^∞ function which coincides with its Taylor series centered at z_0.*

The following result is useful. It states that a power series can be "integrated" term by term.

Corollary 1.7.8 (Integrating a power series)**.** *Let*

$$s(z) = \sum_{n=0}^{\infty} a_n \, (z - z_0)^n,$$

be a power series with radius of convergence $R > 0$. Then for every $c \in \mathbb{C}$, the new series

$$S(z) = c + \sum_{n=0}^{\infty} \frac{a_n}{n+1} \, (z - z_0)^{n+1},$$

have the same radius of convergence $R > 0$.

Moreover, S is a holomorphic function on $B_R(z_0)$ such that

$$S'(z) = \sum_{n=0}^{\infty} a_n \, (z - z_0)^n = s(z), \qquad \text{for } z \in B_R(z_0).$$

Proof. We can rewrite the second power series as

$$c + \sum_{n=0}^{\infty} \frac{a_n}{n+1} \, (z - z_0)^{n+1} = \sum_{n=0}^{\infty} \widetilde{a}_n \, (z - z_0)^n,$$

where

$$\widetilde{a}_n = \begin{cases} c, & \text{if } n = 0, \\ \dfrac{a_{n-1}}{n}, & \text{if } n \geq 1. \end{cases}$$

We then observe that

$$\begin{aligned}\limsup_{n\to\infty} \sqrt[n]{|\widetilde{a}_n|} = \limsup_{n\to\infty} \sqrt[n]{\frac{|a_{n-1}|}{n}} = \limsup_{n\to\infty} \frac{\sqrt[n]{|a_{n-1}|}}{\sqrt[n]{n}} &= \limsup_{n\to\infty} \sqrt[n]{|a_{n-1}|} \\ &= \limsup_{n\to\infty} \sqrt[n]{|a_n|},\end{aligned}$$

thus the two power series have the same radius of convergence. The second part of the statement now follows by appyling Proposition 1.7.5 to the power series

$$\sum_{n=0}^{\infty} \widetilde{a}_n \, (z - z_0)^n.$$

This concludes the proof. □

Example 1.7.9. We can use the previous results to compute explicitly the sum of some remarkable power series. For example, we know that

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \qquad \text{for } |z| < 1.$$

Then the two power series

$$f(z) = \sum_{n=1}^{\infty} n \, z^{n-1} \qquad \text{and} \qquad g(z) = c + \sum_{n=0}^{\infty} \frac{1}{n+1} \, z^{n+1},$$

still have radius of convergence 1 and they are holomorphic functions on the open disk $B_1(0)$. Moreover, we know by Proposition 1.7.5 and Corollary 1.7.8 that

$$f(z) = \frac{d}{dz} \frac{1}{1-z} = \frac{1}{(1-z)^2}, \qquad \text{for } |z| < 1,$$

and

$$g'(z) = \frac{1}{1-z}, \qquad \text{for } |z| < 1.$$

In other words, g is a *primitive* of $1/(1-z)$ in $B_1(0)$. Such a primitive can be computed explicitly, by observing that the function $h(z) = \operatorname{Log}(1-z)$ is holomorphic on

$$A := \mathbb{C} \setminus \{z \in \mathbb{C} \,:\, \operatorname{Re} z \geq 1 \text{ and } \operatorname{Im} z = 0\},$$

with

$$h'(z) = -\frac{1}{1-z}, \qquad z \in A.$$

We thus obtain that

$$g'(z) = -h'(z) \qquad \text{for } z \in A \cap B_1(0) = B_1(0).$$

Since $B_1(0)$ is a connected open set, this means that g and $-h$ coincides on $B_1(0)$ up to a constant (thanks to Lemma 1.4.6). Finally, since $g(0) = c$ and $-h(0) = 0$, this implies that

$$g(z) = c - h(z), \qquad \text{for } |z| < 1$$

and thus

$$c + \sum_{n=0}^{\infty} \frac{1}{n+1}\, z^{n+1} = c - \operatorname{Log}(1-z), \qquad |z| < 1.$$

Observe that we get in particular

$$\sum_{n=1}^{\infty} \frac{z^n}{n} = -\operatorname{Log}(1-z), \qquad |z| < 1.$$

8. Properties of holomorphic functions

Definition 1.8.1. Let $A \subset \mathbb{C}$ be an open connected set, we say that $f : A \to \mathbb{C}$ is *analytic in A* if for every $z_0 \in A$ it admits the Taylor series expansion in every $B_r(z_0) \subset A$, i.e.

$$f(z) = \sum_{n=0}^{\infty} c_n\,(z - z_0)^n, \qquad z \in B_r(z_0),$$

with $c_n = f^{(n)}(z_0)/n!$

Holomorphic functions have the following striking property.

Theorem 1.8.2 (Holomorphic = analytic)**.** *Let $A \subset \mathbb{C}$ be a connected open set and let $f : A \to \mathbb{C}$ be a holomorphic function. Then f is analytic, i.e. for every $z_0 \in A$ we have*

$$f(z) = \sum_{n=0}^{\infty} c_n\,(z - z_0)^n, \qquad \textit{for } |z - z_0| < \operatorname{dist}(z_0, \partial A).$$

Moreover, each coefficient c_n has the following expression

$$c_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\,\pi\, i} \int_\gamma \frac{f(s)}{(s - z_0)^{n+1}}\, ds, \tag{1.8.1}$$

where γ is any positively oriented piecewise regular loop such that $\Gamma_\gamma \subset A$ and such that, if we call D the domain entoured by Γ_γ, we have $z_0 \in D \subset A$.

Proof. Let $z \in A$ be such that $|z - z_0| < \mathrm{dist}(z_0, \partial A)$. We set $r = |z - z_0|$ and set

$$R = \frac{r + \mathrm{dist}(z_0, \partial A)}{2}.$$

We take the positively oriented loop

$$\gamma_R(t) = z_0 + R\, e^{2\,\pi\, i\, t}, \qquad t \in [0, 1],$$

whose image is the circle $\partial B_R(z_0)$ centered at z_0 with radius R. By construction, we have $z \in B_R(z_0) \subset A$, thus we can apply Cauchy's formula (1.6.4). This gives

$$\begin{aligned} f(z) = \frac{1}{2\,\pi\, i} \int_{\gamma_R} \frac{f(s)}{s - z}\, ds &= \frac{1}{2\,\pi\, i} \int_{\gamma_R} \frac{f(s)}{s - z_0 - (z - z_0)}\, ds \\ &= \frac{1}{2\,\pi\, i} \int_{\gamma_R} \frac{f(s)}{s - z_0}\, \frac{1}{1 - \dfrac{z - z_0}{s - z_0}}\, ds. \end{aligned}$$

We now observe that by construction

$$\left| \frac{z - z_0}{s - z_0} \right| = \frac{r}{R} < 1, \qquad \text{for every } s \in \Gamma_{\gamma_R}, \tag{1.8.2}$$

thus we have

$$\frac{1}{1 - \dfrac{z - z_0}{s - z_0}} = \sum_{n=0}^{\infty} \left(\frac{z - z_0}{s - z_0} \right)^n,$$

and the convergence of the series is uniform for $s \in \Gamma_{\gamma_R}$, thanks to (1.8.2). We thus obtain

$$\begin{aligned} f(z) &= \frac{1}{2\,\pi\, i} \int_{\gamma_R} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{s - z_0} \right)^n \frac{f(s)}{s - z_0}\, ds \\ &= \frac{1}{2\,\pi\, i} \sum_{n=0}^{\infty} \left(\int_{\gamma_R} \frac{f(s)}{(s - z_0)^{n+1}}\, ds \right) (z - z_0)^n, \end{aligned}$$

where the exchange between the summation and integral sign has been possible thanks to the uniform convergence of the series[4]. This shows that f is analytic, with coefficients given by (1.8.1) and γ_R the curve whose image is the circle $\partial B_R(z_0)$.

On the other hand, by observing that the function

$$s \mapsto \frac{f(s)}{(s - z_0)^{n+1}},$$

is holomorphic in $A \setminus \{z_0\}$, by Corollary 1.6.13 the integral

$$\int_{\gamma_R} \frac{f(s)}{(s - z_0)^{n+1}}\, ds,$$

is unchanged if γ_R is replaced by the positively oriented loop

$$\gamma_\varrho(t) = z_0 + \varrho\, e^{2\,\pi\, i\, t}, \qquad t \in [0, 1],$$

[4]In other words, we can use Lemma 1.6.11 with the choices

$$g_k(s) = \sum_{n=0}^{k} \left(\frac{z - z_0}{s - z_0} \right)^n, \qquad g(s) = \sum_{n=0}^{\infty} \left(\frac{z - z_0}{s - z_0} \right)^n.$$

where $\varrho > 0$ is any radius such that
$$\varrho < \operatorname{dist}(z_0, \partial A).$$
Thus, if γ is any positively oriented piecewise regular loop as in the statement, by choosing $\varrho > 0$ sufficiently small we get that γ and γ_ϱ satisfy the hypotheses of Corollary 1.6.13. In conclusion we get
$$\int_{\gamma_R} \frac{f(s)}{(s-z_0)^{n+1}}\, ds = \int_{\gamma_\varrho} \frac{f(s)}{(s-z_0)^{n+1}}\, ds = \int_{\gamma} \frac{f(s)}{(s-z_0)^{n+1}}\, ds,$$
and this concludes the proof. □

Remark 1.8.3 (Taylor expansion of the exponential)**.** From the previous result, we get that the entire function $f(z) = e^z$ is analytic in $\mathbb{C}$ and there holds
$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad \text{for } z \in \mathbb{C}.$$

We now have a closer look at the zeros of a holomorphic function. First of all, we need the following

Definition 1.8.4. Let $f : A \to \mathbb{C}$ be a holomorphic function, we say that $z_0 \in A$ is a *zero of order* $m \in \mathbb{N} \setminus \{0\}$ if
$$f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0 \qquad \text{and} \qquad f^{(m)}(z_0) \neq 0.$$
Observe that since f is analytic (Theorem 1.8.2), if it has a zero of order m at z_0, then in a neighborhood of z_0 it admits the Taylor expansion
$$f(z) = \sum_{k=m}^{\infty} \frac{f^{(k)}(z_0)}{k!}\,(z-z_0)^k.$$

Proposition 1.8.5 (Unique continuation principle)**.** *Let $A \subset \mathbb{C}$ be an open connected set and let $f : A \to \mathbb{C}$ be a holomorphic function. The following three facts are equivalent:*

1. *there exists $z_0 \in A$ such that $f^{(n)}(z_0) = 0$, for every $n \in \mathbb{N}$;*
2. *f vanishes identically in $B_r(z_0)$ for some $r > 0$;*
3. *f vanishes identically in A.*

Proof. Of course, we easily have 3. $\Longrightarrow$ 2. $\Longrightarrow$ 1. Also, by using the fact that f is analytic by Theorem 1.8.2, we easily get that 1. $\Longrightarrow$ 2. In order to conclude the proof, it is left to prove that 2. $\Longrightarrow$ 3. This point is delicate and we omit it, the reader can find the proof in [1] or [2]. □

Remark 1.8.6. The previous result asserts in particular that *a holomorphic function can not have a zero of infinite order*, unless it is the trivial function $f \equiv 0$. This is a peculiarity of functions of one complex variable, since for functions of one real variable this could happen. For example, the function
$$f(x) = \begin{cases} \exp\left(-\dfrac{1}{x}\right), & \text{for } x > 0, \\ 0, & \text{for } x \leq 0. \end{cases}$$
is such that $f^{(n)}(0) = 0$ for every $n \in \mathbb{N}$, but it does not reduce to the function identically vanishing on $\mathbb{R}$.

Corollary 1.8.7. *Let $A \subset \mathbb{C}$ be an open connected set. Let $f, g : A \to \mathbb{C}$ be two holomorphic functions such that one of the following properties is satisfied:*

- *there exists* $z_0 \in A$ *and* $r > 0$ *such that* $f = g$ *on* $B_r(z_0)$*;*
- *there exists* $z_0 \in A$ *such that* $f^{(n)}(z_0) = g^{(n)}(z_0)$ *for every* $n \in \mathbb{N}$*.*

Then f *and* g *coincide on* A*.*

Proof. It is sufficient to apply Proposition 1.8.5 to the function $f - g$. □

Proposition 1.8.8. *Let* $A \subset \mathbb{C}$ *be an open connected set and let* $f : A \to \mathbb{C}$ *be a holomorphic function, not identically vanishing. The set*

$$K_f = \{z \in A : f(z) = 0\},$$

is either empty or discrete and made of isolated points, i.e. for every $z_0 \in K_f$ *there exists* $r > 0$ *such that*

$$f(z) \neq 0, \qquad \text{for every } z \in \dot{B}_r(z_0).$$

Moreover, K_f *can not contain any infinite sequence* $\{z_n\}_{n\in\mathbb{N}}$ *such that* $z_n \to w \in A$*.*

Proof. Let us suppose that $K_f \neq \emptyset$, then there exists $z_0 \in A$ such that $f(z_0) = 0$. This zero has finite order $m \in \mathbb{N} \setminus \{0\}$, otherwise by Proposition 1.8.5 we would have $f \equiv 0$ on A. In a neighborhood of z_0 we thus have

$$\begin{aligned} f(z) &= c_m (z - z_0)^m + c_{m+1} (z - z_0)^{m+1} + \dots \\ &= (z - z_0)^m [c_m + c_{m+1} (z - z_0) + \dots] = (z - z_0)^m g(z), \end{aligned}$$

where we set

$$g(z) = \sum_{n=0}^{\infty} c_{n+m} (z - z_0)^n.$$

which is holomorphic in the relevant neighborhood of z_0. We observe that by construction $g(z_0) = c_m \neq 0$ and that g is continuous (since it is holomorphic). By Lemma 1.3.5, there exists $r > 0$ such that in $B_r(z_0)$ we still have $g(z) \neq 0$. This implies that

$$f(z) = (z - z_0)^m g(z) \neq 0, \qquad \text{for every } z \in \dot{B}_r(z_0),$$

as desired.

To prove the last assertion, let us assume that there exists a sequence of zeros $\{z_n\}_{n\in\mathbb{N}} \subset K_f$ converging to some $w \in A$. By continuity of f, we would get

$$0 = \lim_{n\to\infty} f(z_n) = f(w),$$

and thus $w \in K_f$. Since $z_n \in K_f$ is converging to $w \in K_f$, this contradicts the fact that K_f contains only isolated points. □

Remark 1.8.9. We already know that

$$\cos^2 z + \sin^2 z = 1, \qquad \text{for every } z \in \mathbb{C}.$$

Let us reprove this formula by using Proposition 1.8.8. We consider the entire function $f(z) = \cos^2 z + \sin^2 z - 1$. By usual trigonometric formulas, we know that

$$f(x) = \cos^2 x + \sin^2 x - 1 = 0, \qquad \text{for every } x \in \mathbb{R}.$$

This implies that the set of its zeros K_f is not discrete and thus by Proposition 1.8.8 the function f must vanish identically.

Definition 1.8.10 (Analytic continuation)**.** Let $I \subset \mathbb{R}$ be an interval with non-empty interior and let $f : I \to \mathbb{R}$ a real function of one real variable. We say that f admits an *analytic continuation* to the complex plane if there exist an open set $A \subset \mathbb{C}$ and a holomorphic function $F : A \to \mathbb{C}$ such that:

- $I \subset A \cap \{z \in \mathbb{C} : \mathrm{Im}(z) = 0\}$;
- $F(x) = f(x)$, for every $x \in I$.

Remark 1.8.11 (Uniqueness of the analytic continuation)**.** It is easy to see that the analytic continuation is unique, provided it exists. Indeed, let us suppose that $f : I \to \mathbb{R}$ admits two different analytic continuation $F_1 : A \to \mathbb{C}$ and $F_2 : A \to \mathbb{C}$. Then we would get that the difference $F_1 - F_2$ would be a holomorphic function, identically vanishing on the interval I. Since the latter is not discrete, we get $F_1 = F_2$ by Proposition 1.8.8.

Remark 1.8.12 (Existence of the analytic continuation?)**.** We give a sufficient condition for a function $f : I \to \mathbb{R}$ to admit the analytic continuation. Let us suppose that f admits the Taylor expansion on $I = (x_0 - L, x_0 + L)$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n, \qquad |x - x_0| < L,$$

If we set

$$F(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (z - x_0)^n, \qquad z \in B_L(x_0),$$

this defines the analytic continuation of f on $B_L(x_0)$. For example, this gives another way to construct the functions exponential, cosinus, sinus and so on.

Example 1.8.13. The functions

$$z \mapsto e^z, \qquad z \mapsto \cos z, \qquad z \mapsto \sin z,$$

are the analytic continuations of the respective ordinary functions defined on $\mathbb{R}$. The function

$$z \mapsto \operatorname{Log} z,$$

is the analytic continuation of the ordinary logarithm function defined on $(0, +\infty)$.

9. Some remarkable consequences

Theorem 1.9.1 (Liouville's Theorem)**.** *Let f be an entire function. If f is bounded, i.e. if there exists $C > 0$ such that*

$$|f(z)| \le C, \qquad \textit{for every } z \in \mathbb{C},$$

then f is constant.

Proof. We know by Theorem 1.8.2 that f is analytic, i.e.

$$f(z) = \sum_{n=0}^{\infty} c_n\, z^n, \qquad \text{for every } z \in \mathbb{C}. \tag{1.9.1}$$

Moreover, we have the following formula for the coefficients c_n

$$c_n = \frac{1}{2\,\pi\, i} \int_{\gamma_R} \frac{f(s)}{s^{n+1}}\, ds,$$

where γ_R is given by

$$\gamma_R(t) = R\,e^{i\,t}, \qquad t \in [0, 2\,\pi].$$

Observe that since f is entire, the radius R can be chosen arbitrarily large. Since f is bounded, we have

$$\begin{aligned}
|c_n| = \left|\frac{1}{2\,\pi\,i}\int_{\gamma_R}\frac{f(s)}{s^{n+1}}\,ds\right| &= \left|\frac{1}{2\,\pi}\int_0^{2\,\pi}\frac{f(R\,e^{i\,t})}{R^{n+1}\,e^{i\,(n+1)\,t}}\,R\,e^{i\,t}\,dt\right| \\
&\le \frac{1}{2\,\pi}\int_0^{2\,\pi}\left|\frac{f(R\,e^{i\,t})}{R^{n+1}\,e^{i\,n\,t}}\right| R\,dt \\
&\le \frac{C}{2\,\pi\,R^n}\int_0^{2\,\pi} dt = \frac{C}{R^n}, \qquad \text{for every } n \in \mathbb{N}.
\end{aligned}$$

By taking the limit as R goes to $+\infty$, we get

$$c_n = 0, \qquad \text{for every } n \ge 1,$$

Thus from the Taylor series expansion (1.9.1) of f we get the conclusion

$$f(z) = c_0, \qquad \text{for every } z \in \mathbb{C},$$

as desired. □

As a remarkable consequence of Liouville's Theorem, we have the following

Theorem 1.9.2 (Fundamental Theorem of Algebra)**.** *Let*

$$P(z) = a_0 + a_1\,z + \cdots + a_n\,z^n, \qquad z \in \mathbb{C}.$$

be a non-constant polynomial. Then P has at least a root $z_0 \in \mathbb{C}$.

Proof. We can assume without loss of generality that $a_n \neq 0$. The proof is by contradiction. Let us suppose that $P(z) \neq 0$, for every $z \in \mathbb{C}$. Then the function

$$f(z) = \frac{1}{P(z)}, \qquad z \in \mathbb{C},$$

is an entire function. Moreover, f is bounded: indeed, we observe that

$$|f(z)| = \frac{1}{|z|^n\left|\dfrac{a_0}{z^n} + \dfrac{a_1}{z^{n-1}} + \cdots + a_n\right|},$$

so that

$$\lim_{z\to\infty}|f(z)| = 0.$$

This means that there exists $R > 0$ large enough so that

$$|f(z)| \le 1, \qquad \text{for every } |z| > R.$$

On the other hand, by setting

$$\overline{B_R(0)} = \{z \in \mathbb{C} \,:\, |z| \le R\},$$

for every[5] $z \in \overline{B_R(0)}$ we have

$$|f(z)| \le \max_{s\in\overline{B_R(0)}}\frac{1}{|P(s)|} = \frac{1}{\displaystyle\min_{s\in\overline{B_R(0)}}|P(s)|} = C.$$

[5] The set $\{z \in \mathbb{C} \,:\, |z| \le R\}$ is compact and $|P(z)|$ is a continuous function, thus existence of a minimum point is assured by Weierstrass' Theorem.

In conclusion, we obtain

$$|f(z)| \le \max\{1, C\}, \qquad \text{for every } z \in \mathbb{C}.$$

By using Liouville's Theorem, we obtain that f is constant. This in turn implies that P itself is constant, contradicting the hypothesis. $\square$

10. Singularities and the Residue Theorem

Definition 1.10.1. Let $A \subset \mathbb{C}$ be an open set and let $f : A \to \mathbb{C}$ be a holomorphic function. We say that z_0 is an *isolated singularity* for f if

- $z_0 \notin A$;
- there exists $r > 0$ such that $\dot{B}_r(z_0) \subset A$.

Example 1.10.2. The function $f(z) = 1/z$ has an isolated singulatity at $z = 0$. The function $g(z) = 1/((z-1)(z-2))$ has two isolated singularities at $z = 1$ and $z = 2$.

Example 1.10.3. By recalling that the function $f(z) = \operatorname{Log} z$ is defined on $\mathbb{C}^*$ and holomorphic on $\mathbb{C}^{**}$, we get that f has a singularity at every point of the semiaxis of real negative numbers. Observe that these are *not* isolated singularities.

Definition 1.10.4. Let $f : A \to \mathbb{C}$ be an holomorphic function and z_0 an isolated singularity. We say that

- z_0 is *removable* if
$$\lim_{z \to z_0} f(z) = \lambda \in \mathbb{C};$$
- z_0 is a *pole of order* $m \in \mathbb{N} \setminus \{0\}$ if
$$\lim_{z \to z_0} (z - z_0)^m f(z) = \lambda \in \mathbb{C}^*;$$
- z_0 is an *essential singularity* if it is neither removable nor a pole of finite order.

In the case of a pole of order 1, we will also call it *simple pole*.

Example 1.10.5. The function

$$f(z) = \frac{z}{\sin z}, \qquad z \in A = \mathbb{C} \setminus \{k\,\pi \,:\, k \in \mathbb{Z}\},$$

is holomorphic in A, with isolated singularities at the points $k\,\pi$, for $k \in \mathbb{Z}$. We observe that the singularity at $z = 0$ is removable, since (recall (1.5.9))

$$\lim_{z \to 0} \frac{z}{\sin z} = 1.$$

On the other hand, any point of the form $k\,\pi$ with $k \in \mathbb{Z}$ is a simple pole. Indeed, by observing that

$$\begin{aligned}\sin(z) = \sin(z - k\,\pi + k\,\pi) &= \sin(z - k\,\pi)\,\cos(k\,\pi) + \cos(z - k\,\pi)\,\sin(k\,\pi) \\ &= \sin(z - k\,\pi)\,\cos(k\,\pi),\end{aligned}$$

we have

$$\begin{aligned}\lim_{z \to k\,\pi} (z - k\,\pi)\,\frac{z}{\sin z} &= \frac{k\,\pi}{\cos(k\,\pi)} \lim_{z \to k\,\pi} \frac{z - k\,\pi}{\sin(z - k\,\pi)} \\ &= \begin{cases} -k\,\pi, & k \text{ odd}, \\ k\,\pi, & k \text{ even}. \end{cases}\end{aligned}$$

Proposition 1.10.6. *Let $f : A \to \mathbb{C}$ be a holomorphic function with a removable singularity at z_0. If we set*

$$\lambda = \lim_{z \to z_0} f(z),$$

then the function

$$\widetilde{f}(z) = \begin{cases} \lambda, & \text{if } z = z_0, \\ f(z), & \text{if } z \in A, \end{cases}$$

is holomorphic in the new open set $A' = A \cup \{z_0\}$.

Corollary 1.10.7. *Let $f : A \to \mathbb{C}$ be a holomorphic function with a pole of order $m \in \mathbb{N} \setminus \{0\}$ at z_0. We set*

$$\lambda = \lim_{z \to z_0} (z - z_0)^m f(z),$$

then the function

$$F(z) = \begin{cases} \lambda, & \text{if } z = z_0, \\ (z - z_0)^m f(z), & \text{if } z \in A, \end{cases}$$

is holomorphic in the new open set $A' = A \cup \{z_0\}$.

Definition 1.10.8. Let $f : A \to \mathbb{C}$ be a holomorphic function and let z_0 be an isolated singularity of f. We call *residue of f at z_0* the quantity

$$\operatorname{res}(f, z_0) = \frac{1}{2\pi i} \int_\gamma f(z)\, dz,$$

where γ is a positively oriented piecewise regular loop contained in A, whose image entours z_0 (but not other singularities of f).

Remark 1.10.9. By Corollary 1.6.13 we know that this definition is well-posed, since it does not depend on γ.

Example 1.10.10. Let $z_0 \in \mathbb{C}$ and take $f(z) = (z - z_0)^{-n}$ with $n \in \mathbb{N} \setminus \{0\}$. This is holomorphic in $\mathbb{C} \setminus \{z_0\}$ with an isolated singularity (indeed, a pole) at $z = z_0$. We take $\gamma : [0, 2\pi] \to \mathbb{C}$ defined by

$$\gamma(t) = z_0 + e^{i\,t}, \qquad t \in [0, 2\pi],$$

then

$$\begin{aligned} \operatorname{res}(f, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{(z - z_0)^n}\, dz &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{1}{(z_0 + e^{i\,t} - z_0)^n}\, i\, e^{i\,t}\, dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{-i\,t\,(n-1)}\, dt. \end{aligned}$$

We now distinguish two cases: if $n = 1$, then we get

$$\operatorname{res}(f, z_0) = \frac{1}{2\pi} \int_0^{2\pi} dt = 1.$$

On the other hand, if $n \geq 2$, we obtain

$$\operatorname{res}(f, z_0) = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\,t\,(n-1)}\, dt = \frac{1}{2\pi} \left[\frac{e^{-i\,t\,(n-1)}}{-i\,(n-1)} \right]_0^{2\pi} = 0.$$

In conclusion, we obtained

$$\operatorname{res}\left(\frac{1}{(z - z_0)^n}, z_0 \right) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n \geq 2. \end{cases}$$

The following is a remarkable consequence of Cauchy's formula. It permits to compute a residue at a pole just by differentiating a suitable function.

Proposition 1.10.11. *Let f be a holomorphic function with a pole of order m at z_0. Then we have*

$$\operatorname{res}(f,z_0) = \frac{1}{(m-1)!}\lim_{z\to z_0}\left(\frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m f(z)\right)\right). \tag{1.10.1}$$

Proof. We define $g(z) = (z-z_0)^m f(z)$ and observe that this is holomorphic (and thus analytic) in a neighborhood of z_0. We then compute

$$\begin{aligned}\operatorname{res}(f,z_0) = \frac{1}{2\pi i}\int_\gamma f(z)\,dz &= \frac{1}{2\pi i}\int_\gamma \frac{f(z)\,(z-z_0)^m}{(z-z_0)^m}\,dz\\ &= \frac{1}{2\pi i}\int_\gamma \frac{g(z)}{(z-z_0)^m}\,dz\\ &= \frac{g^{(m-1)}(z_0)}{(m-1)!},\end{aligned}$$

where in the last equality we used formula (1.8.1) for the function g. ☐

Remark 1.10.12. By recalling the definition of residue, under the previous assumptions formula (1.10.1) can be written as

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \frac{1}{(m-1)!}\lim_{z\to z_0}\left(\frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m f(z)\right)\right).$$

In other words, we obtained a simple way to compute a line integral, just by differentiating a function!

Corollary 1.10.13. *Let f, g be two holomorphic functions, such that g has a simple zero at z_0 and $f(z_0) \neq 0$. Then f/g has a simple pole at z_0 and we have*

$$\operatorname{res}\left(\frac{f}{g}, z_0\right) = \frac{f(z_0)}{g'(z_0)}. \tag{1.10.2}$$

Proof. We apply (1.10.1) to the function f/g with $m = 1$. We get

$$\operatorname{res}\left(\frac{f}{g}, z_0\right) = \lim_{z\to z_0}(z-z_0)\,\frac{f(z)}{g(z)},$$

then we observe that since $g(z_0) = 0$, the limit can be rewritten as

$$\lim_{z\to z_0}\left(\frac{z-z_0}{g(z)-g(z_0)}\,f(z)\right).$$

The conclusion now follows from the continuity of f and the definition of complex derivative. ☐

We conclude this section with the following

Theorem 1.10.14 (Residue Theorem)**.** *Let $A \subset \mathbb{C}$ be an open connected set and let $f : A \to \mathbb{C}$ be a holomorphic function. For γ a positively oriented piecewise regular loop, we indicate by D the region entoured by Γ_γ. Let $z_1, \ldots, z_k$ be the singularities of f contained in D and suppose that $D \setminus \{z_1, \ldots, z_k\} \subset A$. Then we have*

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \sum_{m=1}^{k}\operatorname{res}(f, z_m).$$

11. Laurent's series expansions

Let $\{a_n\}_{n\in\mathbb{Z}}$ be a sequence indexed over $\mathbb{Z}$ and let $z_0 \in \mathbb{C}$. We call *bylateral series* the expression

$$\sum_{n\in\mathbb{Z}} a_n\,(z-z_0)^n.$$

We say that the bylateral series converges if the two series

$$\sum_{n=0}^{\infty} a_n\,(z-z_0)^n \qquad \text{and} \qquad \sum_{-\infty}^{n=-1} a_n\,(z-z_0)^n,$$

converge. The first series is called *regular part*, while the second one is called *singular* part.

The following convergence result is analogous to the one for power series, see Theorem 1.7.3.

Theorem 1.11.1. *Let $\sum_{n\in\mathbb{Z}} a_n\,(z-z_0)^n$ be a bylateral series with*

$$\limsup_{n\to+\infty} \sqrt[n]{|a_n|} = L_1 < +\infty \qquad \textit{and} \qquad \limsup_{n\to+\infty} \sqrt[n]{|a_{-n}|} = L_2.$$

Let us suppose that $L_2 < 1/L_1$.

i) The power series is totally convergent on every closed annulus

$$\{z \in \mathbb{C} \,:\, \varrho_2 \le |z-z_0| \le \varrho_1\}$$

with radii $\varrho_1 < 1/L_1$ and $\varrho_2 > L_2$ (with the usual convention that if $L_1 = 0$, then $1/L_1 = +\infty$).

ii) The power series does not converge for every z such that $|z-z_0| > 1/L_1$ or $|z-z_0| < L_2$.

Proof. The proof is the same as that of Theorem 1.7.3, it is sufficient to discuss separately the regular and singular parts, i.e.

$$\sum_{n=0}^{\infty} a_n\,(z-z_0)^n \qquad \text{and} \qquad \sum_{-\infty}^{n=-1} a_n\,(z-z_0)^n.$$

For the regular part we can apply directly Theorem 1.7.3, while for the second one we introduce the change of variable

$$w = \frac{1}{z-z_0}.$$

Then the singular part becomes

$$\sum_{-\infty}^{n=-1} a_n\,(z-z_0)^n = \sum_{n=1}^{\infty} a_{-n}\,w^n,$$

which is an ordinary power series, in the new complex variable w. By Theorem 1.7.3, we know that we have total convergence if

$$|w| \le r, \qquad \text{with } r < \frac{1}{L_2},$$

that is

$$|z-z_0| = \frac{1}{|w|} \ge \frac{1}{r}, \qquad \text{with } \frac{1}{r} > L_2.$$

Similarly we prove point ii). We leave the details to the reader. $\square$

Definition 1.11.2 (Inner and outer radius)**.** Let $\{a_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}$ be a sequence such that

$$\limsup_{n\to+\infty} \sqrt[n]{|a_n|} = L_1 \qquad \text{and} \qquad \limsup_{n\to+\infty} \sqrt[n]{|a_{-n}|} = L_2,$$

and

$$L_2 < \frac{1}{L_1}.$$

Then $R_1 = 1/L_1$ is called *outer radius of convergence* of the bylateral series $\sum_{n\in\mathbb{Z}} a_n\,(z-z_0)^n$, while $R_2 = L_2$ is called *inner radius of convergence.* We use the usual conventions:

$$R_1 = \frac{1}{L_1} = +\infty, \qquad \text{if } L_1 = 0,$$

and

$$R_1 = \frac{1}{L_1} = 0, \qquad \text{if } L_1 = +\infty.$$

Remark 1.11.3. We observe that in the region of convergence, the series

$$\sum_{n=1}^{\infty} a_{-n}\, w^n, \qquad \text{with } w = \frac{1}{z-z_0},$$

is a holomorphic function of the variable w, thanks to the results of Subsection 7. Since the function

$$z \mapsto \frac{1}{z-z_0} = w,$$

is holomorphic in $\mathbb{C}\setminus\{z_0\}$, we get that the singular part

$$\sum_{-\infty}^{n=-1} a_n\,(z-z_0)^n,$$

is holomorphic as well, as a composition of holomorphic functions. In conclusion, *a bylateral series is a holomorphic function* in the annular region

$$\{z\in\mathbb{C}\,:\, R_2 < |z-z_0| < R_1\}.$$

The following important result is a sort of converse.

Theorem 1.11.4 (Laurent's Theorem)**.** *Let $f : A \to \mathbb{C}$ be an holomorphic function on the annular region*

$$A = \{z\in\mathbb{C}\,:\, 0 \le R_2 < |z-z_0| < R_1 \le +\infty\}.$$

For every $z\in A$ we have

$$f(z) = \sum_{n\in\mathbb{Z}} c_n\,(z-z_0)^n,$$

with the coefficient c_n given by

$$c_n = \frac{1}{2\,\pi\, i}\int_\gamma \frac{f(s)}{(s-z_0)^{n+1}}\,ds, \qquad \textit{for every } n\in\mathbb{Z}. \tag{1.11.1}$$

Here γ is any positively oriented piecewise regular loop such that $\Gamma_\gamma \subset A$ and such that the region D entoured by Γ_γ contains $B_{R_2}(z_0)$.

Proof. The proof is similar to that of Theorem 1.8.2 and we omit it. The interested reader can find it in [1, Proposizione 4.7-2]. ☐

We observe that if f has an isolated singularity at z_0, then $R_2 = 0$ and we have the Laurent expansion

$$f(z) = \sum_{n\in\mathbb{Z}} c_n\,(z - z_0)^n,$$

in a sufficiently small punctured disk centered at z_0 and from (1.11.1) we get

$$c_{-1} = \frac{1}{2\,\pi\,i}\int_\gamma f(z)\,dz = \operatorname{res}(f, z_0). \tag{1.11.2}$$

Remark 1.11.5. The previous formula also explain the reason for the terminology *residue*. Indeed, if f has an isolated singularity at z_0 and

$$f(z) = \sum_{n\in\mathbb{Z}} c_n\,(z - z_0)^n,$$

then by Theorem 1.6.12 we have

$$\frac{1}{2\,\pi\,i}\int_\gamma (z - z_0)^n\,dz = 0, \qquad \text{for every } n \in \mathbb{N},$$

while by Example 1.10.10 we have

$$\frac{1}{2\,\pi\,i}\int_\gamma (z - z_0)^n\,dz = 0, \qquad \text{for every } n \le -2.$$

Thus by integrating term by term the Laurent series expansion, the term corresponding to $n = -1$ is the only one giving a non-zero integral.

Proposition 1.11.6. *Let $f : A \to \mathbb{C}$ be an holomorphic function and let z_0 be an isolated singularity. Let*

$$f(z) = \sum_{n\in\mathbb{Z}} c_n\,(z - z_0)^n,$$

be its Laurent series in a punctured disk centered at z_0. Then we have:

- *z_0 is removable if and only if $c_n = 0$ for every $n \le -1$;*
- *z_0 is a pole of order m if and only if $c_{-m} \neq 0$ and $c_n = 0$ for every $n \le -m - 1$;*
- *z_0 is essential if and only if the singular part of the Laurent series has infinitely many terms different from 0.*

Proof. If $c_n = 0$ for every $n \le -1$, then

$$f(z) = \sum_{n=0}^{\infty} c_n\,(z - z_0)^n,$$

and thus the limit

$$\lim_{z\to z_0} f(z) = c_0,$$

exists, which means that z_0 is removable. **Viceversa**, if z_0 is removable then the Laurent series must reduce to the Taylor series, i.e. $c_n = 0$ for every $n \le -1$.

If z_0 is a pole of order m, then by Corollary 1.10.7 the function $z \mapsto (z - z_0)^m\,f(z)$ is homolorphic. By Theorem 1.8.2, we thus get

$$(z - z_0)^m\,f(z) = \sum_{n=0}^{\infty} a_n\,(z - z_0)^n, \qquad \text{with } a_0 \neq 0,$$

that is for $z \neq z_0$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{n-m} = \sum_{n=-m}^{\infty} \widetilde{a}_n (z - z_0)^n,$$

where $\widetilde{a}_n = a_{n+m}$ for every $n \geq -m$. This shows that the singular part of the Laurent expansion of f contains only the first m terms (and $\widetilde{a}_{-m} = a_0 \neq 0$). **Viceversa**, if

$$f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n, \qquad \text{with } a_{-m} \neq 0.$$

then we clearly have

$$\lim_{z \to z_0} (z - z_0)^m f(z) = \lim_{z \to z_0} \sum_{n=-m}^{\infty} a_n (z - z_0)^{n+m} = a_{-m} \in \mathbb{C}^*,$$

as desired.

By exclusion, we get the case of an essential singularity. □

Theorem 1.11.7 (Partial fraction decomposition)**.** *Let $P, Q : \mathbb{C} \to \mathbb{C}$ be two polynomials such that*

$$n = \deg(P) < \deg(Q) = m.$$

Let us call $z_1, \dots, z_k$ the zeros of Q, each one having order $m_1, \dots, m_k$ (recall Definition 1.8.4), so that

$$m_1 + \dots + m_k = m.$$

Then the function $f = P/Q$ coincides with the sum of the singular parts of the Laurent series centered at the zeros $z_1, \dots, z_k$. In other words, we have

$$f(z) := \frac{P(z)}{Q(z)} = \sum_{j=1}^{k} \left(\sum_{h=1}^{m_j} \frac{a_{j,h}}{(z - z_j)^h} \right). \tag{1.11.3}$$

Moreover, each coefficient $a_{j,h}$ is given by

$$a_{j,h} = \mathrm{res}\left((z - z_j)^{h-1} \frac{P(z)}{Q(z)}, z_j \right). \tag{1.11.4}$$

Proof. We give a sketch of the proof. Let us set

$$\sigma_j(z) = \sum_{h=1}^{m_j} \frac{a_{j,h}}{(z - z_j)^h}, \qquad z \in \mathbb{C} \setminus \{z_j\},$$

then the function

$$F(z) := f(z) - \sum_{j=1}^{k} \sigma_j(z),$$

is entire, i.e. holomorphic on the whole $\mathbb{C}$. Indeed, in the neighborhood of each pole z_j, the function f can be written as

$$f(z) = \sigma_j(z) + \sum_{n=0}^{\infty} c_n (z - z_j)^n,$$

thanks to Proposition 1.11.6. Moreover, the function F is vanishing at infinity, i.e.

$$\lim_{|z| \to +\infty} F(z) = 0. \tag{1.11.5}$$

By Theorem 1.9.1 (*Liouville's Theorem*), F is constant. By using this information in conjunction with (1.11.5), we finally obtain

$$F(z) = 0, \qquad \text{i. e.} \quad f(z) = \sum_{j=1}^{k} \sigma_j(z).$$

This concludes the proof of (1.11.3).

We now show formula (1.11.4) for the coefficients: for every $\ell = 1, \dots, k$, we take γ_ℓ to be the positively oriented regular loop

$$\gamma_\ell(t) = z_\ell + R\, e^{i\,t}, \qquad t \in [0, 2\,\pi],$$

where the radius $R > 0$ is chosen small enough, in order that all the other zeros of Q falls "outside" the circle $\partial B_R(z_\ell)$. For every $n = 1, \dots, m_\ell$, we thus obtain

$$\begin{aligned}
\operatorname{res}\left((z - z_\ell)^{n-1} \frac{P}{Q}, z_\ell\right) &= \frac{1}{2\,\pi\,i} \int_{\gamma_j} (z - z_\ell)^{n-1} \frac{P(z)}{Q(z)}\, dz \\
&= \sum_{j=1}^{k} \sum_{h=1}^{m_j} \frac{1}{2\,\pi\,i} \int_{\gamma_\ell} (z - z_\ell)^{n-1} \frac{a_{j,h}}{(z - z_j)^h}\, dz \\
&= \frac{1}{2\,\pi\,i} \int_{\gamma_\ell} \frac{a_{\ell,n}}{(z - z_\ell)}\, dz + \sum_{h \neq n} \frac{1}{2\,\pi\,i} \int_{\gamma_\ell} \frac{a_{\ell,h}}{(z - z_\ell)^{h-n+1}}\, dz \\
&+ \sum_{j \neq \ell} \sum_{h=1}^{m_j} \frac{1}{2\,\pi\,i} \int_{\gamma_\ell} (z - z_\ell)^{n-1} \frac{a_{j,h}}{(z - z_j)^h}\, dz.
\end{aligned}$$

We now observe that for every $j \neq \ell$, the function

$$z \mapsto (z - z_\ell)^{n-1} \frac{a_{j,h}}{(z - z_j)^h},$$

is holomorphic inside the region entoured by Γ_{γ_ℓ}, thus by Theorem 1.6.12 (*Cauchy's Theorem*), we have

$$\frac{1}{2\,\pi\,i} \int_{\gamma_\ell} (z - z_\ell)^{n-1} \frac{a_{j,h}}{(z - z_j)^h}\, dz = 0, \qquad \text{for } j \neq \ell.$$

On the other hand, by recalling Example 1.10.10, we have

$$\frac{1}{2\,\pi\,i} \int_{\gamma_\ell} \frac{a_{\ell,n}}{(z - z_\ell)}\, dz = a_{\ell,n} \operatorname{res}\left(\frac{1}{z - z_\ell}, z_\ell\right) = a_{\ell,n}.$$

Finally, for $h \neq n$ we have two possibilities:

- if $h < n$, then $n - 1 \geq h$ and thus we have again that the function

  $$z \mapsto (z - z_\ell)^{n-1} \frac{a_{\ell,h}}{(z - z_\ell)^h},$$

  is holomorphic inside the region entoured by Γ_{γ_ℓ}. As before, by Theorem 1.6.12 (*Cauchy's Theorem*), we have

  $$\frac{1}{2\,\pi\,i} \int_{\gamma_\ell} (z - z_\ell)^{n-1} \frac{a_{\ell,h}}{(z - z_\ell)^h}\, dz = 0;$$

- if $h > n$, then $h - n + 1 \geq 2$ and thus
$$\frac{1}{2\,\pi\,i}\int_{\gamma_\ell}(z-z_\ell)^{n-1}\,\frac{a_{\ell,h}}{(z-z_\ell)^h}\,dz = a_{\ell,h}\,\mathrm{res}\left(\frac{1}{(z-z_\ell)^{h-n+1}}, z_\ell\right) = 0,$$
again by Example 1.10.10, by keeping into account that $h - n + 1 \geq 2$ in this case.

By spending these informations in the chain of equalities above, we get
$$\mathrm{res}\left((z-z_\ell)^{n-1}\,\frac{P}{Q}, z_\ell\right) = a_{\ell,n},$$
as desired. □

Corollary 1.11.8. *Let $P, Q : \mathbb{C} \to \mathbb{C}$ be two polynomials such that*
$$n = \deg(P) < \deg(Q) = m.$$
Let us suppose that all the zeros $z_1, \dots, z_m$ have order 1. Then the formula (1.11.3) above becomes
$$f(z) := \frac{P(z)}{Q(z)} = \sum_{j=1}^{m}\frac{a_j}{(z-z_j)},$$
and each a_j is given by
$$a_j = \mathrm{res}\left(\frac{P}{Q}, z_j\right), \qquad j = 1, \dots, m. \tag{1.11.6}$$

Proof. It is sufficient to observe that each z_j have order 1, thus in formula (1.11.3) we have $m_k = 1$ for every k. □

12. Exercises

Exercise 1.12.1. *Show that for every $z \in \mathbb{C}$ we have*
$$\frac{|\mathrm{Re}\,(z)| + |\mathrm{Im}\,(z)|}{\sqrt{2}} \leq |z| \leq |\mathrm{Re}\,(z)| + |\mathrm{Im}\,(z)|.$$

Solution. Let us write $z = x + i\,y$, then we have to prove that
$$\frac{|x| + |y|}{\sqrt{2}} \leq \sqrt{x^2 + y^2} \leq |x| + |y|. \tag{1.12.1}$$
Let us prove the first inequality. For this, it is sufficient to recall that the function of one real variabile $t \mapsto \sqrt{t}$ is *concave*, that is
$$\sqrt{(1-\lambda)\,t_0 + \lambda\,t_1} \geq (1-\lambda)\,\sqrt{t_0} + \lambda\,\sqrt{t_1}, \qquad \text{for every } t_0, t_1 \geq 0 \text{ and } 0 \leq \lambda \leq 1.$$
By using this inequality with
$$t_0 = x^2, \qquad t_1 = y^2 \qquad \text{and} \qquad \lambda = \frac{1}{2},$$
we obtain
$$\sqrt{\frac{x^2 + y^2}{2}} \geq \frac{|x| + |y|}{2}.$$
After a simplification, we get the first inequality in (1.12.1).

In order to prove the second inequality in (1.12.1), we observe that
$$|x| + |y| = \sqrt{(|x| + |y|)^2} = \sqrt{x^2 + 2\,|x|\,|y| + y^2} \geq \sqrt{x^2 + y^2},$$

where we used that the square rooth is a monotone function and $|x|\,|y| \geq 0$. This gives the desired inequality. □

Exercise 1.12.2. *Let $u : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by*

$$u(x, y) = x^2 - y^2.$$

Verify that this is a harmonic function and find v such that u and v are conjugate harmonic functions.

Write the corresponding holomorphic function $f(z) = u(x, y) + i\,v(x, y)$.

Solution. We first observe that

$$u_{xx} = 2 \qquad \text{and} \qquad u_{yy} = -2,$$

thus the function is harmonic. In order to find v, we need to solve the system

$$v_y = u_x = 2\,x \qquad \text{and} \qquad v_x = -u_y = 2\,y.$$

It is not difficult to see that the choice

$$v(x, y) = 2\,x\,y,$$

is feasible. The corresponding holomorphic function is given by

$$f(z) = (x^2 - y^2) + 2\,i\,x\,y = x^2 + 2\,i\,x\,y + (i\,y)^2 = (x + i\,y)^2 = z^2.$$

This concludes the exercise. □

Exercise 1.12.3. *Let $u : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by*

$$u(x, y) = x^3 - 3\,x\,y^2.$$

Verify that this is a harmonic function and find v such that u and v are conjugate harmonic functions.

Write the corresponding holomorphic function $f(z) = u(x, y) + i\,v(x, y)$.

Solution. By direct computation, we have

$$u_{xx} + u_{yy} = 6\,x - 6\,x = 0.$$

In order to find v, we argue in an indirect way: we observe that

$$u(x, y) = x^3 - 3\,x\,y^2 = \mathrm{Re}(z^3).$$

Indeed, we have

$$z^3 = (x + i\,y)^3 = x^3 + 3\,i\,x^2\,y - 3\,x\,y^2 - i\,y^3 = (x^3 - 3\,x\,y^2) + i\,(3\,x^2\,y - y^3).$$

Then we can choose

$$v(x, y) = \mathrm{Im}(z^3) = 3\,x^2\,y - y^3.$$

By Corollary (1.4.9) we know that u and v are conjugate harmonic functions. Of course, by construction we have

$$f(z) = u(x, y) + i\,v(x, y) = \mathrm{Re}(z^3) + i\,\mathrm{Im}(z^3) = z^3.$$

This concludes the exercise. □

Exercise 1.12.4. *Find the partial fraction decomposition of the rational function*

$$f(z) = \frac{z}{z^2 + z - 6}.$$

Solution. By writing $P(z) = z$ and $Q(z) = z^2 + z - 6$, we have

$$f(z) = \frac{P(z)}{Q(z)},$$

and the function has two simple poles at $z_1 = -3$ and $z_2 = 2$. By using Corollary 1.11.8, we know that

$$\frac{z}{z^2 + z - 6} = \frac{\operatorname{res}(f, -3)}{z + 3} + \frac{\operatorname{res}(f, 2)}{z - 2}.$$

Observe that we have

$$Q(-3) = 0, \quad Q'(-3) \neq 0 \quad \text{and} \quad P(-3) \neq 0,$$

thus we can use formula (1.10.2) and obtain

$$\operatorname{res}(f, -3) = \frac{P(-3)}{Q'(-3)} = \frac{3}{5}.$$

Similarly, we get

$$\operatorname{res}(f, 2) = \frac{P(2)}{Q'(2)} = \frac{2}{5}.$$

In conclusion, we get

$$\frac{z}{z^2 + z - 6} = \frac{3}{5}\frac{1}{z + 3} + \frac{2}{5}\frac{1}{z - 2},$$

as desired. □

Exercise 1.12.5. *Find the partial fraction decomposition of the rational function*

$$f(z) = \frac{z}{(z-1)^2(z-2)}.$$

Solution. By writing $P(z) = z$ and $Q(z) = (z-1)^2(z-2)$, we have

$$f(z) = \frac{P(z)}{Q(z)},$$

and the function has one simple pole at $z_1 = 2$ an a pole of order 2 at $z_2 = 1$. By using formula (1.11.3), we have

$$\frac{z}{(z-1)^2(z-2)} = \frac{a_{1,1}}{z-2} + \frac{a_{2,1}}{(z-1)} + \frac{a_{2,2}}{(z-1)^2},$$

where the coefficients $a_{1,1}, a_{2,1}$ and $a_{2,2}$ are given by formula (1.11.4). Thus we have

$$a_{1,1} = \operatorname{res}\left(\frac{z}{(z-1)^2(z-2)}, 2\right),$$

$$a_{2,1} = \operatorname{res}\left(\frac{z}{(z-1)^2(z-2)}, 1\right),$$

and

$$a_{2,2} = \operatorname{res}\left((z-1)\frac{z}{(z-1)^2(z-2)}, 1\right) = \operatorname{res}\left(\frac{z}{(z-1)(z-2)}, 1\right).$$

We are left with computing these residues. For $a_{1,1}$ we can use formula (1.10.2) and obtain

$$a_{1,1} = \operatorname{res}\left(\frac{z}{(z-1)^2(z-2)}, 2\right) = \frac{2}{1} = 2.$$

Similarly, for $a_{2,2}$ we can still use (1.10.2) and get

$$a_{2,2} = \operatorname{res}\left(\frac{z}{(z-1)\,(z-2)}, 1\right) = \frac{1}{-1} = -1.$$

Finally, in order to compute $a_{2,1}$ we use the formula (1.10.1) of Proposition 1.10.11, with $m = 2$. Thus we get

$$a_{2,1} = \lim_{z\to 1} \frac{d}{dz}\left((z-1)^2 \,\frac{z}{(z-1)^2\,(z-2)}\right) = \lim_{z\to 1} \frac{-2}{(z-2)^2} = -2.$$

This concludes the exercise. □

13. Advanced exercises

Exercise 1.13.1. *Let $A \subset \mathbb{C}$ be a connected open set and let $f : A \to \mathbb{C}$ be an holomorphic function. For every $z_0, z_1 \in A$ we have*

$$f(z_1) = f(z_0) + f'(z_0)\,(z_1 - z_0) + \int_\gamma f''(z)\,(z_1 - z)\,dz,$$

where $\gamma : [a,b] \to \mathbb{C}$ is any piecewise regular curve such that $\Gamma_\gamma \subset A$ and

$$\gamma(b) = z_1 \qquad \textit{and} \qquad \gamma(a) = z_0.$$

Proof. We first observe that, since f is holomorphic, by Theorem 1.8.2 it can de differentiated as many times as we wish. In particular, f'' is well-defined. We now use the definition of curvilinear integral in the complex plane, i.e.

$$\int_\gamma f''(z)\,(z_1 - z)\,dz = \int_a^b f''(\gamma(t))\,(z_1 - \gamma(t))\,\gamma'(t)\,dt.$$

Observe that

$$f''(\gamma(t))\,\gamma'(t) = \frac{d}{dt} f'(\gamma(t)),$$

thus we can use an integration by parts

$$\begin{aligned}\int_a^b f''(\gamma(t))\,(z_1 - \gamma(t))\,\gamma'(t)\,dt &= \Big[f'(\gamma(t))\,(z_1 - \gamma(t))\Big]_a^b + \int_a^b f'(\gamma(t))\,\gamma'(t)\,dt \\ &= -f'(z_0)\,(z_1 - z_0) + \int_\gamma f'(z)\,dz.\end{aligned}$$

Thus, up to now, we obtained

$$f'(z_0)\,(z_1 - z_0) + \int_\gamma f''(z)\,(z_1 - z)\,dz = \int_\gamma f'(z)\,dz.$$

We can now apply Lemma 1.6.7 to the last integral and obtained the desired conclusion. □

Exercise 1.13.2. *Show that the function*

$$\tan z = \frac{\sin z}{\cos z}, \qquad \textit{for } z \in \mathbb{C} \setminus \left\{k\,\frac{\pi}{2} \,:\, k \in \mathbb{Z}\right\},$$

is invertible on the set $\mathcal{S} = \{z \in \mathbb{C} \,:\, -\pi/2 < \operatorname{Re}(z) < \pi/2\}$. Then compute its inverse function

$$z \mapsto \operatorname{Arctan} z,$$

by paying attention to its domain of definition.

Exercise 1.13.3. *By using the Residue Theorem, verify that*

$$\int_0^{2\pi} \cos^2 t\, dt = \pi.$$

Solution. We start by recalling that

$$\cos^2 t = \left(\frac{e^{i\,t} + e^{-i\,t}}{2}\right)^2,$$

thus we get

$$\int_0^{2\pi} \cos^2 t\, dt = \int_0^{2\pi} \frac{(e^{i\,t} + e^{-i\,t})^2}{4}\, dt = \int_0^{2\pi} \frac{(e^{i\,t} + e^{-i\,t})^2}{4\,i\,e^{i\,t}}\, i\, e^{i\,t}\, dt.$$

We now observe that if we introduce the positively oriented curve

$$\gamma(t) = e^{i\,t}, \qquad t \in [0, 2\pi],$$

this is a parametrization of the boundary $\partial B_1(0)$ of the disk of radius 1, centered at the origin and

$$\int_0^{2\pi} \cos^2 t\, dt = \int_0^{2\pi} \frac{(e^{i\,t} + e^{-i\,t})^2}{4\,i\,e^{i\,t}}\, i\, e^{i\,t}\, dt = \int_\gamma \frac{(z + 1/z)^2}{4\,i\,z}\, dz$$
$$= \int_\gamma \frac{(z^2+1)^2}{4\,i\,z^3}\, dz.$$

We now observe that the function

$$f(z) := \frac{(z^2+1)^2}{4\,i\,z^3}, \qquad z \neq 0,$$

has an isolated singularity at $z = 0$ inside $B_1(0)$. More precisely, $z = 0$ is a pole of order 3 and by observing that

$$f(z) = \frac{z}{4\,i} + \frac{1}{2\,i\,z} + \frac{1}{4\,i\,z^3},$$

we get from (1.11.2) that

$$\mathrm{res}(f, 0) = \frac{1}{2\,i}.$$

By appealing to the Residue Theorem, we finally obtain

$$\int_0^{2\pi} \cos^2 t\, dt = 2\pi\, i\, \mathrm{res}(f, 0) = \pi,$$

thus concluding the exercise. □

Exercise 1.13.4 (Fresnel's integrals)**.** *By using Cauchy's Theorem, verify that*

$$\int_0^{+\infty} \cos(t^2)\, dt = \int_0^{+\infty} \sin(t^2)\, dt = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

Solution. We first observe that both integrals have to be intended as follows

$$\int_0^{+\infty} \cos(t^2)\, dt = \lim_{R\to+\infty} \int_0^R \cos(t^2)\, dt, \qquad \int_0^{+\infty} \sin(t^2)\, dt = \lim_{R\to+\infty} \int_0^R \sin(t^2)\, dt.$$

Let us consider the positively oriented piecewise regular loop Γ obtained by gluing

$$\gamma_1(t) = t \qquad t \in [0, R],$$
$$\gamma_2(t) = R\, e^{i\,t}, \qquad t \in [0, \pi/4],$$
$$\gamma_3(t) = (R - t)\, e^{i\,\frac{\pi}{4}}, \qquad t \in [0, R].$$

By Cauchy's Theorem (Theorem 1.6.12) for the holomorphic function $f(z) = e^{-z^2}$ we obtain

$$0 = \int_\Gamma f(z)\,dz = \int_0^R e^{-t^2}\,dt + R\,i\int_0^{\frac{\pi}{4}} e^{-R^2\,e^{2\,i\,t}}\,e^{i\,t}\,dt - \int_0^R e^{-(R-t)^2\,i}\,e^{i\,\frac{\pi}{4}}\,dt$$
$$= \int_0^R e^{-t^2}\,dt + R\,i\int_0^{\frac{\pi}{4}} e^{-R^2\,e^{2\,i\,t}}\,e^{i\,t}\,dt - \int_0^R e^{-i\,t^2}\,e^{i\,\frac{\pi}{4}}\,dt.$$

We now observe that

$$e^{-i\,t^2}\,e^{i\,\frac{\pi}{4}} = e^{i\,\frac{\pi}{4}}\,\left(\cos(t^2) - i\,\sin(t^2)\right),$$

and

$$\lim_{R\to\infty}\int_0^R e^{-t^2}\,dt = \int_0^{+\infty} e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}.$$

Thus from the previous identity we get

$$e^{i\,\frac{\pi}{4}}\lim_{R\to\infty}\int_0^R \left(\cos(t^2) - i\,\sin(t^2)\right)\,dt = \frac{\sqrt{\pi}}{2} + \lim_{R\to\infty}\left|R\,i\int_0^{\frac{\pi}{4}} e^{-R^2\,e^{2\,i\,t}}\,e^{i\,t}\,dt\right|.$$

In order to conclude, we only need to compute the last limit. We start by observing that

$$\left|R\,i\int_0^{\frac{\pi}{4}} e^{-R^2\,e^{2\,i\,t}}\,e^{i\,t}\,dt\right| \le R\int_0^{\frac{\pi}{4}} |e^{-R^2\,e^{2\,i\,t}}|\,dt = R\int_0^{\frac{\pi}{4}} e^{-R^2\cos(2\,t)}\,dt.$$

On the interval $[0,\pi/4]$, the function $t\mapsto \cos(2\,t)$ is concave, thus there exists $c>0$ such that[6]

$$\cos(2\,t) \ge 1 - \frac{4}{\pi}\,t, \qquad \text{for } t\in[0,\pi/4].$$

We obtain

$$R\int_0^{\frac{\pi}{4}} e^{-R^2\cos(2\,t)}\,dt \le R\int_0^{\frac{\pi}{4}} e^{-R^2\,\left(1-\frac{4}{\pi}\,t\right)}\,dt = R\,\frac{\pi}{4}\,\frac{1}{R^2}\,\left[e^{-R^2\,\left(1-\frac{4}{\pi}\,t\right)}\right]_0^{\frac{\pi}{4}}$$
$$= \frac{\pi}{4}\,\frac{1}{R}\,\left[1 - e^{-R^2}\right],$$

and the latter converges to 0 as R goes to ∞. We thus obtained

$$e^{i\,\frac{\pi}{4}}\lim_{R\to\infty}\int_0^R \left(\cos(t^2) - i\,\sin(t^2)\right)\,dt = \frac{\sqrt{\pi}}{2}.$$

By multiplying both sides by $e^{-i\,\pi/4}$, we get

$$\lim_{R\to\infty}\int_0^R \left(\cos(t^2) - i\,\sin(t^2)\right)\,dt = e^{-i\,\frac{\pi}{4}}\,\frac{\sqrt{\pi}}{2}.$$

This gives the desired conclusion. □

Exercise 1.13.5. *By using the Residue Theorem, verify that*

$$\int_0^{2\,\pi} \frac{dt}{1+\sin^2 t} = \sqrt{2}\,\pi.$$

[6] It is easy to see that

$$c = \frac{4}{\pi}.$$

Solution. By using that

$$\sin^2 t = \left(\frac{e^{i\,t} - e^{-i\,t}}{2\,i}\right)^2,$$

we get

$$\int_0^{2\,\pi} \frac{dt}{1+\sin^2 t} = \int_0^{2\,\pi} \frac{1}{1 - \dfrac{(e^{i\,t}-e^{-i\,t})^2}{4}}\,dt = \int_0^{2\,\pi} \frac{4}{4-(e^{i\,t}-e^{-i\,t})^2}\,dt$$
$$= \int_0^{2\,\pi} \frac{4\,i\,e^{i\,t}}{i\,e^{i\,t}\,(4-(e^{i\,t}-e^{-i\,t})^2)}\,dt$$
$$= \frac{4}{i}\int_\gamma \frac{1}{z\left(4-\left(z-\dfrac{1}{z}\right)^2\right)}\,dz,$$

where now

$$\gamma(t) = e^{i\,t}, \qquad t\in[0,2\pi].$$

In other words, γ parametrize the boundary of the disk of radius 1, centered at the origin (with positive orientation, as usual). The function

$$f(z) := \frac{1}{z\left(4-\left(z-\dfrac{1}{z}\right)^2\right)} = \frac{z}{4\,z^2-(z^2-1)^2} = -\frac{z}{z^4-6\,z^2+1},$$

has isolated singularities at the zeros of $6\,z^2 - z^4 - 1$. These are given by

$$z_{1,2} = \pm\sqrt{3+2\sqrt{2}} \qquad\text{and}\qquad z_{3,4} = \pm\sqrt{3-2\sqrt{2}},$$

and they are simple poles. We are only interested in those poles which fall inside $B_1(0)$. We observe that

$$|z_{1,2}|>1 \qquad\text{and}\qquad |z_{3,4}|<1.$$

By the Residue Theorem, we thus obtain

$$\int_0^{2\,\pi}\frac{dt}{1+\sin^2 t} = \frac{4}{i}\,2\,\pi\,i\Big(\mathrm{res}(f,z_3)+\mathrm{res}(f,z_4)\Big) = 8\,\pi\,\Big(\mathrm{res}(f,z_3)+\mathrm{res}(f,z_4)\Big). \tag{1.13.1}$$

We need to compute the residues. We have

$$\mathrm{res}(f,z_3) = \lim_{z\to z_3}(z-z_3)\,f(z) = -\lim_{z\to z_3}\frac{z}{(z^2-3-2\sqrt{2})\,(z-z_4)}$$
$$= -\frac{z_3}{(z_3^2-3-2\sqrt{2})\,(z_3-z_4)}$$
$$= \frac{1}{4\sqrt{2}}\,\frac{z_3}{z_3-z_4},$$

and

$$\mathrm{res}(f,z_4) = \lim_{z\to z_4}(z-z_4)\,f(z) = -\lim_{z\to z_4}\frac{z}{(z^2-3-2\sqrt{2})\,(z-z_3)}$$
$$= -\frac{z_4}{(z_4^2-3-2\sqrt{2})\,(z_4-z_3)}$$
$$= \frac{1}{4\sqrt{2}}\,\frac{z_4}{(z_4-z_3)}.$$

We thus get

$$\operatorname{res}(f, z_3) + \operatorname{res}(f, z_4) = \frac{1}{4\sqrt{2}}\,\frac{z_3 - z_4}{z_3 - z_4} = \frac{1}{4\sqrt{2}}.$$

By spending this information in (1.13.1), we get

$$\int_0^{2\pi} \frac{dt}{1 + \sin^2 t} = \frac{4}{i}\, 2\,\pi\, i\Big(\operatorname{res}(f, z_3) + \operatorname{res}(f, z_4)\Big) = \frac{2}{\sqrt{2}}\,\pi,$$

as desired. □

Exercise 1.13.6. *By using the Residue Theorem, verify that for every $a > 1$ we have*

$$\int_0^{2\pi} \frac{dt}{a + \cos t} = \frac{2\pi}{\sqrt{a^2 - 1}}.$$

Solution. By proceeding as above, we have

$$\begin{aligned}\int_0^{2\pi} \frac{dt}{a + \cos t} = \int_0^{2\pi} \frac{dt}{a + \dfrac{e^{i\,t} + e^{-i\,t}}{2}} &= \int_0^{2\pi} \frac{2\,i\,e^{i\,t}}{(2\,a + e^{i\,t} + e^{-i\,t})\,i\,e^{i\,t}}\,dt \\ &= \int_\gamma \frac{2}{i\,z\,(2\,a + z + 1/z)}\,dz,\end{aligned}$$

where as above we set

$$\gamma(t) = e^{i\,t}, \qquad t \in [0, 2\pi].$$

The integrated function is

$$f(z) := \frac{2}{i\,(2\,a\,z + z^2 + 1)},$$

which is analytic, except that at the singularities

$$z_1 = -a + \sqrt{a^2 - 1} \qquad \text{and} \qquad z_2 = -a - \sqrt{a^2 - 1},$$

which are simple poles. We observe that

$$|z_1| = a - \sqrt{a^2 - 1} < 1 \qquad \text{and} \qquad |z_2| = a + \sqrt{a^2 - 1} > a > 1,$$

thus only the first pole z_1 falls inside the region delimited by γ (which is again the disk of radius 1 and center the origin). We thus obtain

$$\begin{aligned}\int_0^{2\pi} \frac{dt}{a + \cos t} = 2\,\pi\,i\operatorname{res}(f, z_1) &= 2\,\pi\,i\lim_{z\to z_1} \frac{2\,(z - z_1)}{i\,(2\,a\,z + z^2 + 1)} \\ &= 2\,\pi\,i\lim_{z\to z_1} \frac{2\,(z - z_1)}{i\,(z - z_2)\,(z - z_1)} \\ &= 4\,\pi\,\frac{1}{z_1 - z_2} = \frac{2\pi}{\sqrt{a^2 - 1}},\end{aligned}$$

as desired. □

Exercise 1.13.7. *By using the Residue Theorem, verify that we have*

$$\int_0^{2\pi} \frac{dt}{2 + \sin t\,\cos t} = \frac{4\pi}{\sqrt{15}}.$$

Solution. We have

$$\int_0^{2\pi} \frac{dt}{2+\sin t\,\cos t} = \int_0^{2\pi} \frac{dt}{2+\dfrac{e^{2it}-e^{-2it}}{4i}} = 4i\int_0^{2\pi} \frac{i\,e^{it}\,dt}{(8i+e^{2it}-e^{-2it})\,i\,e^{it}}$$
$$= 4\int_\gamma \frac{dz}{z\,(8i+z^2-1/z^2)},$$

where $\gamma(t)=e^{it}$, for $t\in[0,2\pi]$. We study the singularities of the function

$$f(z)=\frac{1}{z\,(8i+z^2-1/z^2)}=\frac{z}{z^4+8\,i\,z^2-1}.$$

This has four simple poles at the roots of $z^4+8\,i\,z^2-1$, i.e.

$$z_1=\sqrt{4+\sqrt{15}}\,e^{-\frac{\pi}{4}i} \qquad \text{and} \qquad z_2=\sqrt{4+\sqrt{15}}\,e^{\frac{3\pi}{4}i},$$
$$z_3=\sqrt{4-\sqrt{15}}\,e^{-\frac{\pi}{4}i} \qquad \text{and} \qquad z_4=\sqrt{4-\sqrt{15}}\,e^{\frac{3\pi}{4}i}.$$

It is not difficult to see that only the second ones fall inside the disk $B_1(0)$ delimited by γ. Indeed,

$$|z_3|=|z_4|=\sqrt{4-\sqrt{15}}<\sqrt{4-\sqrt{9}}=1.$$

We thus obtain

$$\begin{aligned}\int_0^{2\pi}\frac{dt}{2+\sin t\,\cos t} &= 4\cdot 2\pi i\Big(\mathrm{res}(f,z_3)+\mathrm{res}(f,z_4)\Big)\\ &= 8\pi i\lim_{z\to z_3}\frac{z}{(z-z_1)(z-z_2)(z-z_4)}\\ &+8\pi i\lim_{z\to z_4}\frac{z}{(z-z_1)(z-z_2)(z-z_3)}\\ &= 8\pi i\frac{z_3}{(z_3-z_1)(z_3-z_2)(z_3-z_4)}\\ &-8\pi i\frac{z_4}{(z_4-z_1)(z_4-z_2)(z_3-z_4)}.\end{aligned}$$

We observe that for every $z\neq z_1,z_2$, we have

$$(z-z_1)(z-z_2)=z^2+(4+\sqrt{15})\,i,$$

thus by observing that $z_3^2=z_4^2$ we obtain

$$\begin{aligned}\frac{z_3}{(z_3-z_1)(z_3-z_2)(z_3-z_4)}-\frac{z_4}{(z_4-z_1)(z_4-z_2)(z_3-z_4)}\\ =\frac{z_3}{(z_3^2+(4+\sqrt{15})\,i)(z_3-z_4)}\\ -\frac{z_4}{(z_4^2+(4+\sqrt{15})\,i)(z_3-z_4)}\\ =\frac{z_3-z_4}{(z_3^2+(4+\sqrt{15})\,i)(z_3-z_4)}\\ =\frac{1}{z_3^2+(4+\sqrt{15})\,i}=\frac{1}{2\sqrt{15}\,i}.\end{aligned}$$

This gives the conclusion. □

Exercise 1.13.8. *By using the Residue Theorem, verify that for every $n \in \mathbb{N} \setminus \{0,1\}$ we have*

$$\int_0^{+\infty} \frac{dx}{1+x^n} = \frac{\pi}{n} \frac{1}{\sin\left(\frac{\pi}{n}\right)}.$$

Solution. We consider the function

$$f(z) = \frac{1}{1+z^n}, \qquad z \notin \{z_0, \ldots, z_{n-1}\},$$

which is analytic in $\mathbb{C} \setminus \{z_0, \ldots, z_{n-1}\}$, with

$$z_k = e^{i\left(\frac{\pi}{n} + \frac{2\pi}{n} k\right)}, \qquad k = 0, \ldots, n-1.$$

Each z_i is a simple pole and we have

$$\operatorname{res}(f, z_0) = \frac{1}{(1+z^n)'_{|z=z_0}} = \frac{1}{n\, e^{\frac{n-1}{n}\pi i}}.$$

We now fix $R \gg 1$ and integrate the function f on the piecewise C^1 loop Γ_R obtained by joining

$$\gamma_1(t) = t, \qquad t \in [0, R],$$

$$\gamma_2(t) = R\, e^{\frac{2\pi}{n} i t}, \qquad t \in [0,1],$$

$$\gamma_3(t) = (R-t)\, e^{\frac{2\pi}{n} i}, \qquad t \in [0, R].$$

It is not difficult that the interior of Γ_R contains only the pole z_0, thus from the Residue Theorem we obtain

$$\begin{aligned}
\frac{2\pi i}{n\, e^{\frac{n-1}{n}\pi i}} = \int_{\Gamma_R} f(z)\, dz &= \int_{\gamma_1} f(z)\, dz + \int_{\gamma_2} f(z)\, dz + \int_{\gamma_3} f(z)\, dz \\
&= \int_0^R \frac{dt}{1+t^n} + \frac{2\pi}{n} \int_0^1 \frac{R\, i\, e^{\frac{2\pi}{n} i t}}{1+R^n e^{2\pi i t}}\, dt \\
&\quad - \int_0^R \frac{e^{\frac{2\pi}{n} i}}{1+(R-t)^n}\, dt.
\end{aligned}$$

We now observe that

$$\left| \int_0^1 \frac{R\, i\, e^{\frac{2\pi}{n} i t}}{1+R^n e^{2\pi i t}}\, dt \right| \le \int_0^1 \frac{R}{|1+R^n e^{2\pi i t}|}\, dt \le \frac{R}{R^n - 1},$$

and

$$-\int_0^R \frac{e^{\frac{2\pi}{n} i}}{1+(R-t)^n}\, dt = -\int_0^R \frac{e^{\frac{2\pi}{n} i}}{1+s^n}\, ds.$$

Thus we obtain

$$\left(1 - e^{\frac{2\pi}{n} i}\right) \int_0^{+\infty} \frac{1}{1+t^n}\, dt = \lim_{R\to\infty} \int_{\Gamma_R} \frac{1}{1+z^n}\, dz = \frac{2\pi i}{n\, e^{\frac{n-1}{n}\pi i}},$$

that is

$$\int_0^{+\infty} \frac{1}{1+t^n}\, dt = \frac{2\pi i}{n\, e^{\frac{n-1}{n}\pi i}} \frac{1}{1 - e^{\frac{2\pi}{n} i}}.$$

In order to conclude, we observe that

$$\begin{aligned}
e^{\frac{n-1}{n}\pi i}\left(1-e^{\frac{2\pi}{n}i}\right) &= e^{\frac{n-1}{n}\pi i} - e^{\frac{n+1}{n}\pi i} \\
&= \left[\cos\left(\frac{n-1}{n}\,\pi\right) - \cos\left(\frac{n+1}{n}\,\pi\right)\right] \\
&\quad + i\left[\sin\left(\frac{n-1}{n}\,\pi\right) - \sin\left(\frac{n+1}{n}\,\pi\right)\right] \\
&= 2\,\sin\pi\,\sin\left(\frac{\pi}{n}\right) + 2\,i\,\cos\pi\sin\left(-\frac{\pi}{n}\right) \\
&= 2\,i\,\sin\left(\frac{\pi}{n}\right).
\end{aligned}$$

This gives the desired conclusion. □

Exercise 1.13.9. *By using the previous exercise, compute*

$$\int_0^{+\infty} \frac{dx}{8+x^3}.$$

Solution. It is sufficient to use a simple change of variable to reduce the integral to the one computed in the previous exercise. We have

$$\int_0^{+\infty} \frac{dx}{8+x^3} = \frac{1}{8}\int_0^{+\infty} \frac{dx}{1+\left(\frac{x}{2}\right)^3} = \frac{1}{4}\int_0^{+\infty} \frac{dt}{1+t^3}.$$

Since we already knows that

$$\int_0^{+\infty} \frac{dt}{1+t^3} = \frac{\pi}{3}\,\frac{2}{\sqrt{3}},$$

we can conclude. □

Exercise 1.13.10. *By using the Residue Theorem, verify that*

$$\int_{-\infty}^{+\infty} \frac{e^{\frac{x}{3}}}{1+e^x}\,dx = \frac{2\,\pi}{\sqrt{3}}.$$

Solution. We consider the piecewise regular loop Γ obtained by linking the segments

$$\begin{aligned}
\gamma_1(t) &= 2\,R\,t - R, && t\in[0,1], \\
\gamma_2(t) &= R + 2\,\pi\,i\,t, && t\in[0,1], \\
\gamma_3(t) &= 2\,\pi\,i + R - 2\,R\,t, && t\in[0,1], \\
\gamma_4(t) &= -R + 2\,\pi\,i\,t, && t\in[0,1].
\end{aligned}$$

Since the function

$$\frac{e^{\frac{z}{3}}}{1+e^z},$$

has a simple pole at $z_0 = i\,\pi$ and the latter is contained in the bounded region entoured by Γ, we get

$$\int_\Gamma \frac{e^{\frac{z}{3}}}{1+e^z}\,dz = 2\,\pi\,i\,\mathrm{res}\left(\frac{e^{\frac{z}{3}}}{1+e^z},\, i\,\pi\right) = 2\,\pi\,i\,\frac{e^{\frac{i\,\pi}{3}}}{e^{i\,\pi}} = -\pi\,i + \sqrt{3}\,\pi. \tag{1.13.2}$$

We now analyze the integral on the left-hand side. We have

$$\begin{aligned}\int_\Gamma \frac{e^{\frac{z}{3}}}{1+e^z}\,dz &= \sum_{i=1}^{4}\int_{\gamma_i}\frac{e^{\frac{z}{3}}}{1+e^z}\,dz\\
&= 2\,R\int_0^1 \frac{e^{\frac{2\,R\,t-R}{3}}}{1+e^{2\,R\,t-R}}\,dt + 2\,\pi\,i\int_0^1\frac{e^{\frac{R+2\,\pi\,i\,t}{3}}}{1+e^{R+2\,\pi\,i\,t}}\,dt\\
&-2\,R\int_0^1\frac{e^{\frac{2\,\pi\,i+R-2\,R\,t}{3}}}{1+e^{2\,\pi\,i+R-2\,R\,t}}\,dt - 2\,\pi\,i\int_0^1\frac{e^{-\frac{R+2\,\pi\,i\,t}{3}}}{1+e^{-R-2\,\pi\,i\,t}}\,dt\\
&=\int_{-R}^{R}\frac{e^{\frac{s}{3}}}{1+e^s}\,ds - 2\,R\int_0^1\frac{e^{\frac{R-2\,R\,t}{3}}}{1+e^{R-2\,R\,t}}\,dt\\
&+2\,\pi\,i\int_0^1\frac{e^{\frac{R+2\,\pi\,i\,t}{3}}}{1+e^{R+2\,\pi\,i\,t}}\,dt - 2\,\pi\,i\int_0^1\frac{e^{-\frac{R+2\,\pi\,i\,t}{3}}}{1+e^{-R-2\,\pi\,i\,t}}\,dt\\
&=\int_{-R}^{R}\frac{e^{\frac{s}{3}}}{1+e^s}\,ds - 2\,R\,e^{\frac{2\,\pi\,i}{3}}\int_0^1\frac{e^{\frac{R-2\,R\,t}{3}}}{1+e^{R-2\,R\,t}}\,dt\\
&+2\,\pi\,i\int_0^1\frac{e^{\frac{R+2\,\pi\,i\,t}{3}}}{1+e^{R+2\,\pi\,i\,t}}\,dt - 2\,\pi\,i\int_0^1\frac{e^{-\frac{R+2\,\pi\,i\,t}{3}}}{1+e^{-R-2\,\pi\,i\,t}}\,dt.\end{aligned}$$

We now observe that

$$2\,R\,e^{\frac{2\,\pi\,i}{3}}\int_0^1\frac{e^{\frac{R-2\,R\,t}{3}}}{1+e^{R-2\,R\,t}}\,dt = e^{\frac{2\,\pi\,i}{3}}\int_{-R}^{R}\frac{e^{\frac{s}{3}}}{1+e^s}\,ds,$$

while for R large we have

$$\left|\int_0^1\frac{e^{\frac{R+2\,\pi\,i\,t}{3}}}{1+e^{R+2\,\pi\,i\,t}}\,dt\right| + \left|\int_0^1\frac{e^{-\frac{R+2\,\pi\,i\,t}{3}}}{1+e^{-R-2\,\pi\,i\,t}}\,dt\right| \le \frac{e^{\frac{R}{3}}}{e^R-1} + \frac{e^{-\frac{R}{3}}}{1-e^{-R}},$$

and the last quantity goes to 0 as R goes to $+\infty$. By using these informations in (1.13.2), we obtain

$$-\pi\,i+\sqrt{3}\,\pi = \left(1-e^{\frac{2\,\pi\,i}{3}}\right)\lim_{R\to\infty}\int_{-R}^{R}\frac{e^{\frac{s}{3}}}{1+e^s}\,ds.$$

This gives

$$\int_{-\infty}^{+\infty}\frac{e^{\frac{s}{3}}}{1+e^s}\,ds = \pi\,\frac{\sqrt{3}-i}{\dfrac{3}{2}-i\,\dfrac{\sqrt{3}}{2}} = \frac{2\,\pi}{\sqrt{3}},$$

as desired. $\square$

Chapter 2

The $Z-$transform

1. Definitions and examples

Definition 2.1.1. Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ be a sequence. We say that this is $Z-$*transformable* if

$$R := \limsup_{n\to\infty} \sqrt[n]{|x_n|} < +\infty. \tag{2.1.1}$$

Definition 2.1.2. Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ be a $Z-$transformable sequence. We define its $Z-$*transform* by

$$\mathcal{Z}[\{x_n\}](z) = \sum_{n=0}^{\infty} \frac{x_n}{z^n}.$$

By Remark 1.11.3, we know that this is an analytic function on the region $\{z \in \mathbb{C} : |z| > R\}$, with R defined by (2.1.1).

Remark 2.1.3 (Bounded sequences)**.** We observe that if $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ is *bounded*, i.e. there exists $M > 0$ such that

$$|x_n| \le M, \qquad \text{for every } n \in \mathbb{N},$$

then the region of convergence of its $Z-$transform $\mathcal{Z}[\{x_n\}]$ contains the set $\{z \in \mathbb{C} : |z| > 1\}$. Indeed, it is sufficient to observe that in this case R defined in (2.1.1) is smaller than 1, since

$$\limsup_{n\to\infty} \sqrt[n]{|x_n|} \le \lim_{n\to\infty} \sqrt[n]{M} = 1.$$

Let us compute some basic $Z-$transforms.

Example 2.1.4. Let $\{x_n\}_{n\in\mathbb{N}}$ be the constant sequence $x_n = 1$ for every $n \in \mathbb{N}$. This is of course $Z-$transformable, with

$$R = \limsup_{n\to\infty} \sqrt[n]{1} = 1.$$

By recalling the expression for the sum of the geometric series, its $Z-$transform is given by

$$\mathcal{Z}[\{1\}](z) = \sum_{n=0}^{\infty} \frac{1}{z^n} = \frac{1}{1-\dfrac{1}{z}} = \frac{z}{z-1}, \qquad \text{for } |z| > 1.$$

Example 2.1.5 (Delta sequence)**.** We fix $j \in \mathbb{N}$ and consider the sequence $\{\delta_{j,n}\}_{n\in\mathbb{N}}$ defined by

$$\delta_{j,n} = \begin{cases} 1, & \text{if } n = j, \\ 0, & \text{otherwise.} \end{cases}$$

This is of course $Z-$transformable, with

$$R = \limsup_{n\to\infty} \sqrt[n]{|\delta_{j,n}|} = 0.$$

Its $Z-$transform is then given by

$$\mathcal{Z}[\{\delta_{j,n}\}](z) = \sum_{n=0}^{\infty} \frac{\delta_{j,n}}{z^n} = \frac{1}{z^j}, \qquad \text{for } |z| > 0.$$

Observe that for the particular case $j = 0$, we get

$$\mathcal{Z}[\{\delta_{0,n}\}](z) = 1, \qquad \text{for } z \in \mathbb{C}.$$

Example 2.1.6. Let $\{x_n\}_{n\in\mathbb{N}}$ be the sequence

$$x_0 = 0 \qquad \text{and} \qquad x_n = \frac{1}{n}, \qquad \text{for every } n \geq 1.$$

We have

$$R = \limsup_{n\to\infty} \sqrt[n]{\frac{1}{n}} = 1.$$

By recalling Example 1.7.9, we have

$$\sum_{n=1}^{\infty} \frac{s^n}{n} = -\mathrm{Log}(1-s), \qquad |s| < 1,$$

and using this formula with $s = 1/z$, we get

$$\mathcal{Z}[\{1/n\}](z) = \sum_{n=1}^{\infty} \frac{1}{n}\,\frac{1}{z^n} = -\mathrm{Log}\left(1 - \frac{1}{z}\right), \qquad |z| > 1.$$

Example 2.1.7. Let $\{x_n\}_{n\in\mathbb{N}}$ be the sequence $x_n = 1/n!$, for every $n \in \mathbb{N}$. We have[1]

$$R = \limsup_{n\to\infty} \sqrt[n]{\frac{1}{n!}} = 0,$$

thus the $Z-$transform is now a holomorphic function in $\mathbb{C}^*$. By recalling that (see Remark 1.8.3)

$$\sum_{n=0}^{\infty} \frac{s^n}{n!} = e^s, \qquad \text{for } s \in \mathbb{C},$$

and using this formula with $s = 1/z$, we get

$$\mathcal{Z}[\{1/n!\}](z) = \sum_{n=1}^{\infty} \frac{1}{n!}\,\frac{1}{z^n} = e^{\frac{1}{z}}, \qquad z \in \mathbb{C}^*.$$

Observe that this function has an isolated singularity at $z = 0$, which is an *essential singularity* thanks to Proposition 1.11.6.

[1] We use here that

$$\sqrt[n]{n!} \sim \frac{n}{e}, \qquad \text{for } n \to \infty.$$

2. Basic properties

We collect here some important properties of the $Z-$transform. In what follows $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ and $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ are two $Z-$transformable sequences.

Proposition 2.2.1 (Linearity). *Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be two $Z-$transformable sequences, with*

$$r = \limsup_{n\to\infty} \sqrt[n]{|x_n|} \qquad \text{and} \qquad \varrho = \limsup_{n\to\infty} \sqrt[n]{|y_n|}.$$

For every $\alpha, \beta \in \mathbb{C}$, the sequence $\{\alpha\, x_n + \beta, y_n\}_{n\in\mathbb{N}}$ is $Z-$transformable with a radius of convergence

$$R \le \max\{r, \varrho\}.$$

Moreover, we have

$$\mathcal{Z}[\{\alpha\, x_n + \beta\, y_n\}](z) = \alpha\, \mathcal{Z}[\{x_n\}](z) + \beta\, \mathcal{Z}[\{y_n\}](z), \qquad \text{for } |z| > \max\{r, \varrho\}.$$

Proposition 2.2.2 (Time delay). *For every $k \in \mathbb{N} \setminus \{0\}$*

$$\mathcal{Z}[\{x_{n+k}\}](z) = z^k \left(\mathcal{Z}[\{x_n\}](z) - x_0 - \frac{x_1}{z} - \cdots - \frac{x_{k-1}}{z^{k-1}} \right). \tag{2.2.1}$$

Proof. We have

$$\begin{aligned} \mathcal{Z}[\{x_{n+k}\}](z) &= \sum_{n=0}^{\infty} \frac{x_{n+k}}{z^n} = z^k \sum_{n=0}^{\infty} \frac{x_{n+k}}{z^{n+k}} = z^k \sum_{n=k}^{\infty} \frac{x_n}{z^n} \\ &= z^k \left(\sum_{n=0}^{\infty} \frac{x_n}{z^n} - \sum_{n=0}^{k-1} \frac{x_n}{z^n} \right), \end{aligned}$$

which gives the desired conclusion. □

Definition 2.2.3. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ two sequences. We define their *convolution* as the new sequence $\{x_n * y_n\}_{n\in\mathbb{N}}$ such that

$$x_n * y_n = \sum_{k=0}^{n} x_k\, y_{n-k} = \sum_{k=0}^{n} x_{n-k}\, y_k, \qquad \text{for every } n \in \mathbb{N}.$$

Proposition 2.2.4 (Convolution). *Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be two $Z-$transformable sequences, with*

$$r = \limsup_{n\to\infty} \sqrt[n]{|x_n|} \qquad \text{and} \qquad \varrho = \limsup_{n\to\infty} \sqrt[n]{|y_n|}.$$

Then the convolution $\{x_n * y_n\}_{n\in\mathbb{N}}$ is $Z-$transformable and we have

$$\mathcal{Z}[\{x_n * y_n\}](z) = \mathcal{Z}[\{x_n\}](z)\, \mathcal{Z}[\{y_n\}](z), \qquad \text{for } |z| > \max\{r,\, \varrho\}.$$

Proof. By definition of convolution, we have

$$x_n * y_n = \sum_{k=0}^{n} x_k\, y_{n-k}, \qquad \text{for every } n \in \mathbb{N}.$$

It is known that if $\sum_{n=0}^{\infty} x_n$ and $\sum_{n=0}^{\infty} y_n$ are absolutely convergent, then $\sum_{n=0}^{\infty} x_n * y_n$ is absolutely convergent as well and we have

$$\sum_{n=0}^{\infty} x_n * y_n = \left(\sum_{n=0}^{\infty} x_n \right) \left(\sum_{n=0}^{\infty} y_n \right), \tag{2.2.2}$$

see Exercise 2.8.1 below. Thus we get

$$\begin{aligned}\mathcal{Z}[\{x_n * y_n\}](z) = \sum_{n=0}^{\infty} \frac{x_n * y_n}{z^n} &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x_k\, y_{n-k}}{z^n} \\ &= \sum_{n=0}^{\infty} \frac{x_n}{z^n} * \frac{y_n}{z^n}.\end{aligned}$$

We observe that if we take $|z| > \max\{r, \varrho\}$, then

$$\sum_{n=0}^{\infty} \frac{x_n}{z^n} \qquad \text{and} \qquad \sum_{n=0}^{\infty} \frac{y_n}{z^n},$$

are absolutely convergent (and totally, indeed). We can apply (2.2.2) and get the conclusion. $\square$

Proposition 2.2.5 (Derivative). *If $\{x_n\}_{n\in\mathbb{N}}$ is $Z-$transformable, then $\{n\, x_n\}_{n\in\mathbb{N}}$ is $Z-$transformable as well, with*

$$\limsup_{n\to\infty} \sqrt[n]{n\,|x_n|} = \limsup_{n\to\infty} \sqrt[n]{|x_n|}.$$

Moreover, we have

$$\mathcal{Z}[\{n\, x_n\}](z) = -z\, \frac{d}{dz} \mathcal{Z}[\{x_n\}](z). \tag{2.2.3}$$

Proof. We first observe that if $\{x_n\}_{n\in\mathbb{N}}$ is $Z-$transformable, then $\{n\, x_n\}_{n\in\mathbb{N}}$ is $Z-$transformable as well, since

$$\limsup_{n\to\infty} \sqrt[n]{n\,|x_n|} = \limsup_{n\to\infty} \sqrt[n]{|x_n|} < +\infty,$$

where we used that

$$\lim_{n\to\infty} \sqrt[n]{n} = 1.$$

We then compute

$$\begin{aligned}\mathcal{Z}[\{n\, x_n\}](z) = \sum_{n=0}^{\infty} \frac{n\, x_n}{z^n} = z \sum_{n=1}^{\infty} n\, x_n\, z^{-n-1} &= z \sum_{n=1}^{\infty} \frac{d}{dz} \left(-x_n\, z^{-n}\right) \\ &= z\, \frac{d}{dz} \left(-\sum_{n=1}^{\infty} x_n\, z^{-n}\right),\end{aligned}$$

as desired. In order to exchange the summation and the differentiation, we used the fact that a power series is an analytic function on its region of convergence, whose derivative can be computed by differentiating term by term (see Proposition 1.7.5). $\square$

Proposition 2.2.6 (Scaling). *For every $q \in \mathbb{C}^*$, we have*

$$\mathcal{Z}[\{q^n\, x_n\}](z) = \mathcal{Z}[\{x_n\}] \left(\frac{z}{q}\right). \tag{2.2.4}$$

Proof. This is by direct computation, we first have to observe that

$$\limsup_{n\to\infty} \sqrt[n]{|q|^n\,|x_n|} = |q| \limsup_{n\to\infty} \sqrt[n]{|x_n|} =: |q|\, R.$$

Then for $|z| > |q|\, R$ we get

$$\mathcal{Z}[\{q^n\, x_n\}](z) = \sum_{n=0}^{\infty} \frac{q^n\, x_n}{z^n} = \sum_{n=0}^{\infty} \frac{x_n}{(z/q)^n} = \mathcal{Z}[\{x_n\}] \left(\frac{z}{q}\right),$$

as desired. $\square$

Example 2.2.7. As a particular case of the previous result, if we take $q = e^{i\,\vartheta}$ for some $\vartheta \in (-\pi, \pi]$, we obtain

$$\mathcal{Z}[\{e^{i\,n\,\vartheta}\,x_n\}](z) = \mathcal{Z}[\{x_n\}](z\,e^{-i\,\vartheta}), \qquad \text{for } |z| > 1.$$

If $x_n = 1$ for every $n \in \mathbb{N}$, we thus get from Example 2.1.4

$$\mathcal{Z}[\{e^{i\,n\,\vartheta}\}](z) = \frac{z\,e^{-i\,\vartheta}}{z\,e^{-i\,\vartheta} - 1} = \frac{z}{z - e^{i\,\vartheta}}, \qquad \text{for } |z| > 1.$$

Proposition 2.2.8 (Periodic sequences)**.** *Let us suppose that there exists $m \in \mathbb{N} \setminus \{0\}$ such that*

$$x_{n+m} = x_n, \qquad \textit{for every } n \in \mathbb{N}.$$

In this case we say that the sequence is $m-$periodic. We have

$$\mathcal{Z}[\{x_n\}](z) = \frac{z^m}{z^m - 1} \sum_{n=0}^{m-1} \frac{x_n}{z^n}, \qquad \textit{for } |z| > 1. \tag{2.2.5}$$

Proof. We first observe that a periodic sequence is bounded, thus by Remark 2.1.3 is $Z-$transformable and its $Z-$transform is well-defined for $|z| > 1$. By appealing to the definition, we have

$$\begin{aligned}
\mathcal{Z}[\{x_n\}](z) = \sum_{n=0}^{\infty} \frac{x_n}{z^n} &= \sum_{k=0}^{\infty} \sum_{n=k\,m}^{(k+1)\,m-1} \frac{x_n}{z^n} \\
&= \sum_{k=0}^{\infty} \sum_{\ell=0}^{m-1} \frac{x_{\ell+k\,m}}{z^{\ell+k\,m}} = \sum_{k=0}^{\infty} \sum_{\ell=0}^{m-1} \frac{x_\ell}{z^{\ell+k\,m}} \\
&= \sum_{k=0}^{\infty} \left(\frac{1}{z^{k\,m}} \sum_{\ell=0}^{m-1} \frac{x_\ell}{z^\ell} \right).
\end{aligned}$$

In order to conclude, we only need to compute the sum of the geometric series

$$\sum_{j=0}^{\infty} \frac{1}{z^{j\,m}} = \frac{1}{1 - \dfrac{1}{z^m}} = \frac{z^m}{z^m - 1}, \qquad |z| > 1.$$

This conclude the proof. $\square$

Remark 2.2.9. From formula (2.2.5), we can easily see that the $Z-$transform of a $m-$periodic sequence $\{x_n\}_{n\in\mathbb{N}}$ can be extended to the whole

$$\mathbb{C}^* \setminus \{z_0, \dots, z_{m-1}\},$$

where $z_0, \dots, z_{m-1}$ are the solutions (which are all distinct) of $z^m = 1$. By using formula (1.5.2) with $w = 1$, these are given by

$$z_j = e^{\frac{2\,j\,\pi}{m}\,i} = \left(\cos\left(\frac{2\,j\,\pi}{m}\right) + i\,\sin\left(\frac{2\,j\,\pi}{m}\right) \right), \qquad j = 0, 1\dots, m-1.$$

In other words, the function

$$\mathcal{Z}[\{x_n\}](z) = \frac{z^m}{z^m - 1} \sum_{n=0}^{m-1} \frac{x_n}{z^n},$$

has simple poles at $z = e^{\frac{2\,j\,\pi}{n}\,i}$, for $j = 0, \dots, m-1$, and it is otherwise holomorphic for $z \neq 0$.

3. Inversion formula

We now face the problem of how to recover the sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ from the knowledge of its $Z-$transform

$$z \mapsto \mathcal{Z}[\{x_n\}](z), \qquad |z| > R.$$

For this, we need to recall that for a holomorphic function f defined in $\{z \in \mathbb{C} : R < |z|\}$, by *Laurent's Theorem* (see Theorem 1.11.4), we have

$$f(z) = \sum_{k=0}^{\infty} a_k\, z^k + \sum_{k=1}^{\infty} \frac{b_k}{z^k},$$

and the coefficients a_k, b_k are given by

$$a_k = \frac{1}{2\,\pi\, i} \int_\gamma \frac{f(z)}{z^{k+1}}\, dz \qquad \text{and} \qquad b_k = \frac{1}{2\,\pi\, i} \int_\gamma f(z)\, z^{k-1}\, dz,$$

thanks to formula (1.11.1). In particular, by using this information for the function

$$\mathcal{Z}[\{x_n\}](z) = \sum_{k=0}^{\infty} \frac{x_k}{z^k} = x_0 + \sum_{k=1}^{\infty} \frac{x_k}{z^k},$$

we obtain the following relation between a $Z-$transformable sequence $\{x_n\}_{n\in\mathbb{N}}$ and its $Z-$transform

$$x_k = \frac{1}{2\,\pi\, i} \int_\gamma \mathcal{Z}[\{x_n\}](z)\, z^{k-1}\, dz, \qquad k \in \mathbb{N}. \tag{2.3.1}$$

Here γ is any positively oriented piecewise regular loop entirely contained in $\{z \in \mathbb{C} : |z| > R\}$ and entouring the origin. Formula (2.3.1) can be referred to as *inversion formula for the $Z-$transform.*

Proposition 2.3.1 (Injectivity of the $Z-$transform)**.** *Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ and $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ be two $Z-$transformable sequences, with*

$$R_1 = \limsup_{n\to\infty} \sqrt[n]{|x_n|} \qquad \textit{and} \qquad R_2 = \limsup_{n\to\infty} \sqrt[n]{|y_n|}.$$

If there exists $|z_0| > \max\{R_1,\, R_2\}$ and a radius $\varrho > 0$ such that

$$B_\varrho(z_0) \subset \Big\{z \in \mathbb{C} : |z| > \max\{R_1,\, R_2\}\Big\},$$

and

$$\mathcal{Z}[\{x_n\}](z) = \mathcal{Z}[\{y_n\}](z), \qquad \textit{for every } z \in B_\varrho(z_0),$$

then we have

$$x_n = y_n \qquad \textit{for every } n \in \mathbb{N}.$$

Proof. We first observe that if $\mathcal{Z}[\{x_n\}]$ and $\mathcal{Z}[\{y_n\}]$ coincide on the open disk $B_\varrho(z_0)$, then they actually coincide on the whole set

$$\Big\{z \in \mathbb{C} : |z| > \max\{R_1,\, R_2\}\Big\},$$

thanks to Corollary 1.8.7. If we now take γ a positively oriented parametrization of the circle centered at the origin and with radius $r > \max\{R_1,\, R_2\}$, by the inversion formula (2.3.1) we get

$$x_k = \frac{1}{2\,\pi\, i} \int_\gamma \mathcal{Z}[\{x_n\}](z)\, z^{k-1}\, dz = \frac{1}{2\,\pi\, i} \int_\gamma \mathcal{Z}[\{y_n\}](z)\, z^{k-1}\, dz = y_k,$$

for every $k \in \mathbb{N}$. This gives the desired conclusion (and observe that this also proves that $R_1 = R_2$). $\square$

Remark 2.3.2 (Exploiting the Residue Theorem)**.** In the applications, very often we can compute the inverse $Z-$trasform by joining (2.3.1) and the Residue Theorem, i.e. Theorem 1.10.14. Indeed, let us suppose that the $Z-$transform $\mathcal{Z}[\{x_n\}]$ admits an extension to the whole $\mathbb{C}$, with the exception of a finite number of singularities $z_1, \dots, z_\ell$ inside the region entoured by γ. Then we obtain

$$x_k = \frac{1}{2\,\pi\,i}\int_\gamma \mathcal{Z}[\{x_n\}](z)\,z^{k-1}\,dz = \sum_{j=1}^{\ell} \mathrm{res}(\mathcal{Z}[\{x_n\}]\,z^{k-1}, z_j), \qquad k \in \mathbb{N}\setminus\{0\}, \tag{2.3.2}$$

and

$$x_0 = \frac{1}{2\,\pi\,i}\int_\gamma \frac{\mathcal{Z}[\{x_n\}](z)}{z}\,dz = \mathrm{res}\left(\frac{\mathcal{Z}[\{x_n\}]}{z}, 0\right) + \sum_{j=1}^{\ell} \mathrm{res}\left(\frac{\mathcal{Z}[\{x_n\}]}{z}, z_j\right). \tag{2.3.3}$$

These formulas are particularly useful in the case all the singularities z_j and 0 are *poles*. Indeed, in this case

$$\mathrm{res}(\mathcal{Z}[\{x_n\}]\,z^{k-1}, z_j),$$

can be easily computed, by appealing to formula (1.10.1) of Proposition 1.10.11. In this situation, by denoting with m_j the multiplicity of the pole z_j, formulas (2.3.2) and (2.3.3) reduce to

$$x_k = \sum_{j=1}^{\ell} \frac{1}{(m_j-1)!} \lim_{z\to z_j} \left(\frac{d^{m_j-1}}{dz^{m_j-1}}\left((z-z_j)^{m_j}\,\mathcal{Z}[\{x_n\}]\,z^{k-1}\right)\right), \qquad k \in \mathbb{N}\setminus\{0\}.$$

and

$$\begin{aligned} x_0 = {} & \frac{1}{(m_0-1)!} \lim_{z\to 0} \left(\frac{d^{m_0-1}}{dz^{m_0-1}}\left(z^{m_0}\,\frac{\mathcal{Z}[\{x_n\}]}{z}\right)\right) \\ & + \sum_{j=1}^{\ell} \frac{1}{(m_j-1)!} \lim_{z\to z_j} \left(\frac{d^{m_j-1}}{dz^{m_j-1}}\left((z-z_j)^{m_j}\,\frac{\mathcal{Z}[\{x_n\}]}{z}\right)\right). \end{aligned}$$

Example 2.3.3 (Fibonacci's numbers)**.** We want to use the $Z-$transform, in order to determine the sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ inductively defined by

$$\begin{cases} x_{n+2} &=& x_{n+1} + x_n, \\ x_0 &=& 1, \\ x_1 &=& 1. \end{cases}$$

We introduce the $Z-$transform

$$X(z) = \sum_{n=0}^{\infty} x_n\,z^{-n}$$

then from the relation defining $\{x_n\}_{n\in\mathbb{N}}$ and by using property (2.2.1) of the $Z-$transform, we get the relation

$$z^2\,X(z) - z^2 - z = z\,X(z) - z + X(z),$$

that is

$$X(z)\,(z^2 - z - 1) = z^2.$$

This finally gives

$$X(z) = \frac{z^2}{z^2 - z - 1},$$

which is a holomorphic function on $\mathbb{C} \setminus \{z_0, z_1\}$, with two simple poles in

$$z_0 = \frac{1-\sqrt{5}}{2} \qquad \text{and} \qquad z_1 = \frac{1+\sqrt{5}}{2}.$$

From the inversion formula, for $n \geq 2$ we get

$$x_n = \frac{1}{2\,\pi\,i} \int_\gamma X(z)\, z^{n-1}\, dz = \frac{1}{2\,\pi\,i} \int_\gamma \frac{z^{n+1}}{z^2 - z - 1}\, dz,$$

where γ is a positively oriented loop, whose image entours z_0 and z_1. In order to compute the last integral and conclude, it is sufficient to use the Residue Theorem, i.e. formula (2.3.2)

$$\begin{aligned}\frac{1}{2\,\pi\,i} \int_\gamma \frac{z^{n+1}}{z^2 - z - 1}\, dz &= \mathrm{res}\left(\frac{z^{n+1}}{z^2 - z - 1}, z_0\right) + \mathrm{res}\left(\frac{z^{n+1}}{z^2 - z - 1}, z_1\right) \\ &= \frac{z_1^{n+1}}{z_1 - z_0} - \frac{z_0^{n+1}}{z_1 - z_0}.\end{aligned}$$

Observe that we used the formula of Corollary 1.10.13, in order to compute the residues. This finally gives

$$x_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right], \qquad \text{for every } n \in \mathbb{N}.$$

We refer the reader to Section 7 for some further examples.

4. The Initial and Final Value Theorems

Theorem 2.4.1 (Initial value). *Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ be $Z-$transformable, then we have*

$$x_0 = \lim_{|z|\to+\infty} \mathcal{Z}[\{x_n\}](z). \tag{2.4.1}$$

Proof. Let $R < +\infty$ be the radius of convergence, then for every $|z| > R+1$ we have

$$\begin{aligned}\Big|\mathcal{Z}[\{x_n\}](z) - x_0\Big| = \left|\sum_{n=1}^{\infty} \frac{x_n}{z^n}\right| \leq \sum_{n=1}^{\infty} \frac{|x_n|}{|z|^n} &= \sum_{n=1}^{\infty} \frac{|x_n|}{|z|^{n-1}}\,\frac{1}{|z|} \\ &\leq \frac{1}{|z|} \sum_{n=1}^{\infty} \frac{|x_n|}{(R+1)^{n-1}} \\ &= \frac{R+1}{|z|} \sum_{n=1}^{\infty} \frac{|x_n|}{(R+1)^n}.\end{aligned}$$

By assumption, the last series converges and thus by taking the limit as $|z|$ goes to $+\infty$

$$\lim_{|z|\to+\infty} \Big|\mathcal{Z}[\{x_n\}](z) - x_0\Big| \leq \left(\sum_{n=1}^{\infty} \frac{|x_n|}{(R+1)^n}\right) \lim_{|z|\to+\infty} \frac{R+1}{|z|} = 0,$$

we get the desired conclusion □

Remark 2.4.2. The previous result implies in particular that the $Z-$transform is a bounded function at infinity. Observe that this is not in contradiction with Liouville Theorem, since a $Z-$transform is never an entire function (i.e. analytic on the whole $\mathbb{C}$), unless in the trivial case

$$x_n = 0, \qquad \text{for } n \in \mathbb{N} \setminus \{0\}.$$

In this case, we clearly have $\mathcal{Z}[\{x_n\}](z) = x_0$ for every $z \in \mathbb{C}$.

Theorem 2.4.3 (Final value, non-tangential version)**.** *Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ be $Z-$transformable such that*

$$\lim_{n\to\infty} x_n = x_\infty \in \mathbb{C}. \tag{2.4.2}$$

Then we have

$$x_\infty = \lim_{\mathbb{R}\ni x\to 1^+} (x-1)\,\mathcal{Z}[\{x_n\}](x). \tag{2.4.3}$$

Proof. We first recall that by Remark 2.1.3, the function $\mathcal{Z}[\{x_n\}]$ is holomorphic for $|z|>1$. We now write

$$x_n = x_\infty + (x_n - x_\infty),$$

then taking the $Z-$transform and using its linearity we obtain

$$\begin{aligned} \mathcal{Z}[\{x_n\}](z) &= x_\infty\,\mathcal{Z}[\{1\}](z) + \mathcal{Z}[\{x_n - x_\infty\}](z) \\ &= x_\infty\,\frac{z}{z-1} + \mathcal{Z}[\{x_n - x_\infty\}](z), \end{aligned} \tag{2.4.4}$$

where we used Example 2.1.4. We thus obtain

$$\lim_{x\to 1^+} (x-1)\,\mathcal{Z}[\{x_n\}](x) = x_\infty + \lim_{x\to 1^+} (x-1)\,\mathcal{Z}[\{x_n - x_\infty\}](x).$$

Finally, by using Lemma 2.4.4 below with the choice $y_n = x_n - x_\infty$, we get that the last limit is zero. This gives the desired conclusion. $\square$

Lemma 2.4.4. *Let $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ be such that*

$$\lim_{n\to\infty} y_n = 0.$$

Then $\mathcal{Z}[\{y_n\}]$ is holomorphic for $|z|>1$ and

$$\lim_{\mathbb{R}\ni x\to 1^+} (x-1)\,\mathcal{Z}[\{y_n\}](x) = 0. \tag{2.4.5}$$

Proof. We first observe that by hypothesis, we have

$$\limsup_{n\to\infty} \sqrt[n]{|y_n|} \le 1.$$

Thus $z \mapsto \mathcal{Z}[\{y_n\}](z)$ is holomorphic for $|z|>1$. In order to prove (2.4.5), we observe that for every $\varepsilon>0$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$$|y_n| < \varepsilon, \qquad \text{for every } n \ge n_\varepsilon,$$

since y_n converges to 0. Thus we obtain for $z = x \in \mathbb{R}$ with[2] $x > 1$

$$\begin{aligned} \Big|(x-1)\,\mathcal{Z}[\{y_n\}](x)\Big| &\le |x-1|\,\left|\sum_{n=0}^{n_\varepsilon-1} \frac{y_n}{x^n}\right| + \varepsilon\,|x-1| \sum_{n=n_\varepsilon}^{\infty} \frac{1}{x^n} \\ &\le (x-1)\,\left|\sum_{n=0}^{n_\varepsilon-1} \frac{y_n}{x^n}\right| + \varepsilon\,(x-1)\,\frac{x}{x-1}. \end{aligned}$$

[2] In the second inequality we use that

$$\sum_{n=n_\varepsilon+1}^{\infty} \frac{1}{x^n} \le \sum_{n=0}^{\infty} \frac{1}{x^n} = \frac{x}{x-1}.$$

By taking the limit as x goes to 1, we thus get

$$\lim_{x\to 1^+}\Big|(x-1)\,\mathcal{Z}[\{y_n\}](x)\Big|\le\varepsilon.$$

By arbitrariness of $\varepsilon>0$, we get the desired conclusion. □

Remark 2.4.5. If we remove the assumption (2.4.2), Theorem 2.4.3 does not hold anymore. Indeed, if we take the sequence

$$x_n=(-1)^n,$$

we get

$$\mathcal{Z}[\{x_n\}](z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{z^n}=\sum_{n=0}^{\infty}\left(\frac{-1}{z}\right)^n=\frac{1}{1-\left(-\dfrac{1}{z}\right)}=\frac{z}{z+1}.$$

Thus we have

$$\lim_{\mathbb{R}\ni x\to 1^+}(1-x)\,\mathcal{Z}[\{x_n\}](x)=0.$$

On the other hand, the sequence $\{x_n\}_{n\in\mathbb{N}}$ does not converge.

Remark 2.4.6 (Periodic sequences and the Final Value Theorem)**.** Let us consider a $m-$periodic sequence $\{x_n\}_{n\in\mathbb{N}}\subset\mathbb{C}$. Thus we have

$$x_{n+m}=x_n,\qquad \text{for every } n\in\mathbb{N}.$$

Such a sequence can not converge, unless it is constant. Thus in this case we can not apply Theorem 2.4.3. However, by recalling the formula (2.2.5) for its $Z-$transform, we have

$$\mathcal{Z}[\{x_n\}](z)=\frac{z^m}{z^m-1}\sum_{n=0}^{m-1}\frac{x_n}{z^n},\qquad \text{for } |z|>1.$$

In particular, by evaluating this function for $\mathbb{R}\ni x>1$, multiplying by the factor $(x-1)$ and taking the limit, we get

$$\lim_{\mathbb{R}\ni x\to 1^+}(x-1)\,\mathcal{Z}[\{x_n\}](z)=\lim_{\mathbb{R}\ni x\to 1^+}\frac{x^m\,(x-1)}{x^m-1}\sum_{n=0}^{m-1}\frac{x_n}{z^n}=\frac{1}{m}\sum_{n=0}^{m-1}x_n.$$

Observe that we used that

$$\lim_{x\to 1}\frac{x^m-1}{x-1}=m.$$

In other words, for a $m-$periodic sequence, we get

$$\lim_{\mathbb{R}\ni x\to 1^+}(x-1)\,\mathcal{Z}[\{x_n\}](z)=\frac{1}{m}\sum_{n=0}^{m-1}x_n,$$

and observe that the sum on the right-hand side is the *average* of the values assumed by the periodic sequence $\{x_n\}_{n\in\mathbb{N}}$.

5. Relations with Fourier series expansions

Let $\{x_n\}_{n\mathbb{N}} \subset \mathbb{C}$ be a $Z-$transformable sequence, with

$$\limsup_{n\to\infty} \sqrt[n]{|x_n|} = R < +\infty.$$

We have seen that its $Z-$transform is holomorphic in $|z| > R$. In particular, for every $\varrho > R$ the following function of one real variable is well-defined

$$f(t) := \mathcal{Z}[\{x_n\}](\varrho\, e^{i\,t}), \qquad t \in [0, 2\,\pi],$$

and can be periodically extended to the whole $\mathbb{R}$. By definition of $Z-$transform, this is nothing but

$$f(t) = \sum_{n=0}^{\infty} \frac{x_n}{\varrho^n}\, e^{-i\,n\,t}, \qquad t \in [0, 2\,\pi]. \tag{2.5.1}$$

On the other hand, by appealing to the inversion formula (2.3.1) and taking $\gamma = \varrho\, e^{i\,t}$ for $t \in [0, 2\,\pi]$, we have

$$\begin{aligned} x_k &= \frac{1}{2\,\pi\,i} \int_{\partial B_\varrho(0)} \mathcal{Z}[\{x_n\}](z)\, z^{k-1}\, dz \\ &= \frac{1}{2\,\pi\,i} \int_0^{2\,\pi} \mathcal{Z}[\{x_n\}](\varrho\, e^{i\,t})\, \varrho^{k-1}\, e^{i\,(k-1)\,t}\, i\, \varrho\, e^{i\,t}\, dt \\ &= \frac{\varrho^k}{2\,\pi} \int_0^{2\,\pi} f(t)\, e^{i\,k\,t}\, dt. \end{aligned}$$

By inserting this information in (2.5.1), we finally obtain

$$f(t) = \sum_{n=0}^{\infty} \left(\frac{1}{2\,\pi} \int_0^{2\,\pi} f(t)\, e^{i\,n\,t}\, dt \right) e^{-i\,n\,t}, \qquad t \in [0, 2\,\pi], \tag{2.5.2}$$

which is the *Fourier expansion of the periodic function*[3] f. We recall that the latter is the restriction of the $Z-$transform on the circle $\partial B_\varrho(0)$.

6. Applications to signal processing

The main application of the $Z-$transform is in signal processing. It can be used to solve *finite difference linear equations*. These are important since they provide an approximation to solve numerically *ordinary differential equations*. We try to explain the idea with a simple example.

Example 2.6.1 (First order finite differences). Let us consider the linear ordinary differential equation with constant coefficients

$$\begin{cases} y'(t) + a\, y(t) &= b(t), \quad t \geq 0, \\ y(0) &= y_0 \end{cases}$$

[3] We recall that the Fourier expansion of a $(2\,\pi)-$periodic function g is given by

$$g(t) = \sum_{n\in\mathbb{Z}} \widehat{g}(n)\, e^{i\,n\,t}, \quad t \in [0, 2\,\pi], \qquad \text{with } \widehat{g}(n) = \frac{1}{2\,\pi} \int_0^{2\,\pi} g(t)\, e^{-i\,n\,t}\, dt,$$

see the Appendix C. Then formula (2.5.2) implies that in this particular case $\widehat{g}(n) = 0$ for $n \geq 1$.

The idea of the finite difference method is to replace derivatives by incremental ratios, since by Taylor formula

$$y'(t) = \frac{y(t+h) - y(t)}{h} + o(1), \qquad \text{for } 0 < h \ll 1,$$

and discretize the problem. In a nutshell, we fix a time step $\tau > 0$ and look at values of the functions on the regular grid $\{0, \tau, 2\tau, \ldots, n\tau, \ldots\}$. By setting

$$x_n = y(n\,\tau), \qquad b_n = b(n\,\tau), \qquad n \in \mathbb{N},$$

the initial first order differential equation is replaced by

$$\begin{cases} \dfrac{x_{n+1} - x_n}{\tau} + a\,x_n &= b_n, \quad n \in \mathbb{N}, \\ x_0 &= y_0 \end{cases}$$

that is

$$\begin{cases} x_{n+1} &= \tau\, b_n + (1 - a\,\tau)\, x_n, \quad n \in \mathbb{N}, \\ x_0 &= y_0 \end{cases} \tag{2.6.1}$$

The unknown of the problem is now the sequence $\{x_n\}_{n\in\mathbb{N}}$, which by construction is a regular sampling of the values of the original solution y (more precisely, a regular sampling of the values *of an approximation of* y). If we are able to determine these coefficients, then the graph of an approximate solution of the original problem can be obtained by interpolating the points

$$(0, x_0), \quad (\tau, x_1), \quad \ldots, \quad (n\,\tau, x_n), \quad \ldots$$

In order to solve (2.6.1), we can employ the $Z-$transform. Indeed, from (2.6.1), by linearity of the $Z-$transform we get

$$\mathcal{Z}[\{x_{n+1}\}](z) = \tau\,\mathcal{Z}[\{b_n\}](z) + (1 - a\,\tau)\,\mathcal{Z}[\{x_n\}](z).$$

By recalling the translation relation (2.2.1) and taking into account the initial condition, the previous identity becomes

$$z\,\mathcal{Z}[\{x_n\}](z) = z\,y_0 + \tau\,\mathcal{Z}[\{b_n\}](z) + (1 - a\,\tau)\,\mathcal{Z}[\{x_n\}](z),$$

that is

$$\mathcal{Z}[\{x_n\}](z) = \frac{z\,y_0}{z - 1 + a\,\tau} + \frac{\tau}{z - 1 + a\,\tau}\,\mathcal{Z}[\{b_n\}](z).$$

Thus we found the explicit expression of the $Z-$transform. In order to find the coefficients $\{x_n\}_{n\in\mathbb{N}}$, we now have to use the inversion formula (2.3.1).

Example 2.6.2 (Second order finite differences)**.** In the case of

$$\begin{cases} y''(t) + c\,y'(t) + a\,y(t) &= b(t), \quad t \geq 0, \\ y(0) &= y_0 \\ y'(0) &= y_1 \end{cases}$$

we can discretize this problem by observing that

$$\begin{aligned} y''(t) \simeq \frac{y'(t+h) - y'(t)}{h} &\simeq \frac{\dfrac{y(t+2\,h) - y(t+h)}{h} - \dfrac{y(t+h) - y(t)}{h}}{h} \\ &= \frac{y(t+2\,h) - 2\,y(t+h) + y(t)}{h^2}, \end{aligned}$$

and

$$y'(0) \simeq \frac{y(h) - y(0)}{h}.$$

By introducing the time step $\tau > 0$ and setting as before

$$x_n = y(n\,\tau) \qquad \text{and} \qquad b_n = b(n\,\tau),$$

the initial problem can be approximated by

$$\begin{cases} \dfrac{x_{n+2} - 2\,x_{n+1} + x_n}{\tau^2} + c\,\dfrac{x_{n+1} - x_n}{\tau} + a\,x_n &= b_n, \quad n \in \mathbb{N}, \\ x_0 &= y_0 \\ \dfrac{x_1 - x_0}{\tau} &= y_1 \end{cases}$$

This can be also rewritten as

$$\text{(2.6.2)} \qquad \begin{cases} x_{n+2} &= (2 - c\,\tau)\,x_{n+1} + (c\,\tau - 1 - \tau^2\,a)\,x_n + \tau^2\,b_n, \quad n \in \mathbb{N}, \\ x_0 &= y_0 \\ x_1 &= y_0 + \tau\,y_1 \end{cases}$$

In this case as well, one could employ the $Z-$transform in order to solve this initial value problem for the second order finite differences equation.

Let $f : \mathbb{R} \to \mathbb{C}$ be a *causal signal*, i.e. a function such that $f(t) \equiv 0$ for $t < 0$. If we fix a time step $\tau > 0$, we can consider its regular sampling

$$\{f(n\,\tau)\}_{n\in\mathbb{N}} \subset \mathbb{C}.$$

Definition 2.6.3. Let $\tau > 0$, we say that f is *$Z-$transformable with time step τ* if the sequence $\{f(n\,\tau)\}_{n\in\mathbb{N}}$ is $Z-$transformable, i.e. if

$$\limsup_{n\to\infty} \sqrt[n]{|f(n\,\tau)|} < +\infty.$$

Then we call $\mathcal{Z}[\{f(n\,\tau)\}]$ the *$Z-$transform of f with time step τ.*

Remark 2.6.4 (A sufficient condition for transformability)**.** It is easy to see that if the signal f has exponential growth, then it is $Z-$transformable with every time step $\tau > 0$. More precisely, if

$$|f(t)| \leq C\,e^{\alpha\,t},$$

for some $C > 0$ and $\alpha \geq 0$, then it is $Z-$transformable. Indeed, in this case for every time step $\tau > 0$ we have

$$\limsup_{n\to\infty} \sqrt[n]{|f(n\,\tau)|} \leq e^{\alpha\,\tau} \limsup_{n\to\infty} \sqrt[n]{C} = e^{\alpha\,\tau}.$$

Observe that this also gives the following estimate for the radius of convergence

$$R \leq e^{\alpha\,\tau}.$$

Example 2.6.5 (Heaviside step function)**.** Let $H(t)$ be the *Heaviside step function*, defined by

$$H(t) = \begin{cases} 1, & \text{for } t \geq 0, \\ 0, & \text{for } t < 0. \end{cases}$$

For every given time step τ we have $H(n\,\tau) = 1$. Thus it is $Z-$transformable and we have

$$\mathcal{Z}[H(n\,\tau)](z) = \mathcal{Z}[\{1\}](z) = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1 - \dfrac{1}{z}} = \frac{z}{z-1}, \qquad |z| > 1.$$

Example 2.6.6 (Ramp function)**.** Let $t \mapsto R(t)$ be the *ramp function*, defined by

$$R(t) = t\,H(t) = \begin{cases} t, & \text{for } t \geq 0, \\ 0, & \text{for } t < 0. \end{cases}$$

Given the time step $\tau > 0$ we have $R(n\,\tau) = n\,\tau$. Thus it is $Z-$transformable with every time step $\tau > 0$ and we have

$$\mathcal{Z}[\{R(n\,\tau)\}](z) = \mathcal{Z}[\{\tau\,n\}](z) = \tau\,\mathcal{Z}[\{n\}](z).$$

In order to compute the last transform, we observe that by using (2.2.3), we have

$$\mathcal{Z}[\{n\}](z) = \mathcal{Z}[\{n \cdot 1\}](z) = -z\,\frac{d}{dz}\mathcal{Z}[\{1\}](z).$$

In conclusion, for the ramp function we get

$$\mathcal{Z}[\{R(n\,\tau)\}](z) = \frac{\tau\,z}{(z-1)^2}, \qquad \text{for } |z| > 1.$$

Example 2.6.7 (Periodic signals)**.** Let $f : \mathbb{R} \to \mathbb{C}$ be a *positively periodic causal signal*, with period $T > 0$. In other words, we have

$$f(t+T) = f(t), \quad \text{for } t \geq 0.$$

We fix $m \in \mathbb{N} \setminus \{0\}$ and take the time step $\tau = T/m$. Then the regular sampling $\{f(n\,\tau)\}_{n\in\mathbb{N}}$ is a periodic sequence, with period m. Indeed, we have

$$f(n\,\tau + m\,\tau) = f(n\,\tau + T) = f(n\,\tau), \qquad \text{for every } n \in \mathbb{N}.$$

By using (2.2.5), we thus get

$$\mathcal{Z}[\{f(n\,\tau)\}](z) = \frac{z^m}{z^m - 1}\sum_{n=0}^{m-1}\frac{f(n\,\tau)}{z^n}.$$

7. Exercises

Exercise 2.7.1. *Compute the $Z-$transform of the sequence $\{n\}_{n\in\mathbb{N}}$.*

Solution. We first observe that the sequence is $Z-$transformable, since

$$\lim_{n\to\infty} \sqrt[n]{n} = 1.$$

Thus $\mathcal{Z}[\{n\}]$ is an analytic function on $\{z \in \mathbb{C} : |z| > 1\}$, defined by

$$\mathcal{Z}[\{n\}](z) = \sum_{n=0}^{\infty} n\,z^{-n}.$$

By (2.2.3), we know that

$$\mathcal{Z}[\{n\}](z) = \mathcal{Z}[\{n \cdot 1\}](z) = -z\,\frac{d}{dz}\mathcal{Z}[\{1\}](z),$$

thus we only need to compute the $Z-$transform of the constant sequence $x_n = 1$. We have

$$\mathcal{Z}[\{1\}](z) = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}} = \frac{z}{z-1}. \tag{2.7.1}$$

We thus obtain

$$\mathcal{Z}[\{n\}](z) = -z\,\frac{d}{dz}\frac{z}{z-1} = \frac{z}{(z-1)^2}, \qquad |z| > 1. \tag{2.7.2}$$

This concludes the exercise. □

Exercise 2.7.2. *Let us consider the two sequences*

$$x_n = n \qquad \text{and} \qquad y_n = 1, \qquad \text{for every } n \in \mathbb{N}.$$

Compute the convolution $\{x_n * y_n\}_{n\in\mathbb{N}}$ *and its* $Z-$*transform.*

Solution. By definition of convolution, we have

$$x_n * y_n = \sum_{k=0}^{n} x_k\, y_{n-k} = \sum_{k=0}^{n} k = \frac{n\,(n+1)}{2}.$$

By using Proposition 2.2.4, we then obtain

$$\mathcal{Z}\left[\left\{\frac{n\,(n+1)}{2}\right\}\right](z) = \mathcal{Z}[\{n\}](z)\,\mathcal{Z}[\{1\}](z).$$

By recalling that (see previous exercise)

$$\mathcal{Z}[\{n\}](z) = \frac{z}{(z-1)^2}, \qquad |z| > 1,$$

and (see Example 2.1.4)

$$\mathcal{Z}[\{1\}](z) = \frac{z}{z-1}, \qquad |z| > 1,$$

we finally obtain

$$\mathcal{Z}\left[\left\{\frac{n\,(n+1)}{2}\right\}\right](z) = \frac{z^2}{(z-1)^3}, \qquad |z| > 1,$$

thus concluding the exercise. □

Exercise 2.7.3. *Compute the* $Z-$*transform of the sequence* $\{n^2\}_{n\in\mathbb{N}}$.

Solution. We first observe that the sequence is $Z-$transformable, since

$$\lim_{n\to\infty} \sqrt[n]{n^2} = 1.$$

Thus $\mathcal{Z}[\{n\}]$ is an analytic function on $\{z \in \mathbb{C} \,:\, |z| > 1\}$. We can proceed as in the previous exercise, by exploiting (2.2.3). Indeed, we have

$$\mathcal{Z}[\{n^2\}](z) = \mathcal{Z}[\{n\cdot n\}](z) = -z\,\frac{d}{dz}\mathcal{Z}[\{n\}](z),$$

thus we only need to use previous Exercise to compute the $Z-$transform of $\{n\}_{n\in\mathbb{N}}$. We thus obtain

$$\mathcal{Z}[\{n^2\}](z) = -z\,\frac{d}{dz}\,\frac{z}{(z-1)^2} = \frac{z\,(1+z)}{(z-1)^3}, \qquad |z| > 1.$$

This concludes the exercise. □

Exercise 2.7.4. *Let us consider the causal signal* $f(t) = \cos(t)\,H(t)$. *Given a time step* $\tau > 0$, *let us compute the* $Z-$*transform of* f *with time step* τ.

Solution. We first observe that the sequence

$$\{f(n\,\tau)\}_{n\in\mathbb{N}} = \{\cos(n\,\tau)\}_{n\in\mathbb{N}},$$

is bounded, thus by Remark 2.1.3 it is $Z-$transformable and the $Z-$transform is well-defined for $|z| > 1$. In order to compute the $Z-$transform, we observe that

$$\cos(n\,\tau) = \frac{e^{i\,n\,\tau} + e^{-i\,n\,\tau}}{2},$$

thus by linearity of the $Z-$transform we get

$$\mathcal{Z}[\{\cos(n\,\tau)\}](z) = \frac{1}{2}\,\mathcal{Z}[\{e^{i\,n\,\tau}\}](z) + \frac{1}{2}\,\mathcal{Z}[\{e^{-i\,n\,\tau}\}](z).$$

By recalling Example 2.2.7, we get

$$\mathcal{Z}[\{e^{i\,n\,\tau}\}](z) = \frac{z}{z - e^{i\,\tau}} \qquad \text{and} \qquad \mathcal{Z}[\{e^{-i\,n\,\tau}\}](z) = \frac{z}{z - e^{-i\,\tau}}.$$

Thus we obtain

$$\mathcal{Z}[\{\cos(n\,\tau)\}](z) = \frac{1}{2}\,\frac{z}{z - e^{i\,\tau}} + \frac{1}{2}\,\frac{z}{z - e^{-i\,\tau}} = \frac{1}{2}\,\frac{2\,z^2 - z\,(e^{i\,\tau} + e^{-i\,\tau})}{(z - e^{i\,\tau})\,(z - e^{-i\,\tau})}.$$

With simple manipulations, we finally obtain

$$\mathcal{Z}[\{\cos(n\,\tau)\}](z) = \frac{z\,(z - \cos\tau)}{z^2 - 2\,z\,\cos\tau + 1}. \tag{2.7.3}$$

We observe that

$$\lim_{\tau\to 0^+} \cos(n\,\tau) = 1 \qquad \text{and} \qquad \lim_{\tau\to 0^+} \frac{z\,(z - \cos\tau)}{z^2 - 2\,z\,\cos\tau + 1} = \frac{z}{z-1},$$

which agrees with (2.7.1). □

Remark 2.7.5. We point out that even if the causal signal $f(t) = \cos t\,H(t)$ is positively periodic, with period $2\,\pi$, its regular sampling

$$f(n\,\tau),$$

in general *is not* periodic. This is the case if we take the time step $\tau = 2\,\pi/k$, then the sequence $\{f(n\,\tau)\}_{n\in\mathbb{N}}$ is $k-$periodic. Thus from (2.2.5) we would get

$$\mathcal{Z}\left[\left\{f\left(n\,\frac{2\,\pi}{k}\right)\right\}\right](z) = \frac{z^k}{z^k - 1}\sum_{n=0}^{k-1}\cos\left(n\,\frac{2\,\pi}{k}\right)\,z^{-n}, \qquad |z| > 1.$$

By comparing (2.7.3) and the previous expression, we get in particular (for $k \geq 1$)

$$\sum_{n=0}^{k-1}\cos\left(n\,\frac{2\,\pi}{k}\right)\,z^{-n} = \frac{z^k - 1}{z^{k-1}}\,\frac{z - \cos\left(\frac{2\,\pi}{k}\right)}{z^2 - 2\,z\,\cos\left(\frac{2\,\pi}{k}\right) + 1}, \qquad |z| > 1.$$

Observe that for $k \geq 2$, we can take the limit $z \to 1$ on both sides and obtain the well-known relation

$$\sum_{n=0}^{k-1}\cos\left(n\,\frac{2\,\pi}{k}\right) = 0.$$

Exercise 2.7.6. *By using the $Z-$transform, determine the sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ inductively defined by*

$$\begin{cases} x_{n+2} &= 2\,x_{n+1} - x_n, \\ x_0 &= 0, \\ x_1 &= 1. \end{cases}$$

Solution. We introduce the $Z-$transform

$$X(z)=\sum_{n=0}^{\infty}x_n\,z^{-n},$$

then from the relation defining $\{x_n\}_{n\in\mathbb{N}}$ and by using property (2.2.1) of the $Z-$transform, we get the relation

$$z^2\,X(z)-z=2\,z\,X(z)-X(z),$$

that is

$$X(z)\,[z^2-2\,z+1]=z.$$

This finally gives

$$X(z)=\frac{z}{(z-1)^2},$$

which is an analytic function on $\mathbb{C}\setminus\{1\}$, with a pole of order 2 in $z=1$. From formulas (2.3.2) and (2.3.3), for $n\geq 1$ we get

$$\begin{aligned}x_n=\frac{1}{2\,\pi\,i}\int_{\gamma}X(z)\,z^{n-1}\,dz&=\frac{1}{2\,\pi\,i}\int_{\partial B_r(0)}\frac{z^n}{(z-1)^2}\,dz\\&=\mathrm{res}\left(\frac{z^n}{(z-1)^2},1\right)\\&=\lim_{z\to 1}\frac{d}{dz}\left((z-1)^2\,\frac{z^n}{(z-1)^2}\right)=n.\end{aligned}$$

We used the formula of Proposition 1.10.11, in order to compute the residue. This finally gives $x_n=n$ for $n\geq 1$. □

Exercise 2.7.7. *By using the $Z-$transform, determine the sequence $\{x_n\}_{n\in\mathbb{N}}\subset\mathbb{C}$ inductively defined by*

$$\left\{\begin{array}{rcl}x_{n+2}&=&-x_n,\\x_0&=&0,\\x_1&=&2.\end{array}\right.$$

Solution. We introduce the $Z-$transform

$$X(z)=\sum_{n=0}^{\infty}x_n\,z^{-n}$$

then from the relation defining $\{x_n\}_{n\in\mathbb{N}}$ and by using property (2.2.1) of the $Z-$transform, we get the relation

$$z^2\,X(z)-2\,z=-X(z),$$

that is

$$X(z)\,[z^2+1]=2\,z.$$

This finally gives

$$X(z)=\frac{2\,z}{z^2+1},$$

which is an analytic function on $\mathbb{C} \setminus \{-i, i\}$, with two simple poles in $s = \pm i$. From the inversion formula and Remark 2.3.2, for $n \geq 2$ we get

$$\begin{aligned} x_n &= \frac{1}{2\,\pi\,i} \int_\gamma X(z)\, z^{n-1}\, dz = \frac{1}{2\,\pi\,i} \int_\gamma \frac{2\,z^n}{z^2+1}\, dz \\ &= \mathrm{res}\left(\frac{2\,z^n}{z^2+1}, i\right) + \mathrm{res}\left(\frac{2\,z^n}{z^2+1}, -i\right) \\ &= [1 + (-1)^{n+1}]\, i^{n-1}. \end{aligned}$$

We used the formula of Corollary 1.10.13, in order to compute the residue. This finally gives

$$x_n = \begin{cases} 0, & \text{if } n \text{ is even}, \\ 2, & \text{if } n = 2\,k+1 \text{ with } k \text{ even}, \\ -2, & \text{if } n = 2\,k+1 \text{ with } k \text{ odd}, \end{cases}$$

thus concluding the exercise. □

8. Advanced exercises

Exercise 2.8.1. *Let us suppose that $\sum_{n=0}^\infty x_n$ and $\sum_{n=0}^\infty y_n$ are absolutely convergent. Then*

$$\sum_{n=0}^{\infty} |x_n * y_n| < +\infty, \tag{2.8.1}$$

and we have

$$\sum_{n=0}^{\infty} x_n * y_n = \left(\sum_{n=0}^{\infty} x_n\right)\left(\sum_{n=0}^{\infty} y_n\right), \tag{2.8.2}$$

Solution. By hypothesis, we have

$$\sum_{n=0}^{\infty} |x_n| < +\infty \qquad \text{and} \qquad \sum_{n=0}^{\infty} |y_n| < +\infty.$$

This means that the sequences of the associated partial sums are converging, i. e.

$$\lim_{M\to\infty} \sum_{n=0}^{M} |x_n| < +\infty \qquad \text{and} \qquad \lim_{M\to\infty} \sum_{n=0}^{M} |y_n| < +\infty.$$

We first prove (2.8.1). We fix $M \in \mathbb{N}$ and consider the partial sum

$$\sum_{n=0}^{M} |x_n * y_n| = \sum_{n=0}^{M} \left|\sum_{k=0}^{n} x_k\, y_{n-k}\right| \leq \sum_{n=0}^{M}\sum_{k=0}^{n} |x_k|\,|y_{n_k}|. \tag{2.8.3}$$

Now, we would like to exchange the order of the two sums above. At this aim, we observe that the set of indices n, k in the sums can be rewritten as

$$\{(n,k) \in \mathbb{N}\times\mathbb{N} \,:\, 0 \leq n \leq M \text{ and } 0 \leq k \leq n\} = \{(n,k) \in \mathbb{N}\times\mathbb{N} \,:\, 0 \leq k \leq M \text{ and } k \leq n \leq M\}.$$

This implies that

$$\sum_{n=0}^{M}\sum_{k=0}^{n} |x_k|\,|y_{n_k}| = \sum_{k=0}^{M}\sum_{n=k}^{M} |x_k|\,|y_{n_k}| = \sum_{k=0}^{M} |x_k| \left(\sum_{n=k}^{M} |y_{n-k}|\right).$$

In the last sum, we make the change of index $m = n - k$, thus from (2.8.3) we obtain

$$\sum_{n=0}^{M} |x_n * y_n| \leq \sum_{k=0}^{M} |x_k| \left(\sum_{m=0}^{M-k} |y_m|\right) \leq \sum_{k=0}^{M} |x_k| \left(\sum_{m=0}^{M} |y_m|\right) = \left(\sum_{k=0}^{M} |x_k|\right) \left(\sum_{m=0}^{M-k} |y_m|\right). \tag{2.8.4}$$

Observe that $|x_n * y_n|$ is positive, thus the sequence

$$s_M = \sum_{n=0}^{M} |x_n * y_n|,$$

is monotone increasing and it admits limit. By estimate (2.8.4), we get that such a limit is finite, i.e. we proved (2.8.1).

The proof of (2.8.2) can be accomplished by using the same trick of exchanging the order of the sums, we leave the details to the reader (see also [2, Teorema 14.15]). □

Exercise 2.8.2. *Let us consider the Cauchy problem*

$$\begin{cases} y''(t) + y(t) &= 0, \\ y(0) &= 0, \\ y'(0) &= 1. \end{cases}$$

Discretize the problem and find an approximate solution, with the aid of the $Z-$transform.

Solution. Let y be the solution of the Cauchy problem (we consider it to be equal to 0 for $t < 0$), we fix a time step $0 < \tau$ and consider the regular sampling

$$\{x_n\}_{n\in\mathbb{N}} := \{y(n\,\tau)\}_{n\in\mathbb{N}}.$$

From formula (2.6.2) with

$$y_0 = 0, \qquad y_1 = 1, \qquad c = 0 \qquad b = 0 \qquad \text{and} \qquad a = 1,$$

we get

$$\begin{cases} x_{n+2} &= 2\,x_{n+1} - (1 + \tau^2)\,x_n, \quad n \in \mathbb{N}, \\ x_0 &= 0 \\ x_1 &= \tau \end{cases}$$

We introduce the $Z-$transform

$$X(z) = \sum_{n=0}^{\infty} x_n\,z^{-n},$$

by using property (2.2.1) of the $Z-$transform, we get the relation

$$z^2 \left(X(z) - \frac{\tau}{z}\right) = 2\,z\,X(z) - (1 + \tau^2)\,X(z).$$

With some manipulations, we get

$$X(z) = \frac{\tau\,z}{z^2 - 2\,z + 1 + \tau^2}.$$

We observe that X is an olomorphic function on $\mathbb{C} \setminus \{s_1,\, s_2\}$, where

$$s_1 = 1 + \tau\,i \qquad \text{and} \qquad s_2 = 1 - \tau\,i.$$

The function X has two simple poles at this points. By using the inversion formula (2.3.1) and Remark 2.3.2, we obtain

$$\begin{aligned} x_n &= \frac{1}{2\,\pi\,i}\int_\gamma X(z)\,z^{n-1}\,ds = \frac{1}{2\,\pi\,i}\int_\gamma \frac{\tau\,z^n}{z^2-2\,z+1+\tau^2}\,dz \\ &= \mathrm{res}\left(\frac{\tau\,z^n}{z^2-2\,z+1+\tau^2}, s_1\right) + \mathrm{res}\left(\frac{\tau\,z^n}{z^2-2\,z+1+\tau^2}, s_2\right) \\ &= \frac{\tau}{2}\left(\frac{s_1^n}{s_1-1}+\frac{s_2^n}{s_2-1}\right) \\ &= \frac{\tau}{2}\left[\frac{(1+\tau\,i)^n}{\tau\,i}-\frac{(1-\tau\,i)^n}{\tau\,i}\right] \\ &= \frac{(1+\tau\,i)^n-(1-\tau\,i)^n}{2\,i}. \end{aligned}$$

Observe that we used the formula of Corollary 1.10.13, in order to compute the residue. Notice that for $n=0$ and $n=1$ we are back with

$$x_0 = 0 \qquad \text{and} \qquad x_1 = \tau.$$

By using Newton's formula, we get for $n \geq 2$

$$\begin{aligned} \frac{(1+\tau\,i)^n-(1-\tau\,i)^n}{2\,i} &= \frac{1}{2\,i}\left[\sum_{k=0}^n \binom{n}{k}\tau^k\,i^k - \sum_{k=0}^n \binom{n}{k}(-1)^k\,\tau^k\,i^k\right] \\ &= \sum_{k=1}^n \binom{n}{k}\left[\frac{1-(-1)^k}{2}\right]\tau^k\,i^{k-1}. \end{aligned}$$

We observe that $1-(-1)^k=0$ for k even and $i^{k-1}\in\mathbb{R}$ for k odd. Thus $x_n\in\mathbb{R}$ and we have

$$x_n = \sum_{k=1}^n \binom{n}{k}\left[\frac{1-(-1)^k}{2}\right]\tau^k\,i^{k-1}$$

Let us compute the first terms

$$x_0 = 0, \quad x_1 = \tau, \quad x_2 = 2\,\tau, \quad x_3 = 3\,\tau-\tau^3, \quad x_4 = 4\,\tau - 4\,\tau^3$$
$$x_5 = 5\,\tau - 10\,\tau^3 + \tau^5, \quad x_6 = 6\,\tau - 20\,\tau^3 + 6\,\tau^5.$$

see the figure below. □

Exercise 2.8.3. *Let* $\{b_n\}_{n\in\mathbb{N}}\subset\mathbb{C}$ *be a given sequence and let* $\{\delta_{0,n}\}_{n\in\mathbb{N}}$ *be the* Delta sequence centered at 0*, i.e.*

$$\delta_{0,n} = \begin{cases} 1, & \text{if } n=0, \\ 0, & \text{if } n\geq 1. \end{cases}$$

Prove that if $\{y_n\}_{n\in\mathbb{N}}$ *solves*

$$\text{(2.8.5)} \qquad \begin{cases} y_{n+2}+A\,y_{n+1}+B\,y_n &= \delta_{0,n}, \\ y_0 &= 0, \\ y_1 &= 0, \end{cases}$$

then the convolution $\{b_n * y_n\}_{n\in\mathbb{N}}$ *solves*

$$\text{(2.8.6)} \qquad \begin{cases} x_{n+2}+A\,x_{n+1}+B\,x_n &= b_n, \\ x_0 &= 0, \\ x_1 &= 0. \end{cases}$$

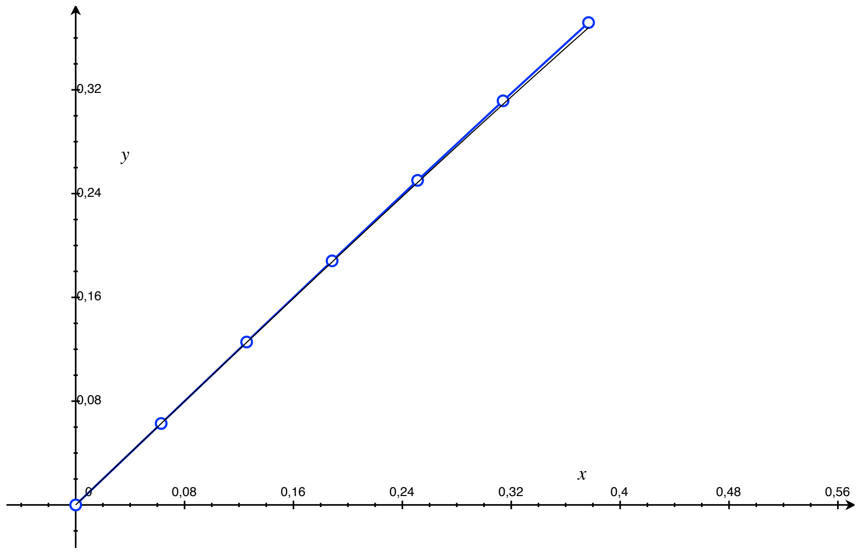

Figure 1. The blue line corresponds to the linear interpolation of the first 7 terms of the sequence $\{(n\,\tau,\, y(n\,\tau))\}_{n\in\mathbb{N}}$ computed in Exercise 2.8.2, with time step $\tau = (2\,\pi)/100$. The black line corresponds to the graph of the exact solution of the ODE, i.e. $y(t) = \sin t$.

Solution. We observe that if $\{y_n\}_{n\in\mathbb{N}}$ is solution of (2.8.5), then its $Z-$transform satisfies

$$z^2\,\mathcal{Z}[\{y_n\}](z) + A\,z\,\mathcal{Z}[\{y_n\}](z) + B\,\mathcal{Z}[\{y_n\}](z) = 1,$$

where we also used that (see Example 2.1.5)

$$\mathcal{Z}[\{\delta_{0,n}\}](z) = 1.$$

In other words, we find

$$\mathcal{Z}[\{y_n\}](z) = \frac{1}{z^2 + A\,z + B}.$$

By using Proposition 2.2.4, we have

$$\mathcal{Z}[\{b_n * y_n\}](z) = \mathcal{Z}[\{b_n\}](z)\,\mathcal{Z}[\{y_n\}](z) = \frac{\mathcal{Z}[\{b_n\}](z)}{z^2 + A\,z + B}. \tag{2.8.7}$$

On the other hand, if $\{x_n\}_{n\in\mathbb{N}}$ solves (2.8.6), then its $Z-$transform must satisfy

$$z^2\,\mathcal{Z}[\{x_n\}](z) + A\,z\,\mathcal{Z}[\{x_n\}](z) + B\,\mathcal{Z}[\{x_n\}](z) = \mathcal{Z}[\{b_n\}](z),$$

that is

$$\mathcal{Z}[\{x_n\}](z) = \frac{\mathcal{Z}[\{b_n\}](z)}{z^2 + A\,z + B}.$$

By comparing this with (2.8.7), we get

$$\mathcal{Z}[\{x_n\}](z) = \mathcal{Z}[\{b_n * y_n\}](z).$$

Thus we get the desired conclusion by Proposition 2.3.1. $\square$

Remark 2.8.4. The sequence $\{y_n\}_{n\in\mathbb{N}}$ in the previous exercise is called *impulse response* for the problem (2.8.6). Observe that it can be explicitly determined, in terms of the coefficients A, B. Indeed, we have

$$\mathcal{Z}[\{y_n\}](z) = \frac{1}{z^2 + A\,z + B},$$

thus by the inversion formula

$$y_n = \frac{1}{2\,\pi\,i} \int_\gamma \frac{z^{n-1}}{z^2 + A\,z + B}\,dz, \qquad \text{for } n \geq 1.$$

Here γ is a positively oriented circle, entouring the two singularities

$$z_0 = \frac{-A - \sqrt{A^2 - 4\,B}}{2} \qquad \text{and} \qquad z_1 = \frac{-A + \sqrt{A^2 - 4\,B}}{2}.$$

Thus, as always, the integral above can be computed by using the Residue Theorem. However, we have to distinguish two cases:

- if $A^2 \neq 4\,B$, then $z_0 \neq z_1$ are two simple poles. Accordingly, we get

$$\begin{aligned} y_n &= \operatorname{res}\left(\frac{z^{n-1}}{z^2 + A\,z + B}, z_0\right) + \operatorname{res}\left(\frac{z^{n-1}}{z^2 + A\,z + B}, z_1\right) \\ &= \frac{z_0^{n-1}}{2\,z_0 + A} + \frac{z_1^{n-1}}{2\,z_1 + A}, \end{aligned}$$

  where we used Corollary 1.10.13, in order to compute the residues;

- if $A^2 = 4\,B$, then $z_0 = z_1 = -A/2$ and this is a pole with multiplicity 2. The integrand now rewrites

$$\frac{z^{n-1}}{z^2 + A\,z + B} = \frac{z^{n-1}}{\left(z + \dfrac{A}{2}\right)^2}.$$

  Accordingly, we get

$$\begin{aligned} y_n &= \operatorname{res}\left(\frac{z^{n-1}}{\left(z + \dfrac{A}{2}\right)^2}, -\frac{A}{2}\right) \\ &= \lim_{z \to -\frac{A}{2}} \frac{d}{dz}\left(\left(z + \frac{A}{2}\right)^2 \frac{z^{n-1}}{\left(z + \dfrac{A}{2}\right)^2}\right) = (n-1)\left(-\frac{A}{2}\right)^{n-2}. \end{aligned}$$

Exercise 2.8.5 (Bessel's equation of order 0)**.** *Find a solution of the following Cauchy problem*

$$\begin{cases} y''(t) + \dfrac{1}{t}\,y'(t) + y(t) &= 0, \quad \textit{for } t \geq 0, \\ y(0) &= 1, \\ y'(0) &= 0. \end{cases}$$

Solution. We look for a solution which can be written as a power series centered at 0, i.e.

$$y(t) = \sum_{k=0}^{\infty} x_k\,t^k, \qquad \text{with } x_0 = y(0) = 1,\ x_1 = y'(0) = 0. \tag{2.8.8}$$

By Corollary 1.7.6, we know that such a function can be differentiated infinitely many times for

$$|t| < R, \qquad \text{where } R = \frac{1}{\limsup_{k\to\infty} \sqrt[k]{|x_k|}}.$$

We now proceed formally to identify the coefficients x_k and then compute the radius of convergence and justify *a posteriori* the computations we will make.

By inserting (2.8.8) in the equation, we get

$$\begin{aligned} y''(t) + \frac{1}{t}\,y'(t) + y(t) &= \sum_{k=2}^{\infty} x_k\,k\,(k-1)\,t^{k-2} + \sum_{k=1}^{\infty} x_k\,k\,t^{k-2} + \sum_{k=0}^{\infty} x_k\,t^k \\ &= \sum_{k=2}^{\infty} x_k\,k\,(k-1)\,t^{k-2} + \sum_{k=2}^{\infty} x_k\,k\,t^{k-2} + \sum_{k=0}^{\infty} x_k\,t^k \\ &= \sum_{m=0}^{\infty} \Big[x_{m+2}\,(m+2)\,(m+1) + x_{m+2}\,(m+2) + x_m\Big]\,t^m \\ &= \sum_{m=0}^{\infty} \Big[x_{m+2}\,(m+2)^2 + x_m\Big]\,t^m. \end{aligned}$$

Thus, if we want y to be a solution, we need to impose that

$$\sum_{m=0}^{\infty} \Big[x_{m+2}\,(m+2)^2 + x_m\Big]\,t^m = 0, \qquad \text{for } t \geq 0,$$

that is we want

$$\begin{cases} x_{m+2}\,(m+2)^2 + x_m &= 0 \\ x_0 &= 1 \\ x_1 &= 0. \end{cases}$$

This means that we are lead to solve a linear recurrence, similar to those already previously solved by means of the $Z-$transform. However, the use of the $Z-$transform now would not give easily the solution. We proceed to determine the sequence $\{x_m\}_{m\in\mathbb{N}}$ directly "by hand".

We first prove that

$$x_{2\,n+1} = 0 \qquad \text{for } n \in \mathbb{N}.$$

This can be proved by induction: indeed, for $n = 0$ this is true by the initial condition. Let us not suppose that $x_{2\,n+1} = 0$ for an index $n \in \mathbb{N}$, we need to prove that this entails that $x_{2\,n+3} = 0$, as well. However, this follows directly from the relation which defines the sequence, indeed

$$x_{2\,n+3}\,(2\,n+3)^2 + x_{2\,n+1} = 0 \qquad \text{that is} \qquad x_{2\,n+3}\,(2\,n+3)^2 = 0,$$

which proves $x_{2\,n+3} = 0$, as desired.

We now prove that

$$x_{2\,n} = \frac{(-1)^n}{(2^n\cdot(n!))^2}, \qquad \text{for } n \in \mathbb{N},$$

We argue again by induction. For $n = 0$, this is true since $x_0 = 1$ by constuction. We now assume that for an index $n \in \mathbb{N}$, we have

$$x_{2\,n} = \frac{(-1)^n}{(2^n\cdot(n!))^2},$$

then by using the recursive relation

$$x_{2\,n+2} = -\frac{1}{(2\,n+2)^2}\,x_{2\,n} = -\frac{1}{(2\cdot(n+1))^2}\cdot\frac{(-1)^n}{(2^n\cdot(n)!)^2} = \frac{(-1)^{n+1}}{(2^{n+1}\cdot(n+1)!)^2},$$

as desired. In conclusion, we get that a solution y to the initial Cauchy problem is given by

$$y(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2^n \cdot (n)!)^2} \, t^{2\,n}.$$

By using that[4]

$$\lim_{n\to\infty} \sqrt[n]{\left| \frac{(-1)^n}{(2^n \cdot (n!))^2} \right|} = 0,$$

we have from Theorem 1.7.3 that the power series above totally converges in $[-r, r]$, for every $r > 0$. □

[4] We use again that

$$\sqrt[n]{n!} \sim \frac{n}{e}, \qquad \text{for } n \to \infty.$$

Chapter 3

Lebesgue integral and L^p spaces

1. A flavour of Lebesgue measure and integration

The construction of the Riemann integral is quite simple and intuitive, but unfortunately it produces a theory which is not "sufficiently rich". Two typical problems are that:

- the class of integrable functions is too narrow;
- the theorems on the exchange between limit and integral signs are quite rigid.

Let us stick for the moment to the case of a positive function $f : [a, b] \to [0, +\infty)$. The idea of Riemann integral is to define

$$\int_a^b f(x)\, dx,$$

by partioning the interval $[a, b]$ through points $t_0 = a < t_1 < t_2 < \cdots < t_n = b$ and approximating the area of the subgraph $\{(x, y) : 0 \leq y \leq f(x)\}$ with rectangles. Roughly speaking, by taking this process to the limit, this is like saying that we are computing the area of the subgraph by summing up the lengths of all its "vertical slices", i.e. the vertical segments in $\mathbb{R} \times \mathbb{R}$ connecting $(x, 0)$ to $(x, f(x))$.

The idea of Lebesgue integration is to change the point of view and compute the area of the subgraph by summing up its "horizontal slices" in correspondence of the values y of the function. These slices are given by the sets

$$\{x \in [a, b] : f(x) > y\} \times \{y\},$$

and observe that, differently from the previous case, these sets *are not* segments (see Figure 1). Indeed, if the function f is very "wild", these sets may be very nasty.

Thus, first of all we need a way to quantify the "length" of these general sets, which extends the ordinary notion of length of a segment (if we are in dimension 1; in general this would be a generalization of the notion of area, volume and so on). This way of measuring is the *Lebesgue*

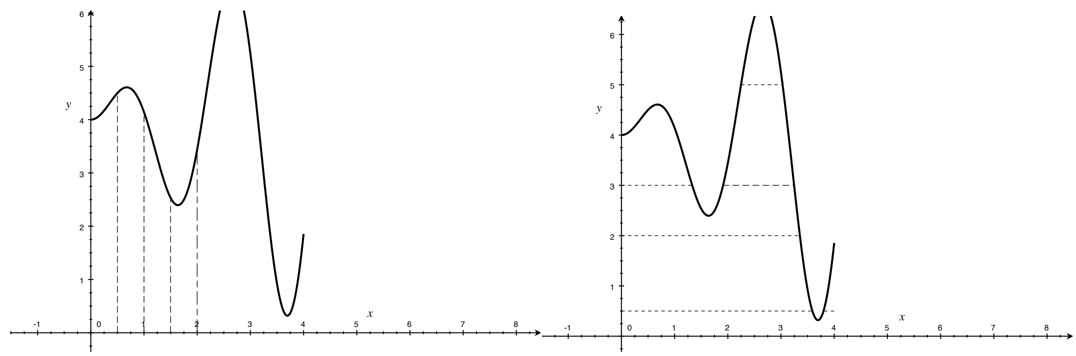

Figure 1. Vertical slices (Riemann) VS. horizontal ones (Lebesgue)

measure of a set: roughly speaking, this is defined through inner and outer approximations through countable unions of intervals.

We do not give here the detailed construction of the $k-$dimensional Lebesgue measure on $\mathbb{R}^k$, we just recall some of its fundamental properties that will be used in what follows. If a set $A \subset \mathbb{R}^k$ is measurable with respect to the $k-$dimensional Lebesgue measure, we will indicate by $|A|$ its measure (this could be $+\infty$). We then have:

- the empty set $\emptyset$ is measurable and $|\emptyset| = 0$;
- if $A \subset \mathbb{R}^k$ is measurable, then $\mathbb{R}^k \setminus A$ is measurable as well;
- $\{A_i\}_{i\in\mathbb{N}} \subset \mathbb{R}^k$ is a countable collection of measurable sets, then their union $\cup_{i\in\mathbb{N}} A_i$ is measurable and
$$\left|\bigcup_{i\in\mathbb{N}} A_i\right| \le \sum_{i\in\mathbb{N}} |A_i|;$$
- if $\{A_i\}_{i\in\mathbb{N}} \subset \mathbb{R}^k$ is a countable collection of measurable *disjoint* sets, we have
$$\left|\bigcup_{i\in\mathbb{N}} A_i\right| = \sum_{i\in\mathbb{N}} |A_i|;$$
- if $A \subset B \subset \mathbb{R}^k$ are measurable sets, then
$$|A| \le |B|;$$
- $A = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_k, b_k]$ is measurable and
$$|A| = (b_1 - a_1)\,(b_2 - a_2)\,\dots\,(b_k - a_k);$$
- for a ball $B_r(x_0) = \{x \in \mathbb{R}^k \,:\, |x - x_0| < r\}$, then $|B_r(x_0)|$ coincides with its $k-$dimensional volume. Thus for example
$$|B_r(x_0)| = \pi\, r^2 \quad \text{for } k = 2 \qquad \text{and} \qquad |B_r(x_0)| = \frac{4}{3}\,\pi\, r^3 \quad \text{for } k = 3;$$
- any affine subspace of dimension $0 \le \ell \le k - 1$ (i.e. a point, a line, a plane etc.) in $\mathbb{R}^k$ is measurable, with measure 0.

Definition 3.1.1 (Measurability)**.** We say that $f : \mathbb{R}^k \to \mathbb{R}$ is *measurable* if for every $\lambda \in \mathbb{R}$ the set

$$E_f(\lambda) = \{x \in \mathbb{R}^k \,:\, f(x) > \lambda\},$$

is (Lebesgue) measurable.

If f is complex-valued, i.e. $f : \mathbb{R}^k \to \mathbb{C}$, then we say that it is measurable if the two real-valued functions $\mathrm{Re}(f) : \mathbb{R}^k \to \mathbb{R}$ and $\mathrm{Im}(f) : \mathbb{R}^k \to \mathbb{R}$ are measurable in the sense precised above.

Definition 3.1.2. We say that a positive measurable function $f : \mathbb{R}^k \to \mathbb{R}_+$ is *summable* if

$$\int_0^{+\infty} |E_f(\lambda)|\, d\lambda < +\infty,$$

where the last integral is intended in the Riemann sense. Indeed, observe that the function $\lambda \mapsto |E_f(\lambda)|$ is monotone decreasing, as $E_f(\lambda_1) \subset E_f(\lambda_2)$ if $\lambda_1 \geq \lambda_2$. In this case, we set

$$\int_{\mathbb{R}^k} f\, dx := \int_0^{+\infty} |E_f(\lambda)|\, d\lambda. \tag{3.1.1}$$

If f is sign-changing, then we say that it is summable if $f_+ = \max\{f, 0\}$ and $f_- = \max\{-f, 0\}$ are summable and in this case we set

$$\int_{\mathbb{R}^k} f\, dx = \int_{\mathbb{R}^k} f_+\, dx - \int_{\mathbb{R}^k} f_-\, dx.$$

Finally, if f is complex-valued we say that it is summable if both real-valued functions $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are summable. In this case, we set

$$\int_{\mathbb{R}^k} f\, dx = \int_{\mathbb{R}^k} \mathrm{Re}(f)\, dx + i \int_{\mathbb{R}^k} \mathrm{Im}(f)\, dx.$$

Remark 3.1.3 ("Much ado about nothing")**.** If a function $f : \mathbb{R}^k \to [0, \infty)$ is Riemann integrable, then it is Lebesgue integrable as well and the value of the integral is unchanged. This is based on the fact that for the case of the Riemann integral one can prove the validity of formula (3.1.1) (which goes under the name of *Cavalieri's principle*).

Remark 3.1.4. If $f : \mathbb{R}^k \to \mathbb{R}$ is summable, then we have that $|f|$ is summable as well, since $|f| = f_+ + f_-$. Moreover, we have the simple but useful inequality

$$\left|\int_{\mathbb{R}^k} f\, dx\right| = \left|\int_{\mathbb{R}^k} f_+\, dx - \int_{\mathbb{R}^k} f_-\, dx\right| \leq \int_{\mathbb{R}^k} f_+\, dx + \int_{\mathbb{R}^k} f_-\, dx = \int_{\mathbb{R}^k} |f|\, dx.$$

The vice versa is true as well, i.e. if $|f|$ is summable, then f is summable. Indeed, observe that for every $\lambda \geq 0$ we have

$$\{|f| > \lambda\} = \{f > \lambda\} \cup \{f < -\lambda\},$$

and the two sets are measurable and disjoints. Thus we get

$$|\{|f| > \lambda\}| = |\{f > \lambda\}| + |\{f < -\lambda\}|,$$

and

$$\begin{aligned} +\infty > \int_0^{+\infty} |\{|f| > \lambda\}|\, d\lambda &= \int_0^{+\infty} |\{f > \lambda\}|\, d\lambda + \int_0^{+\infty} |\{-f > \lambda\}|\, d\lambda \\ &= \int_0^{+\infty} |\{f_+ > \lambda\}|\, d\lambda + \int_0^{+\infty} |\{f_- > \lambda\}|\, d\lambda \\ &= \int_{\mathbb{R}^k} f_+\, dx + \int_{\mathbb{R}^k} f_-\, dx. \end{aligned}$$

This shows that

$$\int_{\mathbb{R}^k} f_+ \, dx < +\infty \qquad \text{and} \qquad \int_{\mathbb{R}^k} f_- \, dx < +\infty,$$

thus f is summable.

By using this, we can also show that $f : \mathbb{R}^k \to \mathbb{C}$ is summable if and only if $|f| : \mathbb{R}^k \to [0, \infty)$ is summable, we leave the details to the reader.

Definition 3.1.5 (Characteristic function)**.** Let $\Omega \subset \mathbb{R}^k$ be a measurable set, we define its *characteristic function* $1_\Omega : \mathbb{R}^k \to \mathbb{R}$ as the function such that

$$1_\Omega(x) = \begin{cases} 1, & \text{if } x \in \Omega, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 3.1.6 (Summability of characteristic functions)**.** It is not difficult to see that Ω measurable entails that 1_Ω is a measurable function. Indeed, we have

$$E_\lambda(1_\Omega) = \begin{cases} \emptyset, & \text{if } \lambda \geq 1, \\ \Omega, & \text{if } 0 \leq \lambda < 1, \\ \mathbb{R}^k, & \text{if } \lambda < 0. \end{cases}$$

thus $E_\lambda(1_\Omega)$ is measurable for every $\lambda \in \mathbb{R}$. Moreover, we have that

$$\begin{aligned} 1_\Omega \text{ summable} \quad &\Longleftrightarrow \quad \int_0^{+\infty} |E_\lambda(1_\Omega)| \, d\lambda < +\infty \\ &\Longleftrightarrow \quad \int_0^1 |E_\lambda(1_\Omega)| \, d\lambda < +\infty \\ &\Longleftrightarrow \quad |\Omega| = \int_0^1 |\Omega| \, d\lambda < +\infty \end{aligned}$$

Remark 3.1.7. For example, the Heaviside function $t \mapsto H(t)$ coincides with the characteristic function of the set $[0, +\infty)$. The latter has not finite measure, thus H is *not* summable.

Definition 3.1.8 (Integral on a set)**.** Let $E \subset \mathbb{R}^k$ be a measurable set, we say that $f : E \to \mathbb{C}$ is *measurable* if its extension by 0 to $\mathbb{R}^k$, i.e.

$$\widetilde{f} = \begin{cases} f(x), & \text{if } x \in E, \\ 0 & \text{otherwise,} \end{cases}$$

is measurable. We say that f is summable on E if $\widetilde{f}$ is summable. In this case, we set

$$\int_E f \, dx = \int_{\mathbb{R}^k} \widetilde{f} \, dx.$$

2. Some results on Lebesgue integration

One of the main advantages of the Lebesgue integral is the greater flexibility in exchanging the limit and integral signs. However, some care is needed in any case. We first need a definition.

Definition 3.2.1. Let $\{f_n\}_{n \in \mathbb{N}}$ be a suquence of (possibly complex-valued) measurable functions defined on a measurable set $E \subset \mathbb{R}^k$. We say that the sequence *converges pointwisely almost every where* if

$$\lim_{n \to \infty} f_n(x) = f(x), \qquad \text{for almost every } x \in E.$$

This means that the set of points $x \in E$ for which the convergence above does not hold has Lebesgue measure zero.

In general, if we only have almost everywhere pointwise convergence, we *can not* take the limit under the integral sign.

Example 3.2.2. Let $\{f_n\}_{n\in\mathbb{N}}$ be the sequence of measurable functions defined on $[0,1]$ by

$$f_n(x) = \begin{cases} n, & \text{if } 0 \le x \le 1/n, \\ 0, & \text{if } 1/n < x \le 1. \end{cases}$$

Then we have

$$\lim_{n\to\infty} f_n(x) = 0, \qquad \text{for a. e. } x \in [0,1],$$

while on the other hand

$$\lim_{n\to\infty} \int_0^1 f_n\,dx = \lim_{n\to\infty} n \int_0^{\frac{1}{n}} dx = 1 > 0.$$

However, we have at least an inequality. This is the content of the first result.

Lemma 3.2.3 (Fatou Lemma)**.** *Let $E \subset \mathbb{R}^k$ be a measurable set and $\{f_n\}_{n\in\mathbb{R}}$ a sequence of non-negative summable functions defined on E. Let us suppose that*

$$\lim_{n\to\infty} f_n(x) = f(x), \qquad \text{for a. e. } x \in E.$$

Then we have

$$\int_E f\,dx \le \liminf_{n\to\infty} \int_E f_n\,dx.$$

Theorem 3.2.4 (Monotone Convergence Theorem)**.** *Let $E \subset \mathbb{R}^k$ be a measurable set and $\{f_n\}_{n\in\mathbb{R}}$ a monotone increasing sequence of non-negative summable functions defined on E, i.e.*

$$0 \le f_0(x) \le f_1(x) \le f_2(x) \le \dots, \qquad \text{for a. e. } x \in E.$$

Let us suppose that

$$\lim_{n\to\infty} f_n(x) = f(x), \qquad \text{for a. e. } x \in E.$$

Then we have

$$\int_E f\,dx = \lim_{n\to\infty} \int_E f_n\,dx.$$

The following result will be extremely important. The hypothesis (3.2.1) below is crucial.

Theorem 3.2.5 (Lebesgue Dominated Convergence Theorem)**.** *Let $E \subset \mathbb{R}^k$ be a measurable set and $\{f_n\}_{n\in\mathbb{R}}$ a sequence of complex-valued measurable functions defined on E. Let us suppose that*

$$\lim_{n\to\infty} f_n(x) = f(x), \qquad \text{for a. e. } x \in E,$$

and that there exists a positive **summable** *function $g : E \to \mathbb{R}$ such that*

$$|f_n(x)| \le g(x), \qquad \text{for every } n \in \mathbb{N}, \text{ for a. e. } x \in E, \tag{3.2.1}$$

Then we have

$$\int_E f\,dx = \lim_{n\to\infty} \int_E f_n\,dx.$$

We now present a couple of results that will be useful in order to exchange the order of integration.

Theorem 3.2.6 (Fubini)**.** *Let $f : \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{C}$ be a summable function on $\mathbb{R}^k \times \mathbb{R}^m$. Then:*

(A) *for a. e. $y \in \mathbb{R}^m$ the function $x \mapsto f(x,y)$ is summable on $\mathbb{R}^k$;*

(B) *the function* $y \mapsto \int_{\mathbb{R}^k} f(x,y)\,dx$ *is summable on* $\mathbb{R}^m$*;*

(C) *there holds*

$$\int_{\mathbb{R}^k\times\mathbb{R}^m} f(x,y)\,dx\,dy = \int_{\mathbb{R}^k}\left(\int_{\mathbb{R}^m} f(x,y)\,dx\right)\,dy.$$

Theorem 3.2.7 (Tonelli)**.** *Let* $f:\mathbb{R}^k\times\mathbb{R}^m\to\mathbb{R}$ *be a positive measurable function, i.e.*

$$f(x,y)\ge 0, \qquad \text{for a e. } (x,y)\in\mathbb{R}^k\times\mathbb{R}^m.$$

Let us suppose that

(A) *for a. e.* $y\in\mathbb{R}^m$ *the function* $x\mapsto f(x,y)$ *is summable on* $\mathbb{R}^k$*;*

(B) *the function* $y \mapsto \int_{\mathbb{R}^k} f(x,y)\,dx$ *is summable on* $\mathbb{R}^m$*.*

Then f *is summable on* $\mathbb{R}^k\times\mathbb{R}^m$*.*

3. L^p spaces

Starting with this section, we will confine ourselves to consider subsets of $\mathbb{R}$ only. However, all the statements that will follow can be easily generalized to $\mathbb{R}^k$.

Let $1<p<+\infty$, we define its *conjugate exponent* p' by

$$\frac{1}{p}+\frac{1}{p'}=1, \qquad \text{that is} \qquad p'=\frac{p}{p-1}.$$

In the extremal cases $p=1$ or $p=+\infty$, we define the conjugate exponent by $p'=+\infty$ and $p'=1$, respectively.

Definition 3.3.1. Let $E\subset\mathbb{R}$ be a measurable set and $1\le p<+\infty$, we define the *space of* $p-$*summable functions on* E by

$$L^p(E)=\left\{f:E\to\mathbb{C}\ \text{measurable}\ :\ \int_E |f|^p\,dx<+\infty\right\}.$$

In the limit case $p=+\infty$, we define

$$L^\infty(E)=\Big\{f:E\to\mathbb{C}\ \text{measurable}\ \ :\ \exists M\ge 0 \text{ s.t. } |f(x)|\le M \text{ for a.e. } x\in E\Big\}.$$

The functions of $L^\infty(E)$ are called *essentially bounded functions on* E.

Let $E\subset\mathbb{R}$ be a measurable set and $f\in L^p(E)$, for $1\le p<+\infty$ we define its L^p *norm*

$$\|f\|_{L^p(E)}=\left(\int_E |f|^p\,dx\right)^{\frac{1}{p}}.$$

In the limit case $p=+\infty$, we set

$$\|f\|_{L^\infty(E)}=\inf\Big\{M\ :\ |f(x)|\le M \text{ for a.e. } x\in E\Big\}.$$

We first need a couple of basic results on convex functions.

Lemma 3.3.2 (Young's inequality)**.** *Let* $1<p<+\infty$*, then for every* $a,b\in\mathbb{R}$ *we have*

$$|a\,b|\le\frac{|a|^p}{p}+\frac{|b|^{p'}}{p'}. \tag{3.3.1}$$

Proof. Without loss of generality, we can suppose that a and b are both positive. Moreover, if $a = 0$ or $b = 0$, then (3.3.1) holds true. Thus, let us assume $a > 0$ and $b > 0$. We fix $b > 0$ and consider the function

$$f(a) = a\,b - \frac{a^p}{p}, \qquad a > 0.$$

By direct computation we see

$$f'(a) \geq 0 \quad \Longleftrightarrow \quad b \geq a^{p-1} \quad \Longleftrightarrow \quad b^{\frac{1}{p-1}} \geq a.$$

This implies that f is increasing on the interval $(0, b^{1/(p-1)}]$ and decreasing on $(b^{1/(p-1)}, +\infty)$. In other words, we obtain

$$f(a) \leq f\left(b^{\frac{1}{p-1}}\right), \qquad \text{for every } a > 0.$$

By using the definition of f, this is the same as

$$a\,b - \frac{a^p}{p} \leq b\,b^{\frac{1}{p-1}} - \frac{1}{p}\,b^{\frac{p}{p-1}} = \left(1 - \frac{1}{p}\right) b^{\frac{p}{p-1}} = \frac{b^{p'}}{p'}, \qquad \text{for every } a > 0,$$

which is (3.3.1). □

Lemma 3.3.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a C^2 function, such that*

$$f''(t) \geq 0, \qquad \textit{for every } t \in \mathbb{R}.$$

Then for every $t_0, t_1 \in \mathbb{R}$ we have

$$f\left(\frac{t_0 + t_1}{2}\right) \leq \frac{1}{2}\,f(t_0) + \frac{1}{2}\,f(t_1).$$

Proof. We recall that for a C^2 function, we have the following Taylor formula with integral remainder term

$$f(t) = f(s) + f'(s)\,(t - s) + \int_s^t f''(\tau)\,(t - \tau)\,d\tau.$$

This can be directly verified, by using an integration by parts in the last integral. By using this formula with

$$t = t_0 \qquad \text{and} \qquad s = \frac{t_0 + t_1}{2},$$

we then get

$$f(t_0) = f\left(\frac{t_0 + t_1}{2}\right) + f'\left(\frac{t_0 + t_1}{2}\right) \frac{t_0 - t_1}{2} + \int_{\frac{t_0+t_1}{2}}^{t_0} f''(\tau)\,(t_0 - \tau)\,d\tau. \tag{3.3.2}$$

We can also use the formula above, with the choices

$$t = t_1 \qquad \text{and} \qquad s = \frac{t_0 + t_1}{2},$$

so to get this time

$$f(t_1) = f\left(\frac{t_0 + t_1}{2}\right) + f'\left(\frac{t_0 + t_1}{2}\right) \frac{t_1 - t_0}{2} + \int_{\frac{t_0+t_1}{2}}^{t_1} f''(\tau)\,(t_1 - \tau)\,d\tau. \tag{3.3.3}$$

By summing up (3.3.2) and (3.3.3), we then get

$$f(t_1) + f(t_0) = 2\,f\left(\frac{t_0 + t_1}{2}\right) + \int_{\frac{t_0+t_1}{2}}^{t_0} f''(\tau)\,(t_0 - \tau)\,d\tau + \int_{\frac{t_0+t_1}{2}}^{t_1} f''(\tau)\,(t_1 - \tau)\,d\tau. \tag{3.3.4}$$

We now suppose without loss of generality that $t_0 \le t_1$. This entails

$$t_0 \le \frac{t_0+t_1}{2} \le t_1,$$

and thus

$$\begin{aligned}\int_{\frac{t_0+t_1}{2}}^{t_0} f''(\tau)\,(t_0-\tau)\,d\tau &+ \int_{\frac{t_0+t_1}{2}}^{t_1} f''(\tau)\,(t_1-\tau)\,d\tau \\ &= -\int_{t_0}^{\frac{t_0+t_1}{2}} f''(\tau)\,(t_0-\tau)\,d\tau + \int_{\frac{t_0+t_1}{2}}^{t_1} f''(\tau)\,(t_1-\tau)\,d\tau \\ &= \int_{t_0}^{\frac{t_0+t_1}{2}} f''(\tau)\,(\tau-t_0)\,d\tau + \int_{\frac{t_0+t_1}{2}}^{t_1} f''(\tau)\,(t_1-\tau)\,d\tau.\end{aligned}$$

By recalling that $f''(\tau) \ge 0$ for every $\tau \in \mathbb{R}$, we obtain that the sum of these two integrals is non-negative. By using this in (3.3.4), we thus get

$$f(t_1) + f(t_0) \ge 2\,f\left(\frac{t_0+t_1}{2}\right),$$

as desired. □

Remark 3.3.4 (A property of convex powers)**.** By using the previous result, we can prove that for every $1 < p < +\infty$ we have

$$\left|\frac{x+y}{2}\right|^p \le \frac{|x|^p}{2} + \frac{|y|^p}{2}, \qquad \text{for every } x, y \in \mathbb{R}. \tag{3.3.5}$$

Indeed, when $p \ge 2$ this follows by using directly Lemma 3.3.3 with $f(t) = |t|^p$. Indeed, this is a C^2 function such that $f'' \ge 0$. For $1 < p < 2$, this function is not C^2, but we can circumvent this problem as follows: we consider $f_\varepsilon(\tau) = (\varepsilon^2 + t^2)^{p/2}$, which is now a C^2 function if $\varepsilon > 0$. Moreover, it is easy to see that

$$f_\varepsilon''(\tau) \ge 0, \qquad \text{for every } \tau \in \mathbb{R}.$$

By applying Lemma 3.3.3, we then get

$$f_\varepsilon\left(\frac{x+y}{2}\right) \le \frac{1}{2}\,f_\varepsilon(x) + \frac{1}{2}\,f_\varepsilon(y), \qquad \text{for every } x, y \in \mathbb{R}.$$

By recalling the definition of f_ε, this is the same as

$$\left(\varepsilon^2 + \left(\frac{x+y}{2}\right)^2\right)^{\frac{p}{2}} \le \frac{(\varepsilon^2+x^2)^{\frac{p}{2}}}{2} + \frac{(\varepsilon^2+y^2)^{\frac{p}{2}}}{2}, \qquad \text{for every } x, y \in \mathbb{R},\ \varepsilon > 0.$$

If we now take the limit as ε goes to 0, we get (3.3.5) in the case $1 < p < 2$, as well.

Proposition 3.3.5 (Hölder's inequality)**.** *Let $E \subset \mathbb{R}$ be a measurable set and let $1 \le p \le +\infty$. For every $f \in L^p(E)$ and $g \in L^{p'}(E)$ we have, $f \cdot g \in L^1(E)$. Moreover, it holds*

$$\left|\int_E f(x)\,g(x)\,dx\right| \le \int_E |f(x)|\,|g(x)|\,dx \le \|f\|_{L^p(E)}\,\|g\|_{L^{p'}(E)}. \tag{3.3.6}$$

Proof. Let us consider the case $1 < p < +\infty$, the remaining cases being simpler. If $f = 0$ or $g = 0$ almost everywhere in E, then there is nothing to prove. Thus let us assume that

$$|\{x \in E\,:\, f(x) \ne 0\}| > 0 \qquad \text{and} \qquad |\{x \in E\,:\, g(x) \ne 0\}| > 0.$$

By (3.3.1) with the choices

$$a = \frac{|f(x)|}{\|f\|_{L^p(E)}} \qquad \text{and} \qquad b = \frac{|g(x)|}{\|g\|_{L^{p'}(E)}},$$

we obtain

$$\left|\frac{|f(x)|}{\|f\|_{L^p(E)}}\,\frac{|g(x)|}{\|g\|_{L^{p'}(E)}}\right| \le \frac{1}{p}\,\frac{|f(x)|^p}{\displaystyle\int_E |f(x)|^p\,dx} + \frac{1}{p'}\,\frac{|g(x)|^{p'}}{\displaystyle\int_E |g(x)|^{p'}\,dx},$$

which is valid for almost every $x \in E$. By taking the integral over E, the previous gives

$$\frac{\displaystyle\int_E |f(x)\,g(x)|\,dx}{\|f\|_{L^p(E)}\,\|g\|_{L^{p'}(E)}} \le \frac{1}{p}\,\frac{\displaystyle\int_E |f(x)|^p\,dx}{\displaystyle\int_E |f(x)|^p\,dx} + \frac{1}{p'}\,\frac{\displaystyle\int_E |g(x)|^{p'}\,dx}{\displaystyle\int_E |g(x)|^{p'}\,dx} = 1.$$

This finally shows that

$$\int_E |f(x)\,g(x)|\,dx \le \|f\|_{L^p(E)}\,\|g\|_{L^{p'}(E)}.$$

In order to conclude, we have to show that

$$\left|\int_E f(x)\,g(x)\,dx\right| \le \int_E |f(x)\,g(x)|\,dx.$$

If f and g are real-valued, then this follows from Remark 3.1.4. If f or g is complex valued, let us set $F = f\,g = u + i\,v$, with u and v real-valued. By definition of modulus of a complex number, we have

$$\left|\int_E F\,dx\right|^2 = \left|\int_E u\,dx\right|^2 + \left|\int_E v\,dx\right|^2. \tag{3.3.7}$$

We can now use Hölder's inequality for real-valued functions as follows:

$$\left|\int_E u\,dx\right|^2 = \left|\int_E \frac{u}{\sqrt{|F|}}\,\sqrt{|F|}\,dx\right|^2 \le \int_E \frac{u^2}{|F|}\,dx\,\int_E |F|\,dx,$$

and

$$\left|\int_E v\,dx\right|^2 = \left|\int_E \frac{v}{\sqrt{|F|}}\,\sqrt{|F|}\,dx\right|^2 \le \int_E \frac{v^2}{|F|}\,dx\,\int_E |F|\,dx.$$

By using these in (3.3.7), we get

$$\begin{aligned}\left|\int_E F\,dx\right|^2 &\le \int_E |F|\,dx\,\left(\int_E \frac{u^2}{|F|}\,dx + \int_E \frac{v^2}{|F|}\,dx\right)\\ &= \left(\int_E |F|\,dx\right)^2.\end{aligned}$$

By recalling that $F = f\,g$ and taking the square root, we get the desired conclusion. □

Remark 3.3.6. Observe that in the previous result we proved that if F is a complex-valued summable function, then

$$\left|\int_E F\,dx\right| \le \int_E |F|\,dx,$$

which generalizes the estimate of Remark 3.1.4 to complex-valued functions.

Proposition 3.3.7 (Minkowski's inequality). *Let $E \subset \mathbb{R}$ be a measurable set and $1 \le p \le +\infty$. For every $f, g \in L^p(E)$ we have $f + g \in L^p(E)$ and*

$$\|f + g\|_{L^p(E)} \le \|f\|_{L^p(E)} + \|g\|_{L^p(E)}. \tag{3.3.8}$$

Proof. Let us focus on the case $1 \le p < +\infty$, the extremal case $p = +\infty$ being simpler. We first prove that

$$f + g \in L^p(E).$$

At this aim, we observe that

$$|f(x) + g(x)|^p \le \Big(|f(x)| + |g(x)|\Big)^p \le 2^{p-1} \left(|f(x)|^p + |g(x)|^p\right), \tag{3.3.9}$$

thanks to the inequality (3.3.5). By integrating (3.3.9) and using that $f, g \in L^p(E)$, we thus get

$$\int_E |f(x) + g(x)|^p \, dx < +\infty,$$

i.e. $f + g \in L^p(E)$.

We now come to the proof of (3.3.8). By using that

$$|z|^p = |z|^{p-2}\, |z|^2 = |z|^{p-2}\, z\, z^*, \qquad \text{for } z \in \mathbb{C},$$

and Hölder's inequality (3.3.6), we have

$$\begin{aligned}
\|f + g\|_{L^p(E)}^p &= \int_E |f + g|^{p-2} \,(f + g)\,(f + g)^* \, dx \\
&= \int_E |f + g|^{p-2} \,(f + g)\, f^* \, dx + \int_E |f + g|^{p-2}\,(f + g)\, g^* \, dx \\
&\le \left(\int_E |f + g|^{(p-1)\,p'} \, dx\right)^{\frac{1}{p'}} \left(\int_E |f|^p \, dx\right)^{\frac{1}{p}} \\
&+ \left(\int_E |f + g|^{(p-1)\,p'} \, dx\right)^{\frac{1}{p'}} \left(\int_E |g|^p \, dx\right)^{\frac{1}{p}} \\
&= \|f + g\|_{L^p(E)}^{p-1} \left(\|f\|_{L^p(E)} + \|g\|_{L^p(E)}\right),
\end{aligned}$$

where we used that $(p-1)\,p' = p$. By simplyfing on both sides the term $\|f + g\|_{L^p(E)}^{p-1}$, we get the conclusion. $\square$

Remark 3.3.8. The previous result permits to infer that $f \mapsto \|f\|_{L^p(E)}$ is a *norm*[1] on the vector space $L^p(E)$.

Definition 3.3.9 (Compactly supported functions). We say that a measurable function $f : \mathbb{R} \to \mathbb{C}$ has *compact support* if there exists a bounded closed interval $[a, b] \subset \mathbb{R}$ such that

$$|f(x)| = 0, \text{ for a. e. } x \in \mathbb{R} \setminus [a, b].$$

[1] This is not fully correct, there is a subtility here. Indeed, the fact that

$$\|f\|_{L^p(E)} = 0,$$

only implies that $f = 0$ **almost everywhere** and not everywhere. The issue is easily fixed by considering $L^p(E)$ as a collection of *equivalence classes* of functions coinciding on E almost everywhere. We will not enter into these details here, which are beyond the scopes of these notes.

Proposition 3.3.10 (Inclusion properties). *Let $E \subset \mathbb{R}$ be a measurable set with finite measure. Let $1 \le p < q \le +\infty$, then we have*

$$L^q(E) \subset L^p(E).$$

More precisely, we have

$$\|f\|_{L^p(E)} \le |E|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(E)}, \qquad \text{for every } f \in L^q(E).$$

Proof. Let us start with the case $q < +\infty$. We observe that if $f \in L^q(E)$, then

$$|f|^p \in L^{\frac{q}{p}}(E).$$

We can now use Hölder's inequality (3.3.6) with conjugate exponents

$$\frac{q}{p} \qquad \text{and} \qquad \left(\frac{q}{p}\right)' = \frac{q}{q-p},$$

so to get

$$\begin{aligned}\int_E |f|^p\,dx = \int_E 1\cdot|f|^p\,dx &\le \left(\int_E 1\,dx\right)^{\frac{q-p}{p}} \left(\int_E |f|^q\right)^{\frac{p}{q}}\\ &\le |E|^{\frac{q-p}{p}}\,\|f\|^p_{L^q(E)}.\end{aligned}$$

By taking the power $1/p$ on both sides, we get the conclusion.

If $f \in L^\infty(E)$ the proof is even simpler, it is sufficient to observe that

$$|f(x)| \le \|f\|_{L^\infty(E)}, \qquad \text{for almost every } x \in E,$$

thus we get

$$\left(\int_E |f|^p\,dx\right)^{\frac{1}{p}} \le |E|^{\frac{1}{p}}\,\|f\|_{L^\infty(E)}.$$

This concludes the proof. □

Remark 3.3.11. The previous inclusion $L^q(E) \subset L^p(E)$ is false if $|E| = +\infty$. Indeed, let us take $E = [1, +\infty)$ and consider the function

$$f(x) = \frac{1}{x}.$$

Then it is easy to see that $f \in L^q(E)$ for every $q > 1$, but $f \notin L^1(E)$. Indeed, we have

$$\int_E |f|^q\,dx = \int_1^{+\infty} \frac{1}{|x|^q}\,dx = \left[\frac{|x|^{1-q}}{q-1}\right]_1^{+\infty} = \frac{1}{q-1},$$

and

$$\int_E |f|\,dx = \int_1^{+\infty} \frac{1}{|x|}\,dx = \lim_{M\to+\infty} \Big[\log x\Big]_1^M = \lim_{M\to+\infty} \log M = +\infty.$$

The following two simple technical results will be useful in the sequel.

Lemma 3.3.12. *Let $g \in L^1(\mathbb{R})$ be such that*

$$\lim_{x\to+\infty} g(x) = L.$$

Then we necessarily have $L = 0$.

Proof. Let us assume by contradiction that $L \neq 0$. By assumption, if we fix $\varepsilon = |L|/2 > 0$ there exists $\Lambda > 0$ such that

$$|g(x) - L| < \frac{|L|}{2}, \qquad \text{for every } x > \Lambda.$$

In particular, by triangle inequality we get

$$\frac{|L|}{2} = |L| - \frac{|L|}{2} < |g(x)|, \qquad \text{for every } x > \Lambda.$$

By using that $g \in L^1(\mathbb{R})$, we would get

$$\int_{\mathbb{R}} |g(x)|\, dx \geq \int_{\Lambda}^{+\infty} |g(x)|\, dx > \frac{|L|}{2} \int_{\Lambda}^{+\infty} dt = +\infty,$$

which is a contradiction with the fact that $g \in L^1(\mathbb{R})$. □

Definition 3.3.13 (Local Lebesgue space)**.** We say that a measurable function $f : \mathbb{R} \to \mathbb{C}$ is *locally summable* if

$$f \in L^1([a, b]),$$

for every couple of real numbers $a < b$. The collection of all locally summable functions will be denoted by $L^1_{\mathrm{loc}}(\mathbb{R})$.

Example 3.3.14. Of course, we have

$$L^1(\mathbb{R}) \subset L^1_{\mathrm{loc}}(\mathbb{R}),$$

but the two spaces does not coincide. For example, the function $x \mapsto x^2$ is in $L^1_{\mathrm{loc}}(\mathbb{R})$, since

$$\int_a^b x^2\, dx = \frac{b^3 - a^3}{3} < +\infty,$$

but of course this does not belong to $L^1(\mathbb{R})$.

Example 3.3.15. Another important example of $L^1_{\mathrm{loc}}(\mathbb{R})$ function is the *cardinal sine* function

$$\mathrm{sinc}\,(x) = \begin{cases} \dfrac{\sin(\pi\, x)}{\pi\, x}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0. \end{cases}$$

Indeed, this is continuous function on $\mathbb{R}$, thus we have $\mathrm{sinc} \in L^\infty([a, b]) \subset L^1([a, b])$, for every $a < b$. On the other side

$$\lim_{L\to\infty} \int_{-L}^{L} \left| \frac{\sin(\pi\, x)}{\pi\, x} \right| dx = +\infty,$$

thus $\mathrm{sinc} \notin L^1(\mathbb{R})$, see Exercise 3.7.3. On the other hand, by using the Cauchy's Theorem for holomorphic functions (see Exercise 3.7.4) one can prove that

$$\lim_{L\to\infty} \int_{-L}^{L} \frac{\sin(\pi\, x)}{\pi\, x}\, dx = 1.$$

Finally, observe that $\mathrm{sinc} \in L^p(\mathbb{R})$ for $p > 1$, since

$$\begin{aligned} \int_{\mathbb{R}} \left| \frac{\sin(\pi\, x)}{\pi\, x} \right|^p dx &= 2 \int_0^1 \left| \frac{\sin(\pi\, x)}{\pi\, x} \right|^p dx + 2 \int_1^\infty \left| \frac{\sin(\pi\, x)}{\pi\, x} \right|^p dx \\ &\leq 2 + \frac{2}{\pi^p} \int_1^\infty \frac{1}{x^p}\, dx = 2 + \frac{2}{(p-1)\, \pi^p}, \end{aligned}$$

where we used that sinc is an even function, smaller than 1.

4. Finer properties of L^p spaces

Definition 3.4.1 (Cauchy sequence)**.** Let $E \subset \mathbb{R}$ be a measurable set and let $1 \le p \le +\infty$. We say that $\{f_n\}_{n\in\mathbb{N}} \subset L^p(E)$ is a *Cauchy sequence* if:

$$\forall \varepsilon,\ \exists n_0 \in \mathbb{N} \text{ such that for every } n, m \ge n_0 \text{ we have } \|f_n - f_m\|_{L^p(E)} < \varepsilon.$$

Theorem 3.4.2 (Riesz-Fischer)**.** *Let $E \subset \mathbb{R}$ be a measurable set. Then for $1 \le p \le +\infty$ the space $L^p(E)$ is a Banach space. In other words, $L^p(E)$ is a normed vector space such that every Cauchy sequence $\{f_n\}_{n\in\mathbb{N}} \subset L^p(E)$ is convergent, i.e. there exists $f \in L^p(E)$ such that*

$$\lim_{n\to\infty} \|f_n - f\|_{L^p(E)} = 0.$$

In what follows, we denote

$$C_0(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} \text{ continuous} : f \text{ compactly supported}\},$$

and more generally for $k \in \mathbb{N} \setminus \{0\}$

$$C_0^k(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} \text{ derivable } k \text{ times} : f, f', \dots, f^{(k)} \text{ continuous compactly supported}\}.$$

Finally, we also set

$$C_0^\infty(\mathbb{R}) = \bigcap_{k\in\mathbb{N}} C_0^k(\mathbb{R}).$$

We then have the following remarkable result (which is *not true* for $p = +\infty$). We omit the proof.

Theorem 3.4.3 (Density Theorem)**.** *Let $1 \le p < +\infty$ and $f \in L^p(\mathbb{R})$. For every $\varepsilon > 0$, there exists $g_\varepsilon \in C_0(\mathbb{R})$ such that*

$$\|f - g_\varepsilon\|_{L^p(\mathbb{R})} < \varepsilon.$$

Thus for every $f \in L^p(\mathbb{R})$, there exists a sequence $\{g_n\}_{n\in\mathbb{N}} \subset C_0(\mathbb{R})$ such that

$$\lim_{n\to\infty} \|f - g_n\|_{L^p(\mathbb{R})} = 0.$$

Remark 3.4.4. The previous result asserts that even if elements of $L^p(\mathbb{R})$ may be very irregular functions, we can always approximate them (in the sense of L^p norm!) with more regular functions. We will see that we can do definitely better and approximate with C^∞ functions, see Theorem 3.5.13 below.

Proposition 3.4.5 (Translations are continuous in L^p)**.** *Let $1 \le p < +\infty$ and $f \in L^p(\mathbb{R})$, for every $h \in \mathbb{R}$ we define the translated function*

$$\mathcal{T}_h f(x) = f(x+h), \qquad x \in \mathbb{R}.$$

Then we have

$$\lim_{h\to 0} \|\mathcal{T}_h f - f\|_{L^p(\mathbb{R})} = 0.$$

Proof. By using the Density Theorem (i.e. Theorem 3.4.3), we know that for every $\varepsilon > 0$ there exists $g_\varepsilon \in C_0(\mathbb{R})$ such that

$$\|f - g_\varepsilon\|_{L^p(\mathbb{R})} < \varepsilon. \tag{3.4.1}$$

By using Minkowski inequality, we have

$$\|\mathcal{T}_h f - f\|_{L^p(\mathbb{R})} \le \|f - g_\varepsilon\|_{L^p(\mathbb{R})} + \|\mathcal{T}_h g_\varepsilon - g_\varepsilon\|_{L^p(\mathbb{R})} + \|\mathcal{T}_h g_\varepsilon - \mathcal{T}_h f\|_{L^p(\mathbb{R})}. \tag{3.4.2}$$

We now observe that by a simple change of variable we have

$$\|\mathcal{T}_h g_\varepsilon - \mathcal{T}_h f\|_{L^p(\mathbb{R})} = \left(\int_{\mathbb{R}} |g_\varepsilon(x+h) - f(x+h)|^p\,dx\right)^{\frac{1}{p}}$$
$$= \left(\int_{\mathbb{R}} |g_\varepsilon(y) - f(y)|^p\,dy\right)^{\frac{1}{p}} = \|g_\varepsilon - f\|_{L^p(\mathbb{R})} < \varepsilon,$$

where in the last inequality we used (3.4.1). By using these information in (3.4.2), we obtained

$$\|\mathcal{T}_h f - f\|_{L^p(\mathbb{R})} \le 2\,\varepsilon + \|\mathcal{T}_h g_\varepsilon - g_\varepsilon\|_{L^p(\mathbb{R})}.$$

We are left with estimating the last term. By continuity we have

$$\lim_{h\to 0} |\mathcal{T}_h g_\varepsilon(x) - g_\varepsilon(x)| = 0, \qquad \text{for every } x \in \mathbb{R}.$$

Moreover, since g_ε is compactly supported, there exists an interval $[a,b]$ such that g_ε identically vanishes outside $[a,b]$. Then for every $|h|<1$ the function $\mathcal{T}_h g_\varepsilon - g_\varepsilon$ identically vanishes outside $[a-1,b+1]$. We can thus infer[2]

$$|\mathcal{T}_h g_\varepsilon(x) - g_\varepsilon(x)|^p \le 2^p\,\|g_\varepsilon\|^p_{L^\infty(\mathbb{R})}\,1_{[a-1,b+1]}(x) \in L^1(\mathbb{R}), \qquad \text{for every } |h|<1.$$

We can apply Lebesgue Dominated Convergence Theorem 3.2.5 and get

$$\lim_{h\to 0} \|\mathcal{T}_h g_\varepsilon - g_\varepsilon\|_{L^p(\mathbb{R})} = 0.$$

In conclusion, this gives

$$\lim_{h\to 0} \|\mathcal{T}_h f - f\|_{L^p(\mathbb{R})} \le 2\,\varepsilon.$$

By arbitrariness of $\varepsilon > 0$, we conclude. □

Remark 3.4.6. The previous result *is false for* $p=+\infty$. Let us take the Heaviside function $H(x)$, for $h>0$ then we have

$$\mathcal{T}_h H(x) - H(x) = \begin{cases} 1, & \text{if } -h \le x < 0, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, we get

$$\|\mathcal{T}_h H - H\|_{L^\infty(\mathbb{R})} = 1, \qquad \text{for every } h>0,$$

and this does not converge to 0.

5. Convolutions

Definition 3.5.1. Let $f,g \in L^1(\mathbb{R})$, we define their *convolution* $f*g$ by

$$f*g(x) = \int_{\mathbb{R}} f(x-y)\,g(y)\,dy, \qquad \text{for a.e. } x \in \mathbb{R}.$$

Observe that by making the change of variable $y = x-t$, the previous definition can also be written as

$$f*g(x) = \int_{\mathbb{R}} f(t)\,g(x-t)\,dt, \qquad \text{for a.e. } x \in \mathbb{R}.$$

[2] We used that $|a-b|^p \le (|a|+|b|)^p \le 2^{p-1}\,(|a|^p + |b|^p)$, which follows from (3.3.5).

Remark 3.5.2 (Causal signals)**.** We observe that if $f, g \in L^1(\mathbb{R})$ are causal, i.e. identically vanishing for $x < 0$, then

$$f * g(x) = \int_0^x f(x-y)\, g(y)\, dy.$$

It is sufficient to observe that by causality

$$g(y) = 0 \ \text{ for } y < 0 \qquad \text{and} \qquad f(x-y) = 0 \ \text{ for } x - y < 0 \text{ i.e. for } y > x.$$

Moreover, $f * g$ is still causal (*exercise: prove the last assertion!*).

Remark 3.5.3 (Convolution of sequences)**.** We have seen in Chapter 2 that the convolution of two sequences $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ and $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ is defined by

$$x_n * y_n = \sum_{k=0}^{n} x_{n-k}\, y_k.$$

We now take two causal signals $f, g \in L^1(\mathbb{R})$ and fix a time step $\tau > 0$. Then we consider the regular samplings

$$x_n = f(n\,\tau) \qquad \text{and} \qquad y_n = g(n\,\tau),$$

and observe that

$$\begin{aligned} f * g(n\,\tau) = \int_0^{n\,\tau} f(n\,\tau - y)\, g(y)\, dy &\sim \tau \sum_{k=0}^{n} f(n\,\tau - k\,\tau)\, g(k\,\tau) \\ &= \tau \sum_{k=0}^{n} x_{n-k}\, y_k = \tau\, (x_n * y_n). \end{aligned}$$

Here we (formally) replaced the integral by a Riemann sum. Thus the convolution of sequences can be seen as a discretized version of the integral definition for causal signals.

It is not difficult to see that for $f, g \in L^1(\mathbb{R})$, the convolution is well-defined and we have $f * g \in L^1(\mathbb{R})$. This follows from the following more general result (just take $p = q = 1$ below).

Proposition 3.5.4 (Young's inequality for convolutions, part I)**.** *Let $f \in L^q(\mathbb{R})$ and $g \in L^p(\mathbb{R})$, for $1 \le p, q \le \infty$. Let us suppose that*

$$\frac{1}{p} + \frac{1}{q} > 1.$$

Then their convolution $f * g$ is well-defined and we have $f * g \in L^r(\mathbb{R})$, with

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1.$$

Moreover, there holds

$$\|f * g\|_{L^r(\mathbb{R})} \le \|f\|_{L^q(\mathbb{R})}\, \|g\|_{L^p(\mathbb{R})}. \tag{3.5.1}$$

Proof. We first observe that by definition of r, we have $r \ge q$ and $r \ge p$. Indeed

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \le \frac{1}{p} \qquad \text{and} \qquad \frac{1}{r} = \frac{1}{p} - 1 + \frac{1}{q} \le \frac{1}{q}.$$

Moreover, we have

$$r = p \iff q = 1 \qquad \text{and} \qquad r = q \iff p = 1.$$

For almost evey $x \in \mathbb{R}$ we have

$$\begin{aligned}|f*g(x)| &= \left|\int_{\mathbb{R}} f(y)\,g(x-y)\,dy\right| \le \int_{\mathbb{R}} |f(y)|\,|g(x-y)|\,dy\\ &= \int_{\mathbb{R}} |f(y)|^{\frac{q}{r}}\,|g(x-y)|^{\frac{p}{r}}\,|f(y)|^{1-\frac{q}{r}}\,|g(x-y)|^{1-\frac{p}{r}}\,dy\\ &\le \left(\int_{\mathbb{R}} |f(y)|^q\,|g(x-y)|^p\,dy\right)^{\frac{1}{r}} \left(\int_{\mathbb{R}} |f(y)|^{\frac{r-q}{r-1}}\,|g(x-y)|^{\frac{r-p}{r-1}}\,dy\right)^{\frac{r-1}{r}},\end{aligned}$$

thanks to Hölder's inequality (3.3.6) with exponents r and $r' = r/(r-1)$. We now observe that by definition of r, we have

$$\frac{r-q}{q\,(r-1)} + \frac{r-p}{p\,(r-1)} = \frac{1}{r-1}\left(\frac{r}{q} - 1 + \frac{r}{p} - 1\right) = \frac{1}{r-1}\,(r-1) = 1,$$

thus we can further use Hölder's inequality in the last integral, i.e.

$$\left(\int_{\mathbb{R}} |f(y)|^{\frac{r-q}{r-1}}\,|g(x-y)|^{\frac{r-p}{r-1}}\,dy\right)^{\frac{r-1}{r}} \le \left(\int_{\mathbb{R}} |f(y)|^q\,dy\right)^{\frac{r-q}{q\,r}} \left(\int_{\mathbb{R}} |g(x-y)|^p\,dy\right)^{\frac{r-p}{p\,r}}.$$

By resuming, we obtained

$$|f*g(x)|^r \le \int_{\mathbb{R}} |f(y)|^q\,|g(x-y)|^p\,dy\,\|f\|_{L^q(\mathbb{R})}^{r-q}\,\|g\|_{L^p(\mathbb{R})}^{r-p}.$$

We now integrate with respect to x and get

$$\int_{\mathbb{R}} |f*g(x)|^r\,dx \le \|f\|_{L^p(\mathbb{R})}^{r-q}\,\|g\|_{L^p(\mathbb{R})}^{r-p} \int_{\mathbb{R}}\int_{\mathbb{R}} |f(y)|^q\,|g(x-y)|^p\,dy\,dx.$$

Observe that the function $(x,y) \mapsto F(x,y) = |f(y)|^q\,|g(x-y)|^p$ is positive on $\mathbb{R}^2$ and satisfies the hypotheses of Tonelli's Theorem: indeed, for almost every $y \in \mathbb{R}$, the function

$$x \mapsto |f(y)|^q\,|g(x-y)|^p,$$

is summable, since $g \in L^p(\mathbb{R})$; moreover, the function

$$y \mapsto \int_{\mathbb{R}} |f(y)|^q\,|g(x-y)|^p\,dx = \|g\|_{L^p(\mathbb{R})}^p\,|f(y)|^q,$$

is summable, since $f \in L^q(\mathbb{R})$ by hypothesis. This implies that $F \in L^1(\mathbb{R}\times\mathbb{R})$ and by Fubini's Theorem we can exchange the order of integration, i.e.

$$\int_{\mathbb{R}} |f*g(x)|^r\,dx \le \|f\|_{L^q(\mathbb{R})}^{r-q}\,\|g\|_{L^p(\mathbb{R})}^{r-p} \int_{\mathbb{R}} |f(y)|^p \left(\int_{\mathbb{R}} |g(x-y)|^p\,dx\right) dy$$

By changing variable in the integral of g, we thus get

$$\int_{\mathbb{R}} |g(x-y)|^p\,dx = \|g\|_{L^p(\mathbb{R}^N)}^p,$$

and finally

$$\int_{\mathbb{R}} |f*g(x)|^r\,dx \le \|f\|_{L^q(\mathbb{R})}^{r-q}\,\|g\|_{L^p(\mathbb{R})}^{r-p}\,\|f\|_{L^q(\mathbb{R})}^{q}\,\|g\|_{L^p(\mathbb{R})}^{p} = \|f\|_{L^q(\mathbb{R})}^{r}\,\|g\|_{L^p(\mathbb{R})}^{r}.$$

By raising to the power $1/r$, we finally get (3.5.1). $\square$

Remark 3.5.5. Very often, we will use the previous result with $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$, for some $1 \le p \le +\infty$. Accordingly, by taking $q = 1$ in the previous result, we get $r = p$ and thus we have

$$f * g \in L^p(\mathbb{R}).$$

When p and q are conjugate, the convolution is a bounded *continuous* function. This is the content of the next result.

Proposition 3.5.6 (Young's inequality for convolutions, part II)**.** *Let $f \in L^p(\mathbb{R})$ and $g \in L^{p'}(\mathbb{R})$, for $1 \le p \le +\infty$. Then their convolution $f * g$ is well-defined and we have $f * g \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$. Moreover, there holds*

$$\|f * g\|_{L^\infty(\mathbb{R})} \le \|f\|_{L^p(\mathbb{R})} \, \|g\|_{L^{p'}(\mathbb{R})}. \tag{3.5.2}$$

Proof. We start with the case $1 < p < \infty$. For almost every $x \in \mathbb{R}$, we have

$$|f * g(x)| = \left| \int_{\mathbb{R}} f(y)\, g(x - y)\, dy \right| \le \left(\int_{\mathbb{R}} |f(y)|^p \, dy \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} |g(x - y)|^{p'} \, dy \right)^{\frac{1}{p'}},$$

thanks to Hölder's inequality (3.3.6). By observing that with a simple change of variable we have

$$\left(\int_{\mathbb{R}} |g(x - y)|^{p'} \, dy \right)^{\frac{1}{p'}} = \left(\int_{\mathbb{R}} |g(y)|^{p'} \, dy \right)^{\frac{1}{p'}},$$

we thus obtain

$$|f * g(x)| \le \|f\|_{L^p(\mathbb{R})} \, \|g\|_{L^{p'}(\mathbb{R})}, \qquad \text{for a. e. } x \in \mathbb{R}.$$

This shows at the same time that $f * g \in L^\infty(\mathbb{R})$ and (3.5.2). We now prove that $f * g$ is continuous. We take $x \in \mathbb{R}$ and $h \in \mathbb{R}$, then we have

$$\begin{aligned} |f * g(x + h) - f * g(x)| &= \left| \int_{\mathbb{R}} f(y)\, g(x + h - y)\, dy - \int_{\mathbb{R}} f(y)\, g(x - y)\, dy \right| \\ &= \left| \int_{\mathbb{R}} f(y)\, [g(x + h - y) - g(x - y)]\, dy \right| \\ &\le \left(\int_{\mathbb{R}} |f(y)|^p \, dy \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} |g(x + h - y) - g(x - y)|^{p'} \, dy \right)^{\frac{1}{p'}}. \end{aligned}$$

As above, with a simple change of variable we get

$$\left(\int_{\mathbb{R}} |g(x + h - y) - g(x - y)|^{p'} \, dy \right)^{\frac{1}{p'}} = \left(\int_{\mathbb{R}} |g(h + t) - g(t)|^{p'} \, dt \right)^{\frac{1}{p'}} = \|\mathcal{T}_h g - g\|_{L^{p'}(\mathbb{R})}.$$

In conclusion, we get

$$\lim_{h \to 0} |f * g(x + h) - f * g(x)| \le \|f\|_{L^p(\mathbb{R})} \lim_{h \to 0} \|\mathcal{T}_h g - g\|_{L^{p'}(\mathbb{R})} = 0,$$

thanks to Proposition 3.4.5.

The cases $p = 1$ or $p = \infty$ are even simpler. For example, if $p = 1$ then $g \in L^\infty(\mathbb{R})$ and we have

$$|f * g(x)| = \left| \int_{\mathbb{R}} f(y)\, g(x - y)\, dy \right| \le \|g\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |f(y)|\, dy,$$

which gives again the desired conclusion. The continuity is proved as above, we leave the details to the reader. □

Corollary 3.5.7. *The convolution of two $L^2(\mathbb{R})$ functions is in $L^\infty(\mathbb{R}) \cap C(\mathbb{R})$.*

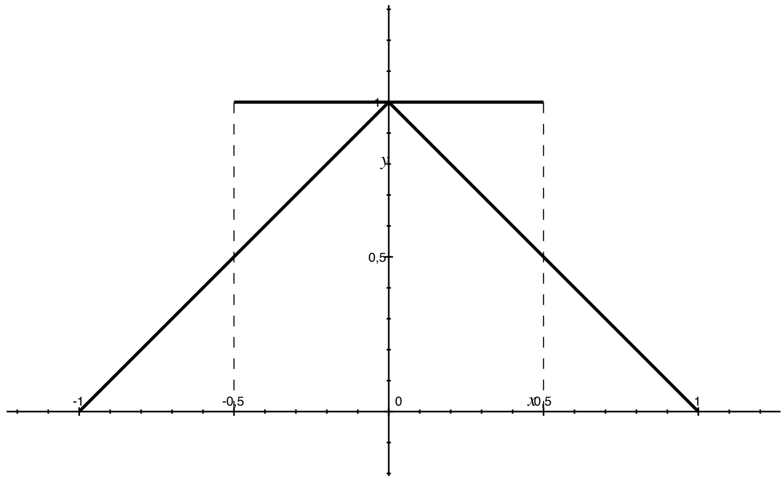

Figure 2. The graphs of the functions $x \mapsto \mathrm{rect}(x)$ and $x \mapsto \mathrm{tri}(x) = \mathrm{rect} * \mathrm{rect}(x)$.

Example 3.5.8 (Rectangular and triangular functions)**.** Let us consider the *rectangular function* defined by

$$\mathrm{rect}(x) = \begin{cases} 1, & \text{if } x \in \left[-\frac{1}{2}, \frac{1}{2}\right], \\ 0, & \text{otherwise}, \end{cases}$$

i.e. this is the characteristic function of the interval $[-1/2, 1/2]$. We want to compute the convolution $\mathrm{rect} * \mathrm{rect}$. Observe that rect is comptacly supported and belongs to $L^p(\mathbb{R})$ for every $1 \leq p \leq \infty$. Thus we already know that $\mathrm{rect} * \mathrm{rect}$ is a bounded continuous function by Proposition 3.5.6.

By using the definition of convolution and a change of variable, we have

$$\begin{aligned} \mathrm{rect} * \mathrm{rect}(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{rect}(x-y)\, dy &= \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \mathrm{rect}(y)\, dy \\ &= \begin{cases} 0, & \text{if } x \geq 1, \\ 1-x, & \text{if } 0 \leq x < 1, \\ 1+x, & \text{if } -1 \leq x < 0, \\ 0, & \text{if } x \leq -1, \end{cases} \end{aligned}$$

Observe that this function is indeed continuous. The convolution $\mathrm{rect} * \mathrm{rect}$ can be written in compact form

$$\mathrm{rect} * \mathrm{rect}(x) = \begin{cases} 0, & \text{if } |x| \geq 1, \\ 1-|x|, & \text{if } |x| < 1. \end{cases}$$

This function is called *triangular function*, we use the notation $x \mapsto \mathrm{tri}(x)$.

From the previous result, we get in particular that the convolution between $f \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$ is a continuous bounded function. We can define the convolution also for functions $f \in L^1_{\mathrm{loc}}$, by enforcing a bit the hypotheses on the second function g. Rather than giving the most general result, we give some particular cases which will be particularly useful.

Proposition 3.5.9 (A first regularization result)**.** *Let $f \in L^1_{\rm loc}(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$. Let us suppose that g has compact support, i.e. there exists a bounded closed interval $[a,b]$ such that*

$$|g| = 0, \qquad \textit{for a. e. } x \in \mathbb{R} \setminus [a,b].$$

Then the convolution $f * g$ is well-defined and is a continuous function. Moreover, we have the estimate

$$|f * g(x)| \le \|g\|_{L^\infty(\mathbb{R})} \int_{x-b}^{x-a} |f|\, dy, \qquad \textit{for a. e. } x \in \mathbb{R}. \tag{3.5.3}$$

Proof. We first show that the convolution is well-defined. By definition of convolution and thanks to the hypothesis on g, for almost every $x \in \mathbb{R}$ we have

$$|f(x-y)\, g(y)| \le \|g\|_{L^\infty(\mathbb{R})}\, |f(x-y)|\, 1_{[a,b]}(y), \qquad y \in \mathbb{R},$$

and the last function is in $L^1(\mathbb{R})$, since f is locally integrable. This also shows the validity of the estimate (3.5.3).

Let $x \in \mathbb{R}$, we want to show that

$$\lim_{h \to 0} |f * g(x+h) - f * g(x)| = 0.$$

For every $|h| < (b-a)$ we have

$$\begin{aligned}
|f * g(x+h) - f * g(x)| &= \left| \int_{\mathbb{R}} f(x+h-y)\, g(y)\, dy - \int_{\mathbb{R}} f(x-y)\, g(y)\, dy \right| \\
&= \left| \int_{\mathbb{R}} \Big(f(x+h-y) - f(x-y) \Big)\, g(y)\, dy \right| \\
&\le \int_{\mathbb{R}} \Big| f(x+h-y) - f(x-y) \Big|\, |g(y)|\, dy \\
&\le \|g\|_{L^\infty(\mathbb{R})} \int_a^b \Big| f(x+h-y) - f(x-y) \Big|\, dy.
\end{aligned}$$

With a simple change of variable $x - y = t$, this gives

$$|f * g(x+h) - f * g(x)| \le \|g\|_{L^\infty(\mathbb{R})} \int_{x-b}^{x-a} \Big| f(t+h) - f(t) \Big|\, dt.$$

We now introduce the new function $F(t) = f(t)\, 1_{[x+a-2\,b, x+b-2\,a]}(t)$, this is in $L^1(\mathbb{R})$ by hypothesis. Observe that by construction

$$[x-b-h, x-a-h] \subset [x+a-2\,b, x+b-2\,a], \qquad \text{for every } |h| < b-a,$$

thus in particular for every $|h| < b-a$ we get

$$F(t+h) - F(t) = f(t+h) - f(t), \qquad t \in [x-b, x-a].$$

Thus we get

$$\int_{x-b}^{x-a} \Big| f(t+h) - f(t) \Big|\, dt = \int_{x-b}^{x-a} \Big| F(t+h) - F(t) \Big|\, dt \le \int_{\mathbb{R}} \Big| F(t+h) - F(t) \Big|\, dt.$$

Thus in particular we obtained

$$|f * g(x+h) - f * g(x)| \le \|g\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \Big| F(t+h) - F(t) \Big|\, dt.$$

If we now pass to the limit as h goes to 0 and use Proposition 3.4.5 in the right-hand side, we get the conclusion. $\square$

Remark 3.5.10. Under the assumptions of Proposition 3.5.9, the convolution $f * g$ in general is not in $L^p(\mathbb{R})$, for any $1 \le p \le \infty$. Indeed, if we take $f(x) = x$ and $g(x) = \mathrm{rect}(x)$, then we have

$$\begin{aligned} f * g(x) = \int_{\mathbb{R}} f(x-y)\, g(y)\, dy &= \int_{-\frac{1}{2}}^{\frac{1}{2}} (x-y)\, dy \\ &= \frac{1}{2}\, \left[-(x-y)^2\right]_{-\frac{1}{2}}^{\frac{1}{2}} \\ &= \frac{1}{2}\left[\left(x+\frac{1}{2}\right)^2 - \left(x-\frac{1}{2}\right)^2\right] = x \notin L^p(\mathbb{R}). \end{aligned}$$

Proposition 3.5.11. *Let $f \in L^1_{loc}(\mathbb{R})$ and let $g \in C^1_0(\mathbb{R})$. Then the convolution $f * g$ is a C^1 function. Moreover, we have*

$$\frac{d}{dx}(f * g) = f * \frac{d}{dx} g. \tag{3.5.4}$$

Proof. By the previous result, we already know that $f * g$ is well-defined and is continuous (indeed, observe that g has compact support and $C^1_0(\mathbb{R}) \subset L^\infty(\mathbb{R})$). We only need to show that $f * g$ is derivable and formula (3.5.4) holds, then continuity of its derivative will follow again from Proposition 3.5.9, since g' is $L^\infty(\mathbb{R})$ with compact support.

Let $x \in \mathbb{R}$, for every $|h| < 1$ we have

$$\frac{f * g(x+h) - f * g(x)}{h} = \int_{\mathbb{R}} f(y)\, \frac{g(x+h-y) - g(x-y)}{h}\, dy.$$

We have

$$\lim_{h \to 0} \frac{g(x+h-y) - g(x-y)}{h} = g'(x-y),$$

in order to pass the limit under the integral sign, we need to find a domination with an L^1 function. We first observe that by the Mean Value Theorem[3]

$$\left| \frac{g(x+h-y) - g(x-y)}{h} \right| = |g'(\xi)| \le \|g'\|_{L^\infty(\mathbb{R})},$$

where ξ in a point belonging to interval $(x-y, x-y+h)$. Moreover, if g is supported in $[a, b]$, for every $|h| < 1$ the function

$$\frac{g(x+h-y) - g(x-y)}{h},$$

has compact support contained in $[x-1-b, x+1-a]$. In conclusion, for every $|h| < 1$ we get

$$\left| f(y)\, \frac{g(x+h-y) - g(x-y)}{h} \right| \le \|g'\|_{L^\infty(\mathbb{R})}\, |f(y)|\, 1_{[x-1-b, x+1-a]} \in L^1(\mathbb{R}).$$

We can apply Lebesgue Dominated Convergence Theorem and obtain

$$\lim_{h \to 0} \frac{f * g(x+h) - f * g(x)}{h} = \int_{\mathbb{R}} f(y)\, g'(x-y)\, dy = f * g'(x).$$

This shows at the same time that $f * g$ is derivable and that formula (3.5.4) holds. $\square$

[3] In italian "*Teorema di Lagrange*".

By iterating the previous result, we get the following.

Corollary 3.5.12. *Let $f \in L^1_{loc}(\mathbb{R})$ and let $g \in C^k_0(\mathbb{R})$ for some $k \geq 1$. Then the convolution $f * g$ is a C^k function. Moreover, we have*

$$\frac{d^m}{dx^m}(f * g) = f * \frac{d^m}{dx^m} g, \qquad \text{for every } m = 1, \dots, k.$$

Theorem 3.5.13 (Smooth approximations by convolution)**.** *Let $k \geq 1$ and let $g \in C^k_0(\mathbb{R})$ be a function such that $\int_{\mathbb{R}} g\, dx = 1$. For every $\varepsilon > 0$, we define*

$$g_\varepsilon(x) = \frac{1}{\varepsilon}\, g\left(\frac{x}{\varepsilon}\right).$$

If $f \in L^p(\mathbb{R})$ for some $1 \leq p < \infty$, we have $f_\varepsilon = f * g_\varepsilon \in C^k(\mathbb{R}) \cap L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and

$$\lim_{\varepsilon \to 0^+} \|f_\varepsilon - f\|_{L^p(\mathbb{R})} = 0.$$

Proof. Since $L^p(\mathbb{R}) \subset L^1_{\mathrm{loc}}(\mathbb{R})$, from the previous Corollary we already know that $f_\varepsilon \in C^k(\mathbb{R})$. Moreover, since

$$g_\varepsilon \in C^k_0(\mathbb{R}) \subset L^1(\mathbb{R}) \cap L^{p'}(\mathbb{R}),$$

we can apply Propositions 3.5.4 and 3.5.6 and get $f_\varepsilon \in L^\infty(\mathbb{R}) \cap L^p(\mathbb{R})$ as well, with

$$\|f_\varepsilon\|_{L^p(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})}\, \|g_\varepsilon\|_{L^1(\mathbb{R})},$$

and

$$\|f_\varepsilon\|_{L^\infty(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})}\, \|g_\varepsilon\|_{L^{p'}(\mathbb{R})}.$$

We now compute

$$\begin{aligned}
\|f_\varepsilon - f\|^p_{L^p(\mathbb{R})} &= \int_{\mathbb{R}} |f * g_\varepsilon - f|^p\, dx = \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(x-y) g_\varepsilon(y)\, dy - f(x) \right|^p dx \\
&= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \Big[f(x-y) - f(x) \Big] g_\varepsilon(y)\, dy \right|^p dx \\
&\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \Big| f(x-y) - f(x) \Big|\, |g_\varepsilon(y)|\, dy \right)^p dx
\end{aligned}$$

where we used that

$$\int_{\mathbb{R}} g_\varepsilon(y)\, dy = \frac{1}{\varepsilon} \int_{\mathbb{R}} g\left(\frac{y}{\varepsilon}\right)\, dy = \int_{\mathbb{R}} g\, dy = 1.$$

Moreover, by using Hölder's inequality

$$\begin{aligned}
\int_{\mathbb{R}} \Big| f(x-y) - f(x) \Big|\, |g_\varepsilon(y)|\, dy &= \int_{\mathbb{R}} \Big| f(x-y) - f(x) \Big|\, |g_\varepsilon(y)|^{\frac{1}{p}}\, |g_\varepsilon(y)|^{\frac{1}{p'}}\, dy \\
&\leq \|g_\varepsilon\|^{\frac{1}{p'}}_{L^1(\mathbb{R})} \left(\int_{\mathbb{R}} \Big| f(x-y) - f(x) \Big|^p\, |g_\varepsilon(y)|\, dy \right)^{\frac{1}{p}}.
\end{aligned}$$

By observing that

$$\|g_\varepsilon\|_{L^1(\mathbb{R})} = \|g\|_{L^1(\mathbb{R})}$$

we thus obtain

$$\begin{aligned}
\|f_\varepsilon - f\|^p_{L^p(\mathbb{R})} &\leq \|g\|^{p-1}_{L^1(\mathbb{R})} \int_{\mathbb{R}} \int_{\mathbb{R}} \Big| f(x-y) - f(x) \Big|^p\, |g_\varepsilon(y)|\, dy\, dx \\
&= \|g\|^{p-1}_{L^1(\mathbb{R})} \int_{\mathbb{R}} \|\mathcal{T}_{-\varepsilon\, t}\, f - f\|^p_{L^p(\mathbb{R})}\, |g(t)|\, dt.
\end{aligned}$$

Finally, by Proposition 3.4.5 we know that

$$\lim_{\varepsilon\to 0} \|\mathcal{T}_{-\varepsilon\, t}\, f - f\|^p_{L^p(\mathbb{R})} = 0,$$

and

$$\|\mathcal{T}_{-\varepsilon\, t}\, f - f\|^p_{L^p(\mathbb{R})}\, |g(t)| \le 2^p\, \|f\|^p_{L^p(\mathbb{R})}\, |g(t)| \in L^1(\mathbb{R}).$$

We thus conclude by applying Lebesgue Dominated Convergence Theorem and taking the limit under the integral sign. □

Remark 3.5.14. Of course, if in the previous Theorem we take $g \in C^\infty_0(\mathbb{R})$, then $f_\varepsilon = f * g_\varepsilon$ is C^∞ as well. An important instance of function $g \in C^\infty_0(\mathbb{R})$ which is used very often is the *standard mollifier*

$$g(x) = \left\{ \begin{array}{rl} c\, \exp\left(-\dfrac{1}{1-x^2}\right), & \text{if } |x| < 1, \\ 0, & \text{otherwise,} \end{array} \right.$$

where the constant $c > 0$ is chosen so that $\int_{\mathbb{R}} g\, dx = 1$. Observe that $g \in C^\infty_0(\mathbb{R})$.

The following simple result will be useful in the sequel.

Lemma 3.5.15. *Let us that $f, g \in L^1(\mathbb{R})$ both have compact support. Then $f * g$ has compact support as well.*

Proof. Let us assume that

$$|f(x)| = 0, \qquad \text{for a. e. } x \in \mathbb{R} \setminus [a, b],$$

and

$$|g(x)| = 0, \qquad \text{for a. e. } x \in \mathbb{R} \setminus [c, d].$$

By definition of convolution, we have

$$f * g(x) = \int_{\mathbb{R}} f(x-y)\, g(y)\, dy = \int_c^d f(x-y)\, g(y)\, dy.$$

We now observe that

$$|f(x-y)| = 0, \qquad \text{for a. e. } y \in \mathbb{R} \setminus [x-b, x-a].$$

This implies that for every $x \in \mathbb{R}$ such that

$$x - a \le c \qquad \text{or} \qquad x - b \ge d,$$

we have $f(x-y) = 0$ for almost every $y \in [c, d]$ and thus

$$f * g(x) = \int_{\mathbb{R}} f(x-y)\, g(y)\, dy = \int_c^d f(x-y)\, g(y)\, dy = 0,$$

in this case. Thus we proved that

$$|f * g(x)| = 0, \qquad \text{for a. e. } x \le c + a \quad \text{or} \quad x \ge b + d.$$

In other words, $f * g$ vanishes almost everywhere in $\mathbb{R} \setminus [a + c, b + d]$. □

6. Exercises

Exercise 3.6.1. *Let us take $a < b$ and consider the generalized rectangular function $1_{[a,b]}$. Compute the convolution $1_{[a,b]} * 1_{[a,b]}$ and verify that we have*

$$1_{[a,b]} * 1_{[a,b]}(x) = (b-a)\,\mathrm{tri}\left(\frac{x-a-b}{b-a}\right). \tag{3.6.1}$$

Solution. In order to do this, we first observe that

$$1_{[a,b]}(x) = \mathrm{rect}\left(\frac{x-a}{b-a} - \frac{1}{2}\right), \tag{3.6.2}$$

thus we obtain

$$\begin{aligned} 1_{[a,b]} * 1_{[a,b]}(x) &= \int_{\mathbb{R}} 1_{[a,b]}(x-y)\,1_{[a,b]}(y)\,dy \\ &= \int_{\mathbb{R}} \mathrm{rect}\left(\frac{x-y-a}{b-a} - \frac{1}{2}\right) \mathrm{rect}\left(\frac{y-a}{b-a} - \frac{1}{2}\right) dy \\ &= \int_{\mathbb{R}} \mathrm{rect}\left(\frac{x-2\,a}{b-a} - 1 - \left(\frac{y-a}{b-a} + \frac{1}{2}\right)\right) \mathrm{rect}\left(\frac{y-a}{b-a} - \frac{1}{2}\right) dy. \end{aligned}$$

If we now perform the change of variable

$$y' = \frac{y-a}{b-a} - \frac{1}{2},$$

the previous chain of identities gives

$$\begin{aligned} 1_{[a,b]} * 1_{[a,b]}(x) &= (b-a) \int_{\mathbb{R}} \mathrm{rect}\left(\frac{x-2\,a}{b-a} - 1 - y'\right) \mathrm{rect}\left(y'\right)\,dy' \\ &= (b-a)\,\mathrm{rect} * \mathrm{rect}\left(\frac{x-a-b}{b-a}\right) \\ &= (b-a)\,\mathrm{tri}\left(\frac{x-a-b}{b-a}\right), \end{aligned}$$

where in the last identity we used Example 3.5.8. □

Remark 3.6.2. For example, if in (3.6.1) we take $a = 0$ and $b > 0$, we get

$$1_{[0,b]} * 1_{[0,b]}(x) = b\,\mathrm{tri}\left(\frac{x-b}{b}\right) = \begin{cases} b - |x-b|, & \text{if } 0 < x < 2\,b. \\ 0, & \text{otherwise.} \end{cases}$$

If instead $a = -L$ and $b = L$, then

$$1_{[-L,L]} * 1_{[-L,L]}(x) = 2\,L\,\mathrm{tri}\left(\frac{x}{2\,L}\right).$$

Exercise 3.6.3. *Let $a < b$ and $c < d$ be real numbers. Generalize the previous exercise and compute the convolution $1_{[a,b]} * 1_{[c,d]}$.*

Exercise 3.6.4. *Compute the convolution* $\mathrm{tri} * \mathrm{tri}$.

Proof. By using that tri vanishes outside the interval $[-1, 1]$, it is easily seen that for $|x| > 2$ we have

$$\mathrm{tri}(x-t)\,\mathrm{tri}(t) = 0, \qquad \text{for every } t \in \mathbb{R}.$$

Thus we have

$$\mathrm{tri} * \mathrm{tri}(x) = 0, \qquad \text{for every } |x| > 2.$$

We also observe that by using $\mathrm{tri}(-t) = \mathrm{tri}(t)$, we have

$$\begin{aligned}\mathrm{tri} * \mathrm{tri}(-x) = \int_{\mathbb{R}} \mathrm{tri}(-x-t)\,\mathrm{tri}(t)\,dt &= \int_{\mathbb{R}} \mathrm{tri}(x+t)\,\mathrm{tri}(-t)\,dt \\ &= \int_{\mathbb{R}} \mathrm{tri}(x-s)\,\mathrm{tri}(s)\,ds = \mathrm{tri} * \mathrm{tri}(x),\end{aligned}$$

where we used the change of variable $-t = s$. The last identity shows that $\mathrm{tri} * \mathrm{tri}$ is an even function. Finally, we take $x \in [0,2]$, observe that

$$\mathrm{tri}(x-t) \neq 0 \qquad \Longleftrightarrow \qquad |x-t| < 1 \qquad \Longleftrightarrow \qquad t \in [-1+x, 1+x]$$

and compute the convolution:

$$\begin{aligned}\mathrm{tri} * \mathrm{tri}(x) = \int_{\mathbb{R}} \mathrm{tri}(x-t)\,\mathrm{tri}(t)\,dt &= \int_{[-1,1]\cap[-1+x,1+x]} (1-|x-t|)\,(1-|t|)\,dt \\ &= \int_{-1+x}^{1} (1-|x-t|)\,(1-|t|)\,dt.\end{aligned}$$

We now distinguish two cases: $x \in [0,1]$ and $x \in [1,2]$. In the first case, we have $-1+x \le 0$, thus

$$\begin{aligned}\int_{-1+x}^{1} (1-|x-t|)\,(1-|t|)\,dt &= \int_{-1+x}^{0} (1-x+t)\,(1+t)\,dt + \int_{0}^{x} (1-x+t)\,(1-t)\,dt \\ &\quad + \int_{x}^{1} (1-t+x)\,(1-t)\,dt \\ &= \left[(1-x)\,t + (2-x)\,\frac{t^2}{2} + \frac{t^3}{3}\right]_{-1+x}^{0} + \left[(1-x)\,t + x\,\frac{t^2}{2} - \frac{t^3}{3}\right]_{0}^{x} \\ &\quad + \left[(1+x)\,t - (2+x)\,\frac{t^2}{2} + \frac{t^3}{3}\right]_{x}^{1} \\ &= (1-x)^2 - (2-x)\,\frac{(x-1)^2}{2} - \frac{(x-1)^3}{3} + (1-x)\,x + \frac{x^3}{2} - \frac{x^3}{3} \\ &\quad + (1+x) - \frac{(2+x)}{2} + \frac{1}{3} - (1+x)\,x + (2+x)\,\frac{x^2}{2} - \frac{x^3}{3}.\end{aligned}$$

After some simplifications, we get

$$\mathrm{tri} * \mathrm{tri}(x) = \frac{2}{3} - x^2 + \frac{x^3}{2}, \qquad \text{for } x \in [0,1].$$

In the second case, we have $1 \ge -1+x \ge 0$ and thus

$$\begin{aligned}\int_{-1+x}^{1} (1-|x-t|)\,(1-|t|)\,dt &= \int_{-1+x}^{1} (1-x+t)\,(1-t)\,dt \\ &= \left[(1-x)\,t + x\,\frac{t^2}{2} - \frac{t^3}{3}\right]_{-1+x}^{1} \\ &= (1-x) + \frac{x}{2} - \frac{1}{3} + (x-1)^2 - x\,\frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}.\end{aligned}$$

With some algebraic manipulations, we then get

$$\mathrm{tri} * \mathrm{tri}(x) = \frac{4}{3} - 2\,x + x^2 - \frac{x^3}{6}, \qquad \text{for } x \in [1,2].$$

By putting all the informations together, we thus finally found

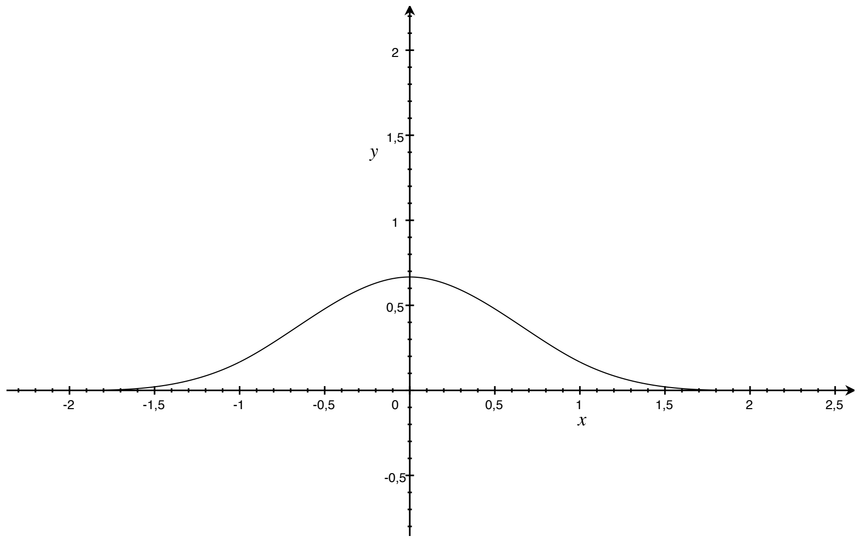

Figure 3. The graph of the convolution $\mathrm{tri} * \mathrm{tri}$

$$\mathrm{tri} * \mathrm{tri}(x) = \begin{cases} \dfrac{2}{3} - x^2 + \dfrac{|x|^3}{2}, & \text{if } |x| \le 1, \\[2ex] \dfrac{4}{3} - 2\,|x| + x^2 - \dfrac{|x|^3}{6}, & \text{if } 1 \le |x| \le 2, \\[2ex] 0, & \text{otherwise.} \end{cases}$$

This concludes the exercise. □

Exercise 3.6.5. *Let H be the Heaviside function and $g(x) = e^{-|x|}$, justify that the convolution $H * g$ is well-defined and prove that*

$$H * g(x) = \begin{cases} e^x, & \textit{if } x < 0, \\ 2 - e^{-x}, & \textit{if } x \ge 0. \end{cases}$$

Solution. We observe that $H \in L^\infty(\mathbb{R})$ and $g \in L^1(\mathbb{R})$, since

$$\int_{\mathbb{R}} e^{-|x|}\, dx = 2 \int_0^{+\infty} e^{-x}\, dx = 2 < +\infty.$$

Then $H * g$ is well-defined and $H * g \in L^\infty(\mathbb{R})$, by Proposition 3.5.6. We now compute the convolution: we have

$$H * g(x) = \int_{\mathbb{R}} H(x-y)\, g(y)\, dy = \int_{-\infty}^{x} e^{-|y|}\, dy,$$

thus if $x < 0$

$$\int_{-\infty}^{x} e^{-|y|}\, dy = \int_{-\infty}^{x} e^{y}\, dy = e^x,$$

while for $x \ge 0$

$$\int_{-\infty}^{x} e^{-|y|}\, dy = \int_{-\infty}^{0} e^{y}\, dy + \int_0^x e^{-y}\, dy = 1 + (-e^{-x} + 1) = 2 - e^{-x},$$

as desired. □

7. Advanced exercises

Exercise 3.7.1. *Let $a < b$ be two real numbers and let $f \in L^\infty([a,b])$. Prove that*

$$\lim_{p \nearrow +\infty} \|f\|_{L^p([a,b])} = \|f\|_{L^\infty([a,b])}.$$

Solution. We first observe that

$$\|f\|_{L^p([a,b])} \le (b-a)^{\frac{1}{p}} \, \|f\|_{L^\infty([a,b])},$$

which follows from Proposition 3.3.10, with $q = +\infty$. This implies that

$$\limsup_{p \to +\infty} \|f\|_{L^p([a,b])} \le \|f\|_{L^\infty([a,b])} \, \limsup_{p \to +\infty} (b-a)^{\frac{1}{p}} = \|f\|_{L^\infty([a,b])}.$$

On the other hand, by definition of L^∞ norm, for every $\varepsilon > 0$ the set

$$E_\varepsilon = \Big\{ x \in [a,b] \,:\, |f(x)| \ge \|f\|_{L^\infty([a,b])} - \varepsilon \Big\},$$

has positive measure. Thus we get

$$\begin{aligned} \|f\|_{L^p([a,b])} = \left(\int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}} &\ge \left(\int_{E_\varepsilon} |f(x)|^p \, dx \right)^{\frac{1}{p}} \\ &\ge \Big(\|f\|_{L^\infty([a,b])} - \varepsilon \Big) \left(\int_{E_\varepsilon} dx \right)^{\frac{1}{p}} \\ &= |E_\varepsilon|^{\frac{1}{p}} \, \Big(\|f\|_{L^\infty([a,b])} - \varepsilon \Big). \end{aligned}$$

This implies

$$\begin{aligned} \liminf_{p \to +\infty} \|f\|_{L^p([a,b])} &\ge \Big(\|f\|_{L^\infty([a,b])} - \varepsilon \Big) \liminf_{p \to +\infty} |E_\varepsilon|^{\frac{1}{p}} \\ &= \|f\|_{L^\infty([a,b])} - \varepsilon. \end{aligned}$$

By arbitrariness of $\varepsilon > 0$, we obtain

$$\liminf_{p \to +\infty} \|f\|_{L^p([a,b])} \ge \|f\|_{L^\infty([a,b])}.$$

This concludes the proof. □

Exercise 3.7.2 (Interpolation inequality)**.** *Let $f \in L^q(\mathbb{R}) \cap L^p(\mathbb{R})$, for $1 \le q < p \le +\infty$. Prove that $f \in L^r(\mathbb{R})$ for every $q < r < p$ and we have*

$$\|f\|_{L^r(\mathbb{R})} \le \|f\|_{L^p(\mathbb{R})}^{1-\vartheta} \, \|f\|_{L^q(\mathbb{R})}^{\vartheta}, \tag{3.7.1}$$

where the exponent $\vartheta \in (0,1)$ is given by

$$\vartheta = \begin{cases} \dfrac{q}{r} \, \dfrac{p-r}{p-q}, & \text{if } p < +\infty, \\[2ex] \dfrac{q}{r}, & \text{if } p = +\infty. \end{cases}$$

Solution. We first consider the case $p = +\infty$, which is simpler. In this case we have

$$\int_{\mathbb{R}} |f(x)|^r \, dx = \int_{\mathbb{R}} |f(x)|^{r-q} \, |f(x)|^q \, dx \le \|f\|_{L^\infty(\mathbb{R})}^{r-q} \int_{\mathbb{R}} |f(x)|^q \, dx,$$

which implies

$$\|f\|_{L^r(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(x)|^r\,dx\right)^{\frac{1}{r}} \le \|f\|_{L^\infty(\mathbb{R})}^{\frac{r-q}{r}} \left(\int_{\mathbb{R}} |f(x)|^q\,dx\right)^{\frac{1}{r}} = \|f\|_{L^\infty(\mathbb{R})}^{\frac{r-q}{r}}\,\|f\|_{L^q(\mathbb{R})}^{\frac{q}{r}}.$$

This proves (3.7.1) for $p = +\infty$.

We now suppose $p < +\infty$. We observe that if $q < r < p$ then there exists $\alpha \in (0,1)$ such that

$$r = q + \alpha\,(p - q).$$

With a simple computation, we find

$$\alpha = \frac{r-q}{p-q}.$$

We now write

$$\int_{\mathbb{R}} |f(x)|^r\,dx = \int_{\mathbb{R}} |f(x)|^{\alpha\,p}\,|f(x)|^{q\,(1-\alpha)}\,dx,$$

then we use Hölder inequality with conjugate exponents

$$\frac{1}{\alpha} \qquad \text{and} \qquad \frac{1}{1-\alpha}.$$

This gives

$$\int_{\mathbb{R}} |f(x)|^r\,dx \le \left(\int_{\mathbb{R}} |f(x)|^p\,dx\right)^{\alpha} \left(\int_{\mathbb{R}} |f(x)|^q\,dx\right)^{1-\alpha}.$$

By taking the power $1/r$ on both sides and recalling the definition of α, we get the desired conclusion (3.7.1) for $p < +\infty$, as well. □

Exercise 3.7.3. *Show that the function*

$$x \mapsto \frac{\sin(\pi\,x)}{\pi\,x},$$

does not belong to $L^1(\mathbb{R})$.

Solution. We show that

$$\lim_{k\to\infty} \int_{-k}^{k} \left|\frac{\sin(\pi\,x)}{\pi\,x}\right|\,dx = +\infty.$$

Since the integrand is even, this is the same as

$$\lim_{k\to\infty} \int_{0}^{k} \left|\frac{\sin(\pi\,x)}{\pi\,x}\right|\,dx = +\infty.$$

We now write for every $k \ge 1$

$$\int_{0}^{k} \left|\frac{\sin(\pi\,x)}{\pi\,x}\right|\,dx = \sum_{n=0}^{k} \int_{n-1}^{n} \left|\frac{\sin(\pi\,x)}{\pi\,x}\right|\,dx,$$

and observe that

$$\frac{1}{|\pi\,x|} \ge \frac{1}{\pi\,n}, \qquad \text{for } x \in [n-1, n].$$

Thus we get

$$\int_{0}^{k} \left|\frac{\sin(\pi\,x)}{\pi\,x}\right|\,dx \ge \frac{1}{\pi}\sum_{n=0}^{k} \frac{1}{n}\int_{n-1}^{n} |\sin(\pi\,x)|\,dx.$$

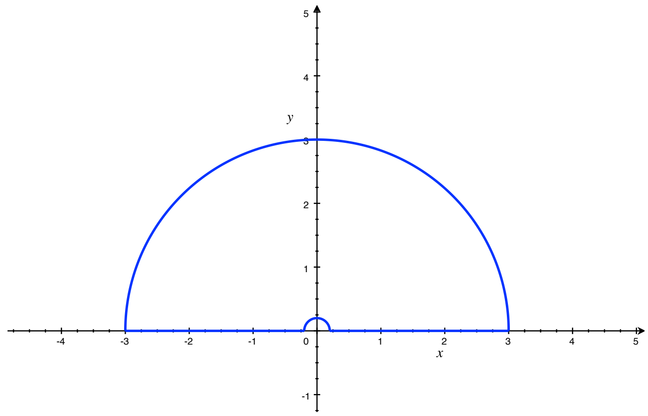

Figure 4. The positively oriented loop Γ_R of Exercise 3.7.4.

On the other hand, since the function $x \mapsto |\sin(\pi x)|$ is $\pi-$periodic, we have

$$\int_{n-1}^{n} |\sin(\pi x)|\, dx = \int_0^1 |\sin(\pi x)|\, dx = \int_0^1 \sin(\pi x)\, dx = \left[-\frac{\cos(\pi x)}{\pi}\right]_0^1 = \frac{2}{\pi}.$$

In conclusion, we get

$$\int_0^k \left|\frac{\sin(\pi x)}{\pi x}\right| dx \geq \frac{2}{\pi^2} \sum_{n=0}^{k} \frac{1}{n}.$$

By recalling that the harmonic series is divergent, we obtain

$$\lim_{k\to\infty} \int_0^k \left|\frac{\sin(\pi x)}{\pi x}\right| dx \geq \frac{2}{\pi^2} \lim_{k\to\infty} \sum_{n=0}^{k} \frac{1}{n} = +\infty,$$

as desired. □

Exercise 3.7.4. *Show that*

$$\int_{\mathbb{R}} \frac{\sin(\pi x)}{\pi x}\, dx = 1.$$

Solution. We first observe that if we change variable $\pi x = t$, we can equivalently prove that

$$\int_{\mathbb{R}} \frac{\sin t}{t}\, dx = \pi.$$

We fix $0 < \varepsilon \ll 1$ and $R \gg 1$. We consider the positively oriented piecewise regular loop Γ_R obtained by glueing the following regular simple curves

$$\gamma_1(t) = t, \qquad t \in [\varepsilon, R],$$

$$\gamma_2(t) = R\, e^{i\,t}, \qquad t \in [0, \pi],$$

$$\gamma_3(t) = t, \qquad t \in [-R, -\varepsilon],$$

$$\gamma_4(t) = -\varepsilon\, e^{-i\,t}, \qquad t \in [0, \pi],$$

We then consider the function of a complex variable $f(z) = e^{i\,z}/z$, which is holomorphic in $\mathbb{C}^*$, with a simple pole at $z = 0$. The region entoured by Γ_R does not contain the origin, thus by Cauchy's Theorem (see Theorem 1.6.12) we have

$$
\begin{aligned}
0 = \int_{\Gamma_R} \frac{e^{i\,z}}{z}\,dz = \int_\varepsilon^R \frac{e^{i\,t}}{t}\,dt + i \int_0^\pi e^{i\,R\,e^{i\,t}}\,dt \\
+ \int_{-R}^{-\varepsilon} \frac{e^{i\,t}}{t}\,dt - i \int_0^\pi e^{-i\,\varepsilon\,e^{-i\,t}}\,dt.
\end{aligned} \tag{3.7.2}
$$

We now recall that $e^{i\,t} = \cos t + i\,\sin t$, thus the first and third integral above give

$$
\begin{aligned}
\int_\varepsilon^R \frac{e^{i\,t}}{t}\,dt + \int_{-R}^{-\varepsilon} \frac{e^{i\,t}}{t}\,dt &= \int_\varepsilon^R \frac{\cos t}{t}\,dt + i \int_\varepsilon^R \frac{\sin t}{t}\,dt \\
&\quad + \int_{-R}^{-\varepsilon} \frac{\cos t}{t}\,dt + i \int_{-R}^{-\varepsilon} \frac{\sin t}{t}\,dt \\
&= 2\,i \int_\varepsilon^R \frac{\sin t}{t}\,dt,
\end{aligned}
$$

thanks to the fact that $\sin t/t$ is even, while $\cos t/t$ is odd, thus we have a cancellation. From (3.7.2) we thus obtained

$$
\begin{aligned}
2 \int_\varepsilon^R \frac{\sin t}{t}\,dt &= -\int_0^\pi e^{i\,R\,e^{i\,t}}\,dt + \int_0^\pi e^{-i\,\varepsilon\,e^{-i\,t}}\,dt \\
&= -\int_0^\pi e^{i\,R\,\cos t}\,e^{-R\,\sin t}\,dt + \int_0^\pi e^{-i\,\varepsilon\,\cos t}\,e^{-\varepsilon\,t}\,dt.
\end{aligned}
$$

We now pass to the limit as ε goes to 0. Observe that

$$
|e^{-i\,\varepsilon\,\cos t}\,e^{-\varepsilon\,t}| = e^{-\varepsilon\,t} \le 1, \qquad t \in [0, \pi]
$$

and

$$
\lim_{\varepsilon \to 0} e^{-i\,\varepsilon\,\cos t}\,e^{-\varepsilon\,t} = 1, \qquad \text{for } t \in [0, 1],
$$

thus by using Lebesgue Dominated Convergence Theorem we obtain

$$
2 \int_0^R \frac{\sin t}{t}\,dt = -\int_0^\pi e^{i\,R\,\cos t}\,e^{-R\,\sin t}\,dt + \pi. \tag{3.7.3}
$$

Finally, we want to take the limit as R goes to $+\infty$. Observe that

$$
\left| \int_0^\pi e^{i\,R\,\cos t}\,e^{-R\,\sin t}\,dt \right| \le \int_0^\pi e^{-R\,\sin t}\,dt,
$$

and

$$
\lim_{R \to \infty} e^{-R\,\sin t} = 0, \qquad \text{for a. e. } t \in [0, \pi],
$$

$$
e^{-R\,\sin t} \le 1, \qquad t \in [0, \pi].
$$

Thus again by Lebesgue Dominated Convergence Theorem

$$
\lim_{R \to +\infty} \left| \int_0^\pi e^{i\,R\,\cos t}\,e^{-R\,\sin t}\,dt \right| \le \lim_{R \to +\infty} \int_0^\pi e^{-R\,\sin t}\,dt = 0.
$$

By (3.7.3) we thus get

$$
\lim_{R \to +\infty} \int_0^R \frac{\sin t}{t}\,dt = \frac{\pi}{2}.
$$

By recalling that $\sin t/t$ is an even function, we get the desired conclusion. $\square$

Remark 3.7.5. In the previous exercise the use of the Lebesgue Dominated Convergence Theorem could be avoided. We prefer to use it, in order to shorten the presentation.

Exercise 3.7.6. *Let $f \in L^\infty(\mathbb{R})$ be a compactly supported function. Show that for every $\alpha < 0$, there exists $C_\alpha > 0$ such that*

$$|f(t)| \le C_\alpha\, e^{\alpha\,|t|}, \qquad \text{for every } t \in \mathbb{R}.$$

Solution. In order to show the claimed estimate, we set

$$M = \|f\|_{L^\infty(\mathbb{R})},$$

and we suppose that

$$|f(t)| = 0, \qquad \text{for a. e. } t \in \mathbb{R} \setminus [-T, T].$$

We then observe that by definition

$$|f(t)| \le M, \qquad \text{for a. e. } t \in [0, T].$$

On the other hand, by using that for $\alpha < 0$ the function $t \mapsto e^{\alpha\, t}$ is decreasing, we have

$$e^{\alpha\, t} \ge e^{\alpha\, T}, \qquad \text{for } t \in [0, T].$$

We then obtain

$$|f(t)| \le M = \frac{M}{e^{\alpha\, T}}\, e^{\alpha\, T} \le \left(\frac{M}{e^{\alpha\, T}}\right) e^{\alpha\, t}, \qquad \text{for a. e. } t \in [0, T].$$

By recalling that $|f(t)| = 0$ for $t > T$, we then conclude that

$$|f(t)| \le \left(\frac{M}{e^{\alpha\, T}}\right) e^{\alpha\, t}, \qquad \text{for a. e. } t \ge 0. \tag{3.7.4}$$

We are left to prove the upper bound for $t \le 0$. However, this is similar: indeed, for $\alpha < 0$ the function $t \mapsto e^{-\alpha\, t}$ is increasing, thus we get

$$e^{-\alpha\, t} \ge e^{\alpha\, T}, \qquad \text{for } t \in [-T, 0].$$

As above, this entails

$$|f(t)| \le M = \frac{M}{e^{\alpha\, T}}\, e^{\alpha\, T} \le \left(\frac{M}{e^{\alpha\, T}}\right) e^{-\alpha\, t}, \qquad \text{for a. e. } t \in [-T, 0]$$

By recalling that $|f(t)| = 0$ for $t < -T$, we then conclude that

$$|f(t)| \le \left(\frac{M}{e^{\alpha\, T}}\right) e^{-\alpha\, t}, \qquad \text{for a. e. } t \le 0. \tag{3.7.5}$$

By keeping together (3.7.4), (3.7.5) and defining $C_\alpha = M/e^{\alpha\,, T}$, we finally get the desired estimate. □

Exercise 3.7.7 (The Kallman-Rota inequality)**.** *Let $1 \le p \le +\infty$ and let $f \in C^2(\mathbb{R})$ be such that $f, f'' \in L^p(\mathbb{R})$. Prove that we have $f' \in L^p(\mathbb{R})$, as well. Moreover, show that we have the inequality*

$$\|f'\|_{L^p(\mathbb{R})} \le 2\,\sqrt{\|f\|_{L^p(\mathbb{R})}\,\|f''\|_{L^p(\mathbb{R})}}. \tag{3.7.6}$$

Solution. Let $s > 0$ and $t \in \mathbb{R}$, by using an integration by parts we have

$$\begin{aligned}\int_0^s (s - \tau)\, f''(t + \tau)\, d\tau &= \Big[(s - \tau)\, f'(t + \tau)\Big]_0^s + \int_0^s f'(t + \tau)\, d\tau \\ &= -s\, f'(t) + f(t + s) - f(t).\end{aligned}$$

This identity can be rewritten as

$$s\,f'(t) = f(t+s) - f(t) - \int_0^s (s-\tau)\,f''(t+\tau)\,d\tau,$$

that is

$$s\,f'(t) = \mathcal{T}_s f(t) - f(t) - \int_0^s (s-\tau)\,\mathcal{T}_\tau f''(t)\,d\tau.$$

As always we denote by $\mathcal{T}_h$ the translation operator. We now take the L^p norm (with respect to the variable t) on both sides and use *Minkowski inequality*, so to get

$$s\,\|f'\|_{L^p(\mathbb{R})} \le \|\mathcal{T}_s f\|_{L^p(\mathbb{R})} + \|f\|_{L^p(\mathbb{R})} + \int_0^s (s-\tau)\,\|\mathcal{T}_\tau f''\|_{L^p(\mathbb{R})}\,d\tau. \tag{3.7.7}$$

This already shows that $f' \in L^p(\mathbb{R})$.

We now prove inequality (3.7.6). By using the definition of translation operator and a simple change of variable, it is not difficult to see that

$$\|\mathcal{T}_s f\|_{L^p(\mathbb{R})} = \|f\|_{L^p(\mathbb{R})} \qquad \text{and} \qquad \|\mathcal{T}_\tau f''\|_{L^p(\mathbb{R})} = \|f''\|_{L^p(\mathbb{R})}.$$

Thus from (3.7.7) we get

$$s\,\|f'\|_{L^p(\mathbb{R})} \le 2\,\|f\|_{L^p(\mathbb{R})} + \|f''\|_{L^p(\mathbb{R})} \int_0^s (s-\tau)\;d\tau = 2\,\|f\|_{L^p(\mathbb{R})} + \frac{s^2}{2}\,\|f''\|_{L^p(\mathbb{R})}.$$

This in particular gives

$$s\,\|f'\|_{L^p(\mathbb{R})} - \frac{s^2}{2}\,\|f''\|_{L^p(\mathbb{R})} \le 2\,\|f\|_{L^p(\mathbb{R})}, \tag{3.7.8}$$

which is valid for every $s > 0$. We now observe that the function

$$s \mapsto s\,\|f'\|_{L^p(\mathbb{R})} - \frac{s^2}{2}\,\|f''\|_{L^p(\mathbb{R})},$$

is maximal for

$$s = \frac{\|f'\|_{L^p(\mathbb{R})}}{\|f''\|_{L^p(\mathbb{R})}}.$$

By making such a choice above in (3.7.8), we end up with

$$\frac{1}{2}\,\frac{\|f'\|^2_{L^p(\mathbb{R})}}{\|f''\|_{L^p(\mathbb{R})}} \le 2\,\|f\|_{L^p(\mathbb{R})}.$$

This finally gives the desired inequality (3.7.6), up to some simple algebraic manipulations. $\square$

Chapter 4

The Laplace Transform

1. Definition and first properties

In this chapter, we will use the following notation

$$\mathbb{R}_+ = [0, +\infty) \qquad \text{and} \qquad \mathbb{R}_- = (-\infty, 0).$$

We will also use repeatedly the following fact: by recalling formula (1.5.3), for every $z \in \mathbb{C}$ we have

$$|e^z| = e^{\mathrm{Re}(z)}.$$

We recall that this is just a plain consequence of the definition of complex exponential.

Definition 4.1.1. Let $f : \mathbb{R} \to \mathbb{C}$ be a *causal signal*, i.e. a measurable function such that

$$f(t) = 0, \qquad \text{for } t < 0.$$

We say that f is $L-$*transformable* if there exists $\alpha \in \mathbb{R}$ such that

$$e^{-\alpha\, t}\, f(t) \in L^1(\mathbb{R}_+).$$

In this case, we define its *Laplace transform* by

$$\mathcal{L}[f](z) := \int_0^{+\infty} e^{-z\, t}\, f(t)\, dt, \qquad z \in \mathbb{C} \text{ such that } \mathrm{Re}(z) \geq \alpha.$$

Remark 4.1.2. We observe that the definition is well-posed. Indeed, for every $z \in \mathbb{C}$ such that $\mathrm{Re}(z) \geq \alpha$, we have

$$|e^{-z\, t}\, f(t)| = |e^{-\mathrm{Re}(z)\, t}\, e^{-i\, \mathrm{Im}(z)\, t}\, f(t)| = e^{-\mathrm{Re}(z)\, t}\, |f(t)| \leq e^{-\alpha\, t}\, |f(t)|, \qquad \text{for } t \geq 0.$$

By observing that the last function is in $L^1(\mathbb{R}_+)$ by hypothesis, we then get $e^{-z\, t}\, f(t) \in L^1(\mathbb{R}_+)$ and thus $\mathcal{L}[f]$ is well-defined.

Definition 4.1.3. Let $f : \mathbb{R} \to \mathbb{C}$ be an $L-$transformable causal signal, we define

$$\sigma_f = \inf\{\alpha \in \mathbb{R} \,:\, e^{-\alpha\, t}\, f(t) \in L^1(\mathbb{R}_+)\}.$$

Then its Laplace transform is well-defined on the right half-plane

$$\{z \in \mathbb{C} \,:\, \mathrm{Re}(z) > \sigma_f\}.$$

The number σ_f is called *abscissa of convergence*. The axis

$$\{z \in \mathbb{C} \,:\, \mathrm{Re}(z) = \sigma_f\},$$

is also called *critical axis*.

Remark 4.1.4. It is not difficult to see that for an $L-$transformable causal signal f, we have

$$e^{-\alpha\, t}\, f \in L^1(\mathbb{R}_+), \qquad \text{for every } \alpha > \sigma_f. \tag{4.1.1}$$

Indeed, if $\alpha > \sigma_f$, we can take

$$\varepsilon = \frac{\alpha - \sigma_f}{2} > 0.$$

Then, by definition of infimum, we have that there exists $\alpha_\varepsilon < \sigma_f + \varepsilon$ such that

$$e^{-\alpha_\varepsilon\, t}\, f \in L^1(\mathbb{R}_+).$$

Observe that by construction, we have

$$\alpha_\varepsilon < \sigma_f + \varepsilon = \sigma_f + \frac{\alpha - \sigma_f}{2} = \frac{\alpha + \sigma_f}{2} < \alpha.$$

This implies that

$$\int_0^{+\infty} e^{-\alpha\, t}\, |f(t)|\, dt \le \int_0^{+\infty} e^{-\alpha_\varepsilon\, t}\, |f(t)|\, dt < +\infty,$$

and thus the claimed property (4.1.1).

Example 4.1.5 (Laplace transform of the Heaviside function)**.** Let us consider the causal signal given by the Heaviside function H. We observe

$$e^{-\alpha\, t}\, H(t) = \begin{cases} e^{-\alpha\, t}, & t \ge 0, \\ 0, & t < 0, \end{cases}$$

and this function is in $L^1(\mathbb{R}_+)$ if and only if $\alpha > 0$. Indeed, we have

$$\int_0^{+\infty} e^{-\alpha\, t}\, dt = \begin{cases} +\infty, & \text{if } \alpha \le 0, \\ 1/\alpha, & \text{if } \alpha > 0. \end{cases}$$

Thus the Heaviside function is $L-$transformable, with $\sigma_H = 0$. Its Laplace transform is thus the function of a complex variable defined on $\{z \in \mathbb{C} \,:\, \mathrm{Re}(z) > 0\}$ by

$$\mathcal{L}[H](z) = \int_0^{+\infty} e^{-z\, t}\, dt = \left[-\frac{e^{-z\, t}}{z}\right]_0^{+\infty} = \frac{1}{z}, \qquad \text{for } \mathrm{Re}(z) > 0.$$

Observe that in order to compute the last integral, we used that for $\mathrm{Re}(z) > 0$

$$\lim_{t\to\infty} \left|\frac{e^{-z\, t}}{z}\right| = \lim_{t\to\infty} \frac{e^{-\mathrm{Re}(z)\, t}}{|z|} = 0.$$

Remark 4.1.6 (Link with the $Z-$transform)**.** Let f be an $L-$transformable causal signal, with abscissa of convergence σ_f. Let us fix a time step $\tau > 0$ and consider the regular grid

$$\{0, \tau, 2\,\tau, \dots, n\,\tau, \dots\}.$$

We can imagine to discretize the integral defining the Laplace transform by using *Riemann integral sums*, i.e.

$$\int_0^{+\infty} e^{-z\, t}\, f(t)\, dt \simeq \sum_{n=0}^{\infty} e^{-z\, n\, \tau}\, f(n\, \tau)\, \tau, \qquad \mathrm{Re}(z) > \sigma_f.$$

If in the last sum we make the change of variable $e^{z\,\tau} = w \in \mathbb{C}$, we thus get

$$\int_0^{+\infty} e^{-z\,t}\, f(t)\, dt \simeq \tau \sum_{n=0}^{\infty} \frac{f(n\,\tau)}{w^n},$$

and the last expression is exactly the $Z-$transform of f with time step $\tau > 0$, in the complex variable w. Observe that the map

$$z \mapsto e^{z\,\tau} = w,$$

send the half-plane $\{z \in \mathbb{C} : \mathrm{Re}(z) > \sigma_f\}$ into the annular set $\{w \in \mathbb{C} : |w| > e^{\sigma_f\,\tau}\}$, by recalling property (1.5.5).

Lemma 4.1.7 (A necessary condition for transformability)**.** *Let f be a $L-$transformable causal signal. Then for every $T > 0$ we have $f \in L^1([0,T])$.*

Proof. By hypothesis, there exists $\alpha \in \mathbb{R}$ such that

$$\int_0^{+\infty} e^{-\alpha\,t}\, |f(t)|\, dt < +\infty.$$

In particular, we get

$$+\infty > \int_0^{+\infty} e^{-\alpha\,t}\, |f(t)|\, dt \geq \int_0^{T} e^{-\alpha\,t}\, |f(t)|\, dt \geq \min\left\{1, e^{-\alpha\,T}\right\} \int_0^{T} |f(t)|\, dt,$$

where we used that the real exponential is a monotone function. By observing that

$$\min\left\{1, e^{-\alpha\,T}\right\} > 0,$$

we finally obtain

$$\int_0^{T} |f(t)|\, dt < +\infty,$$

as desired. □

Example 4.1.8. For example, the causal signal

$$f(t) = \frac{1}{t}\, H(t),$$

is *not* $L-$transformable. Indeed, we have

$$\int_0^1 |f(t)|\, dt = \int_0^1 \frac{1}{t}\, dt = +\infty.$$

2. $L-$transformable signals

On the other hand, the condition of Lemma 4.1.7 is only a necessary one and does not guarantee that a causal signal with that property is $L-$transformable.

Example 4.2.1. Take the causal signal

$$f(t) = e^{t^2}\, H(t).$$

It is easy to see that this function belongs to $L^1([0,T])$ for every $T > 0$. However, for every $\alpha \in \mathbb{R}$ the function

$$e^{-\alpha\,t}\, f(t) = e^{t^2 - \alpha\,t}\, H(t),$$

is positive and such that

$$\lim_{t\to+\infty} e^{-\alpha\,t}\, f(t) = +\infty.$$

Thus $e^{-\alpha t} f \notin L^1(\mathbb{R}_+)$, for every $\alpha \in \mathbb{R}$. This shows that the set

$$\{\alpha \in \mathbb{R} \,:\, e^{-\alpha t} f \in L^1(\mathbb{R}_+)\},$$

is empty and f is not $L-$transformable.

Proposition 4.2.2 (A sufficient condition for transformability, I)**.** *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ be a causal signal having exponential order, i.e. such that there exists $C, T > 0$ and $\beta \in \mathbb{R}$ such that*

$$|f(t)| \le C\, e^{\beta t}, \qquad \text{for a. e. } t \ge T.$$

Then f is $L-$transformable and $\sigma_f \le \beta$. Moreover, we have the estimate

$$\Big|\mathcal{L}[f](z)\Big| \le \int_0^T e^{-\beta t}\, |f(t)|\, dt + \frac{C}{\mathrm{Re}(z) - \beta}\, e^{(\beta - \mathrm{Re}(z))\, T}, \qquad \text{for } \mathrm{Re}(z) > \beta. \tag{4.2.1}$$

Proof. We prove that

$$\text{if } \alpha > \beta, \text{ then } e^{-\alpha t} f \in L^1(\mathbb{R}_+). \tag{4.2.2}$$

Indeed, we first observe that for almost every $t \in [0, T]$ and every $\alpha \in \mathbb{R}$, we have

$$e^{-\alpha t} \le \max\Big\{1, e^{-\alpha T}\Big\}.$$

This entails that

$$\int_0^T e^{-\alpha t}\, |f(t)|\, dt \le \max\Big\{1, e^{-\alpha T}\Big\}\, \|f\|_{L^1([0,T])} < +\infty,$$

thanks to the fact that $f \in L^1_{\mathrm{loc}}(\mathbb{R})$. On the other hand, by using the assumption on f, for $\alpha > \beta$, we have

$$e^{-\alpha t}\, |f(t)| \le C\, e^{-\alpha t}\, e^{\beta t}, \qquad \text{for a. e. } t \ge T,$$

and the last function is in $L^1([T, +\infty))$, thanks to the fact that $\beta - \alpha < 0$. The last two estimates show (4.2.2), thus we get in particular that f is $L-$transformable.

In order to prove the estimate on the abscissa of convergence, it is sufficient to observe that (4.2.2) implies the following

$$\{\alpha \in \mathbb{R} \,:\, e^{-\alpha t} f \in L^1(\mathbb{R}_+)\} \supset (\beta, +\infty).$$

By taking the infimum of both sets we would get

$$\sigma_f = \inf\{\alpha \in \mathbb{R} \,:\, e^{\alpha t} f \in L^1(\mathbb{R}_+)\} \le \inf(\beta, +\infty) = \beta.$$

as desired.

Finally, we come to the proof of (4.2.1). For $\mathrm{Re}(z) > \beta$, we estimate

$$\begin{aligned}
\Big|\mathcal{L}[f](z)\Big| = \left|\int_0^{+\infty} e^{-z t}\, f(t)\, dt\right| &\le \int_0^{+\infty} e^{-\mathrm{Re}(z)\, t}\, |f(t)|\, dt \\
&= \int_0^T e^{-\mathrm{Re}(z)\, t}\, |f(t)|\, dt + \int_T^{+\infty} e^{-\mathrm{Re}(z)\, t}\, |f(t)|\, dt \\
&\le \int_0^T e^{-\beta t}\, |f(t)|\, dt + C \int_T^{+\infty} e^{(\beta - \mathrm{Re}(z))\, t}\, dt.
\end{aligned}$$

By computing the last two integrals, we get the desired estimate. □

Corollary 4.2.3 (Compactly supported causal signals)**.** *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ be a compactly supported causal signal. Then f is $L-$transformable and $\sigma_f = -\infty$, i. e. its Laplace transform is defined on the whole $\mathbb{C}$.*

Proof. Since f has compact support, there exists $T > 0$ such that

$$|f(t)| = 0, \qquad \text{for a. e. } t \geq T.$$

This means that f satisfies the assumptions of the previous result, for every $\alpha < 0$. Thus we conclude that $\sigma_f \leq \alpha$ for every $\alpha < 0$, i.e. $\sigma_f = -\infty$. □

Example 4.2.4. Let $L > 0$, let us compute the Laplace transform of the function

$$1_{[0,L)}(t) = H(t) - H(t - L).$$

Then we get

$$\mathcal{L}[1_{[0,L)}](z) = \int_0^L e^{-z\,t}\,dt = \left[-\frac{e^{-z\,t}}{z}\right]_0^L = \frac{1 - e^{-L\,z}}{z}.$$

The results is apparently in contrast with Corollary 4.2.3, since we have a singularity at $z = 0$. But this is indeed *removable*, since

$$\lim_{z\to 0} \frac{1 - e^{-L\,z}}{z} = L \lim_{z\to 0} \frac{1 - e^{-L\,z}}{L\,z} = L \lim_{w\to 0} \frac{e^w - 1}{w} = L.$$

Thus the Laplace transform is entire.

Proposition 4.2.5 (A sufficient condition for transformability, II)**.** *Let f be a causal signal such that $f \in L^p(\mathbb{R})$, for some $1 \leq p \leq \infty$. Then f is $L-$transformable and $\sigma_f \leq 0$. Moreover, we have:*

- *if $1 < p \leq \infty$, it holds*

$$\left|\mathcal{L}[f](z)\right| \leq \left(\frac{1}{p'\,\mathrm{Re}(z)}\right)^{\frac{1}{p'}} \|f\|_{L^p(\mathbb{R})}, \qquad \textit{for } \mathrm{Re}(z) > 0; \tag{4.2.3}$$

- *if $p = 1$, then $\mathcal{L}[f]$ can be extended up to the imaginary axis $\{z \in \mathbb{C} : \mathrm{Re}(z) = 0\}$ and*

$$\left|\mathcal{L}[f](z)\right| \leq \|f\|_{L^1(\mathbb{R})}, \qquad \textit{for } \mathrm{Re}(z) \geq 0. \tag{4.2.4}$$

Proof. It is sufficient to observe that for every $\alpha > 0$, the function $t \mapsto e^{-\alpha\,t}$ belongs to $L^q(\mathbb{R}_+)$, for every $1 \leq q \leq \infty$. Indeed, for $1 \leq q < \infty$ we have

$$\int_0^{+\infty} e^{-\alpha\,q\,t}\,dt = \left[-\frac{e^{-\alpha\,q\,t}}{\alpha\,q}\right]_0^{\infty} = \frac{1}{\alpha\,q} < +\infty, \tag{4.2.5}$$

while for $q = \infty$ we just observe that

$$0 \leq e^{-\alpha\,t} \leq 1, \qquad \text{for every } t \geq 0.$$

By choosing $q = p'$, we thus get by Hölder's inequality (Proposition 3.3.5) that $e^{-\alpha\,t}\,f(t) \in L^1(\mathbb{R}_+)$ for every $\alpha > 0$, i.e.

$$\int_0^{+\infty} e^{-\alpha\,t}\,|f(t)|\,dt \leq \|f\|_{L^p(\mathbb{R}_+)}\,\|e^{-\alpha\,t}\|_{L^{p'}(\mathbb{R}_+)} < +\infty.$$

This shows that f is $L-$transformable and that $\sigma_f \leq 0$, since $\alpha > 0$ is arbitrary.

Let us now suppose $1 < p \leq \infty$, then we have

$$\begin{aligned}\left|\mathcal{L}[f](z)\right| = \left|\int_0^{+\infty} e^{-z\,t}\,f(t)\,dt\right| &\leq \int_0^{+\infty} e^{-\mathrm{Re}(z)\,t}\,|f(t)|\,dt\\ &\leq \|f\|_{L^p(\mathbb{R}_+)}\,\|e^{-\mathrm{Re}(z)\,t}\|_{L^{p'}(\mathbb{R}_+)},\end{aligned}$$

again thanks to Hölder inequality. By using formula (4.2.5) above, with $q = p'$ and $\alpha = \mathrm{Re}(z)$, we get the estimate (4.2.3).

Finally, we take $f \in L^1(\mathbb{R}_+)$ and prove the last part of the statement. This is a plain consequence of the Dominated Convergence Theorem 3.2.5. Indeed, we have

$$\lim_{x\to 0^+} \mathcal{L}[f](x+i\,y) = \lim_{x\to 0^+} \int_0^{+\infty} e^{-x\,t-i\,t\,y}\, f(t)\,dt. \tag{4.2.6}$$

We now observe that for every $x \ge 0$ we have

$$|e^{-x\,t-i\,t\,y}\, f(t)| = e^{-x\,t}\,|f(t)| \le |f(t)|, \qquad \text{for every } t \ge 0,$$

and the latter is in L^1 and independent of the parameter x. We can thus pass the limit under the integral sign in (4.2.6) and obtain the desired conclusion. The estimate (4.2.4) is left to the reader. □

3. Properties of the Laplace transform

Theorem 4.3.1. *Let f be an $L-$transformable causal signal, with abscissa of convergence σ_f. Then for every $\sigma_0 > \sigma_f$ its Laplace transform $\mathcal{L}[f]$ is bounded and continuous on $\mathrm{Re}(z) \ge \sigma_0$. Moreover, we have*

$$\lim_{\mathrm{Re}(z)\to+\infty} \mathcal{L}[f](z) = 0, \tag{4.3.1}$$

and

$$\lim_{|\mathrm{Im}(z)|\to+\infty} \mathcal{L}[f](z) = 0, \qquad \textit{for } \mathrm{Re}(z) > \sigma_f. \tag{4.3.2}$$

Proof. Let us fix $\sigma_0 > \sigma_f$, then for every $\mathrm{Re}(z) \ge \sigma_0$ we have

$$|\mathcal{L}[f](z)| \le \int_0^{+\infty} e^{-\mathrm{Re}(z)\,t}\,|f(t)|\,dt \le \int_0^{+\infty} e^{-\sigma_0\,t}\,|f(t)|\,dt < +\infty,$$

where the last term is finite thanks to the definition of abscissa of convergence.

We now prove that $\mathcal{L}$ is continuous on $\mathrm{Re}(z) \ge \sigma_0$, for every $\sigma_0 > \sigma_f$. Let us fix $z \in \mathbb{C}$ such that $\mathrm{Re}(z) \ge \sigma_0$, we need to prove that

$$\lim_{h\to 0} \Big|\mathcal{L}[f](z+h) - \mathcal{L}[f](z)\Big| = 0.$$

For every $h \in \mathbb{C}$ such that

$$|h| \le \frac{\mathrm{Re}(z) - \sigma_f}{2}, \tag{4.3.3}$$

we have

$$\begin{aligned}\Big|\mathcal{L}[f](z+h) - \mathcal{L}[f](z)\Big| &= \left|\int_0^{+\infty} \left[e^{-(z+h)\,t} - e^{-z\,t}\right] f(t)\,dt\right| \\ &\le \int_0^{+\infty} e^{-\mathrm{Re}(z)\,t}\,|f(t)|\,|e^{-h\,t} - 1|\,dt.\end{aligned}$$

In order to conclude, it would be sufficient to pass the limit as $h \to 0$ under the integral sign. Indeed, observe that

$$\lim_{h\to 0} e^{-\mathrm{Re}(z)\,t}\,|f(t)|\,|e^{-h\,t} - 1| = 0, \qquad \text{for a. e. } t \ge 0.$$

We want to use the Dominated Convergence Theorem: observe that by triangle inequality and recalling (4.3.3)

$$\begin{aligned} e^{-\mathrm{Re}(z)\,t}\,|f(t)|\,|e^{-h\,t}-1| &\le e^{-\mathrm{Re}(z)\,t}\,|f(t)|\,\left(e^{-\mathrm{Re}(h)\,t}+1\right) \\ &= e^{-(\mathrm{Re}(z)+\mathrm{Re}(h))\,t}|f(t)| + e^{-\mathrm{Re}(z)\,t}\,|f(t)| \\ &\le e^{-\frac{\mathrm{Re}(z)+\sigma_f}{2}\,t}|f(t)| + e^{-\mathrm{Re}(z)\,t}\,|f(t)|. \end{aligned}$$

In the last inequality we used that, thanks to (4.3.3), we have

$$|\mathrm{Re}(h)| \le |h| \le \frac{\mathrm{Re}(z)-\sigma_f}{2},$$

which implies that

$$\mathrm{Re}(h) \ge -\frac{\mathrm{Re}(z)-\sigma_f}{2},$$

and thus

$$\mathrm{Re}(z)+\mathrm{Re}(h) \ge \frac{\mathrm{Re}(z)+\sigma_f}{2}. \tag{4.3.4}$$

Observe that the function above

$$t \mapsto e^{-\frac{\mathrm{Re}(z)+\sigma_f}{2}\,t}|f(t)| + e^{-\mathrm{Re}(z)\,t}\,|f(t)|,$$

is independent of h and is in L^1, since

$$\frac{\mathrm{Re}(z)+\sigma_f}{2} > \sigma_f \qquad \text{and} \qquad \mathrm{Re}(z) > \sigma_f.$$

Thus we can apply the Dominated Convergence Theorem and obtain

$$\lim_{h\to 0}\left|\mathcal{L}[f](z+h)-\mathcal{L}[f](z)\right| \le \lim_{h\to 0}\int_0^{+\infty} e^{-\mathrm{Re}(z)\,t}\,|f(t)|\,|e^{-h\,t}-1|\,dt = 0,$$

as desired.

In order to prove (4.3.1), we observe that $z = x + i\,y$ we have

$$|\mathcal{L}[f](x+i\,y)| \le \int_0^{+\infty} e^{-x\,t}\,|f(t)|\,dt.$$

and for $x \ge \sigma_0$,

$$e^{-x\,t}\,|f(t)| \le e^{-\sigma_0\,t}\,|f(t)| \in L^1(\mathbb{R}_+).$$

Moreover, we have that

$$\lim_{x\to+\infty} e^{-x\,t}\,|f(t)| = 0.$$

We can use Lebesgue Dominated Convergence Theorem and get the conclusion.

At last, we prove (4.3.2). We recall that

$$e^{x+i\,y} = -e^{x+i\,(y+\pi)},$$

thanks to the definition of complex exponential. Thus for every $x > \sigma_f$ and $y > 0$ we have

$$\mathcal{L}[f](x+i\,y) = \int_0^{+\infty} e^{-(x+i\,y)\,t}\,f(t)\,dt,$$

and also

$$\begin{aligned}\mathcal{L}[f](x+i\,y) &= -\int_0^{+\infty} e^{-(x\,t+i\,y\,t+i\,\pi)}\,f(t)\,dt\\ &= -\int_0^{+\infty} e^{-x\,t-i\,y\,\left(t+\frac{\pi}{y}\right)}\,f(t)\,dt\\ &= -\int_{\frac{\pi}{y}}^{+\infty} e^{-x\,\left(\tau-\frac{\pi}{y}\right)}\,e^{-i\,y\,\tau}\,f\left(\tau-\frac{\pi}{y}\right)\,d\tau\\ &= -\int_0^{+\infty} e^{-x\,\left(\tau-\frac{\pi}{y}\right)}\,e^{-i\,y\,\tau}\,f\left(\tau-\frac{\pi}{y}\right)\,d\tau\end{aligned}$$

Observe that in the last equality we used that, by causality, we have

$$f\left(\tau-\frac{\pi}{y}\right)=0,\qquad \text{for } 0\le\tau\le\frac{\pi}{y}.$$

By summing up the two expressions above, we obtain for $x>\sigma_f$ and $y\neq 0$

$$\mathcal{L}[f](x+i\,y)=\frac{1}{2}\int_0^{+\infty} e^{-i\,y\,t}\,\left[e^{-x\,t}\,f(t)-e^{-x\,\left(t-\frac{\pi}{y}\right)}\,f\left(t-\frac{\pi}{y}\right)\right]\,dt.$$

By taking the modules on both sides, we obtain

$$\begin{aligned}\left|\mathcal{L}[f](x+i\,y)\right| &\le \frac{1}{2}\int_0^{+\infty}\left|e^{-x\,t}\,f(t)-e^{-x\,\left(t-\frac{\pi}{y}\right)}\,f\left(t-\frac{\pi}{y}\right)\right|\,dt\\ &= \frac{1}{2}\left\|e^{-x\,t}\,f-\mathcal{T}_{-\frac{\pi}{y}}\left(e^{-x\,t}\,f\right)\right\|_{L^1(\mathbb{R}_+)},\end{aligned}\tag{4.3.5}$$

where we used the usual notation for the translations, i.e.

$$\mathcal{T}_h g(t)=g(t+h).$$

It is only left to observe that if $y\to+\infty$, then $-\pi/y\to 0$, thus we get the desired conclusion by applying Theorem 3.4.5 in (4.3.5). In order to prove (4.3.2) for $y\to-\infty$, as well, we can reproduce the proof above, by using this time

$$e^{x+i\,y}=-e^{x+i\,(y-\pi)}.$$

We leave the details to the reader. □

Remark 4.3.2. The result (4.3.2) goes under the name of *Riemann-Lebesgue Lemma* for the Laplace transform.

Before proving further properties of the Laplace transform, we need to record the following technical result.

Lemma 4.3.3. *For every $z\in\mathbb{C}^*$, we have*

$$\left|\frac{e^z-1}{z}\right|\le e^{|z|}.$$

Proof. Recall that we have

$$e^z=\sum_{k=0}^{\infty}\frac{z^k}{k!},\qquad z\in\mathbb{C}.$$

Thus, by taking the modulus and using the triangle inequality, we get for every $z \in \mathbb{C}^*$

$$\left|\frac{e^z - 1}{z}\right| = \left|\sum_{k=1}^{\infty} \frac{z^{k-1}}{k!}\right| \le \sum_{k=1}^{\infty} \frac{|z|^{k-1}}{k!} \le \sum_{k=1}^{\infty} \frac{|z|^{k-1}}{(k-1)!} = \sum_{m=0}^{\infty} \frac{|z|^m}{m!} = \frac{e^{|z|} - 1}{|z|}.$$

This concludes the proof. □

Theorem 4.3.4. *Let f be an $L-$transformable causal signal. Its Laplace transform $\mathcal{L}[f]$ is a holomorphic function on the half-plane $\mathrm{Re}(z) > \sigma_f$. Moreover, the function $t \mapsto t\, f(t)$ is still $L-$transformable with the same abscissa of convergence and we have*

$$\frac{d}{dz}\mathcal{L}[f](z) = -\mathcal{L}[t\, f](z), \qquad \text{for } \mathrm{Re}(z) > \sigma_f. \tag{4.3.6}$$

Proof. We divide the proof in 3 steps, for ease of readability.

• `Step 1`. In this step, we prove that $t \mapsto t\, f(t)$ is still $L-$transformable, with abscissa of convergence

$$\sigma_{t\, f} = \sigma_f. \tag{4.3.7}$$

Let $\alpha > \sigma_f$ and fix

$$\varepsilon = \frac{\alpha - \sigma_f}{2} > 0.$$

Observe that by definition we still have

$$\alpha - \varepsilon = \frac{\alpha + \sigma_f}{2} > \sigma_f,$$

thus we know that $e^{-(\alpha-\varepsilon)\, t}\, f \in L^1(\mathbb{R}_+)$, thanks to Remark 4.1.4. We now observe that

$$e^{-\alpha\, t}\, t\, f(t) = \left(e^{-(\alpha-\varepsilon)\, t}\, f(t)\right)(t\, e^{-\varepsilon\, t}). \tag{4.3.8}$$

We now use that $\varepsilon > 0$, thus the function

$$t \mapsto t\, e^{-\varepsilon\, t},$$

is bounded on $\mathbb{R}_+$. On the other hand, we already observed that

$$t \mapsto t\, e^{-(\alpha-\varepsilon)\, t}\, f(t),$$

is in $L^1(\mathbb{R}_+)$. In conclusion, from (4.3.8) we get that

$$t \mapsto e^{-\alpha\, t}\, t\, f(t),$$

is still in $L^1(\mathbb{R}_+)$, for every $\alpha > \sigma_f$. By resuming this discussion, we obtained

$$\text{for every } \alpha > \sigma_f, \text{we have } e^{-\alpha\, t}\, t\, f(t) \in L^1(\mathbb{R}_+). \tag{4.3.9}$$

We point out that this already shows that $t \mapsto t\, f(t)$ is $L-$transformable. Moreover, (4.3.9) shows that

$$(\sigma_f, +\infty) \subset \{\alpha \in \mathbb{R} \,:\, e^{-\alpha\, t}\, t\, f(t) \in L^1(\mathbb{R}_+)\}.$$

Thus by taking the infima of the two sets, we have

$$\sigma_{t\, f} = \inf\{\alpha \in \mathbb{R} \,:\, e^{-\alpha\, t}\, t\, f(t) \in L^1(\mathbb{R}_+)\} \le \inf(\sigma_f, +\infty) = \sigma_f.$$

In order to conclude this step of the proof, we only need to show the reverse inequality

$$\sigma_{t\,f} \geq \sigma_f.$$

This would eventually show (4.3.7). We take $\beta > \sigma_{t\,f}$, thus we have $e^{-\beta\,t}\,t\,f \in L^1(\mathbb{R}_+)$, again by Remark 4.1.4. By using the elementary inequality

$$1 \leq t + 1_{[0,1]}(t), \qquad t \in \mathbb{R}_+,$$

we get

$$|e^{-\beta\,t}\,f(t)| \leq e^{-\beta\,t}\,t\,|f(t)| + e^{-\beta\,t}\,|f(t)|\,1_{[0,1]}(t), \qquad t \in \mathbb{R}_+.$$

Both functions in the right-hand side are in $L^1(\mathbb{R}_+)$ (for the second one, we can use Lemma 4.1.7), thus this is true for $e^{-\beta\,t}\,f$ as well. Since this holds for every $\beta > \sigma_{t\,f}$, we proved the inclusion

$$(\sigma_{t\,f}, +\infty) \subset \{\beta \in \mathbb{R} \,:\, e^{-\beta\,t}\,f(t) \in L^1(\mathbb{R}_+)\}.$$

By taking the infimaof the two setw, we finally obtain

$$\sigma_f = \inf\{\beta \in \mathbb{R} \,:\, e^{-\beta\,t}\,f(t) \in L^1(\mathbb{R}_+)\} \leq \inf(\sigma_{t\,f}, +\infty) = \sigma_{t\,f},$$

and thus (4.3.7).

- `Step 2`. In this step, we prove that $\mathcal{L}[f]$ is derivable and formula (4.3.6) holds.

  Let z be such that $\mathrm{Re}(z) > \sigma_f$, for $h \in \mathbb{C}$ such that

$$|h| \leq \frac{\mathrm{Re}(z) - \sigma_f}{2}, \tag{4.3.10}$$

we still have $\mathrm{Re}(z + h) > \sigma_f$, thanks to (4.3.4). Then we consider

$$\frac{\mathcal{L}[f](z + h) - \mathcal{L}[f](z)}{h} = \int_0^{+\infty} \frac{e^{-h\,t} - 1}{h}\,e^{-z\,t}\,f(t)\,dt.$$

Observe that $z \mapsto e^{-z\,t}$ is holomorphic and

$$\lim_{h\to 0} \frac{e^{-h\,t} - 1}{h} = -t \lim_{h\to 0} \frac{e^{-h\,t} - 1}{-h\,t} = -t.$$

Thus, in order to conclude we need to pass the limit under the integral sign. We would like to use Lebesgue Dominated Convergence Theorem (see Theorem 3.2.5), thus we need to find a summable domination for

$$\left|\frac{e^{-h\,t} - 1}{h}\,e^{-z\,t}\,f(t)\right| = \left|\frac{e^{-h\,t} - 1}{-h\,t}\right|\,e^{-\mathrm{Re}(z)\,t}\,|t\,f(t)|,$$

independent of h satisfying (4.3.10). By using Lemma 4.3.3, we obtain

$$\begin{aligned}\left|\frac{e^{-h\,t} - 1}{h}\,e^{-z\,t}\,f(t)\right| &\leq e^{|h|\,t}\,e^{-\mathrm{Re}(z)\,t}\,|t\,f(t)| \leq e^{\frac{\mathrm{Re}(z)-\sigma_f}{2}\,t}\,e^{-\mathrm{Re}(z)\,t}\,|t\,f(t)| \\ &= e^{-\frac{\mathrm{Re}(z)+\sigma_f}{2}\,t}\,|t\,f(t)|.\end{aligned}$$

By observing that

$$\frac{\mathrm{Re}(z) + \sigma_f}{2} > \sigma_f = \sigma_{t\,f},$$

we obtain that the last function above is summable on $\mathbb{R}_+$ (by Remark 4.1.4) and independent of h. By keeping everything together, we get for every h verifying (4.3.10)

$$\left|\frac{e^{-h\,t} - 1}{h}\,e^{-z\,t}\,f(t)\right| \leq e^{-\frac{\mathrm{Re}(z)+\sigma_f}{2}\,t}\,|t\,f(t)|, \qquad t \geq 0. \tag{4.3.11}$$

We can thus conclude by applying Lebesgue Dominated Convergence Theorem as described above and obtain

$$\begin{aligned}\lim_{h\to 0}\frac{\mathcal{L}[f](z+h)-\mathcal{L}[f](z)}{h} &= \lim_{h\to 0}\int_0^{+\infty}\frac{e^{-h\,t}-1}{h}\,e^{-z\,t}\,f(t)\,dt\\ &= -\int_0^{+\infty}e^{-z\,t}\,t\,f(t)\,dt=-\mathcal{L}[t\,f](z).\end{aligned}$$

Thus we have shown that $\mathcal{L}[f]$ is derivable and formula (4.3.6) holds.

• `Step 3`. In order to prove that $\mathcal{L}[f]$ is holomorphic, we only need to prove that its complex derivative

$$\frac{d}{dz}\mathcal{L}[f],$$

is continuous. From formula (4.3.6), we know that this derivative coincides with the Laplace transform of a causal signal, i.e. $t\mapsto t\,f(t)$, thus Theorem 4.3.1 implies that this is continuous. □

By recalling that a holomorphic function can be derived infinitely many times (see Theorem 1.8.2), we can iterate the previous result and get

Corollary 4.3.5. *Let f be an $L-$transformable causal signal. For every $n\in\mathbb{N}\setminus\{0\}$ the function $t\mapsto t^n\,f(t)$ is still $L-$transformable with the same abscissa of convergence and we have*

$$\frac{d^n}{dz^n}\mathcal{L}[f](z)=(-1)^n\,\mathcal{L}[t^n\,f](z),\qquad \textit{for } \mathrm{Re}(z)>\sigma_f. \tag{4.3.12}$$

Example 4.3.6 (Unitary ramp)**.** We consider the unitary ramp function $R(t)=t\,H(t)$. By Theorem 4.3.4, this is still $L-$transformable and the abscissa of convergence is

$$\sigma_R=\sigma_H=0.$$

Its Laplace transform is given by

$$\mathcal{L}[R](z)=\mathcal{L}[t\,H](z)=-\frac{d}{dz}\mathcal{L}[H](z)=-\frac{d}{dz}\frac{1}{z}=\frac{1}{z^2}\qquad \text{for } \mathrm{Re}(z)>0,$$

thanks to formula (4.3.6).

Example 4.3.7. More generally, for $k\in\mathbb{N}\setminus\{0\}$ we consider the causal signal $t\mapsto t^k\,H(t)$. By formula (4.3.12), we get

$$\mathcal{L}[t^k\,H](z)=(-1)^k\,\frac{d^k}{dz^k}\mathcal{L}[H](z)=(-1)^k\,\frac{d^k}{dz^k}\frac{1}{z},\qquad \text{for } \mathrm{Re}(z)>0.$$

If we now observe that

$$\frac{d^k}{dz^k}\frac{1}{z}=(-1)^k\,\frac{k!}{z^{k+1}},$$

we obtain

$$\mathcal{L}[t^k\,H](z)=\frac{k!}{z^{k+1}},\qquad \text{for } \mathrm{Re}(z)>0.$$

4. Remarkable formulas

Proposition 4.4.1 (Linearity). *Let f, g be two $L-$transformable causal signal, with abscissa of convergence σ_f and σ_g. Then for every $c_1, c_2 \in \mathbb{C}$ the causal signal $c_1\, f + c_2\, g$ is $L-$transformable and*

$$\mathcal{L}[c_1\, f + c_2\, g] = c_1\, \mathcal{L}[f] + c_2\, \mathcal{L}[g], \qquad \mathrm{Re}(z) > \max\{\sigma_f, \sigma_g\}. \tag{4.4.1}$$

Proof. We just observe that for every z such that $\mathrm{Re}(z) > \max\{\sigma_f, \sigma_g\}$ we have

$$\left| e^{-z\,t}\,(c_1\, f(t) + c_2\, g(t)) \right| \le e^{-\mathrm{Re}(z)\,t}\, (|c_1|\,|f(t)| + |c_2|\,|g(t)|) \in L^1(\mathbb{R}_+),$$

thus the linear combination is $L-$transformable, with abscissa of convergence smaller than or equal to $\max\{\sigma_f, \sigma_g\}$. Formula (4.4.1) follows from linearity of the integral. □

Proposition 4.4.2 (Temporal dilations). *Let f be an $L-$transformable causal signal. For $\lambda > 0$, we define $f_\lambda(t) = f(\lambda\, t)$. Then f_λ is $L-$transformable with abscissa of convergence $\sigma_{f_\lambda} = \lambda\, \sigma_f$ and we have*

$$\mathcal{L}[f_\lambda](z) = \frac{1}{\lambda}\, \mathcal{L}[f] \left(\frac{z}{\lambda}\right), \qquad \textit{for } \mathrm{Re}(z) > \lambda\, \sigma_f.$$

Proof. By definition of Laplace transform and a change of variables we have

$$\mathcal{L}[f_\lambda](z) = \int_0^{+\infty} e^{-z\,t}\, f(\lambda\, t)\, dt = \frac{1}{\lambda} \int_0^{+\infty} e^{-\frac{z}{\lambda}\, s}\, f(s)\, ds,$$

and observe that

$$e^{-\frac{z}{\lambda}\, s}\, f(s) \in L^1(\mathbb{R}_+) \qquad \text{if } \mathrm{Re}\left(\frac{z}{\lambda}\right) > \sigma_f, \text{ that is if } \mathrm{Re}(z) > \lambda\, \sigma_f.$$

This concludes the proof. □

Proposition 4.4.3 (Phase multiplication). *Let f be an $L-$transformable causal signal. For $a \in \mathbb{C}$, the function $e^{a\,t}\, f(t)$ is still $L-$transformable, with abscissa of convergence given by $\sigma_f + \mathrm{Re}(a)$. We have*

$$\mathcal{L}[e^{a\,t}\, f](z) = \mathcal{L}[f](z - a), \qquad \textit{for } \mathrm{Re}(z) > \sigma_f + \mathrm{Re}(a).$$

Proof. For every $z \in \mathbb{C}$ such that $\mathrm{Re}(z) > \sigma_f + \mathrm{Re}(a)$, we have

$$|e^{-z\,t}\, e^{a\,t}\, f(t)| = e^{-(\mathrm{Re}(z) - \mathrm{Re}(a))\,t}\, |f(t)| \in L^1(\mathbb{R}_+),$$

thanks to the definition of σ_f and the fact that $\mathrm{Re}(z) - \mathrm{Re}(a) > \sigma_f$. We thus have for every $\mathrm{Re}(z) > \sigma_f + \mathrm{Re}(a)$

$$\mathcal{L}[e^{a\,t}\, f](z) = \int_0^{+\infty} e^{-z\,t}\, e^{a\,t}\, f(t)\, dt = \int_0^{\infty} e^{-(z-a)\,t}\, f(t)\, dt = \mathcal{L}[f](z - a),$$

as desired. □

Example 4.4.4. Let $a \in \mathbb{C}$, by Proposition 4.4.3 the Laplace transform of the causal signal $t \mapsto e^{a\,t}\, H(t)$ is given by

$$\mathcal{L}[e^{a\,t}\, H](z) = \mathcal{L}[H](z - a) = \frac{1}{z - a}, \qquad \mathrm{Re}(z) > \mathrm{Re}(a). \tag{4.4.2}$$

Example 4.4.5. Let $k \in \mathbb{N}$ and $a \in \mathbb{C}$. By using Proposition 4.4.3 and recalling Example 4.3.7, we get

$$\mathcal{L}\left[\frac{e^{a\,t}\,t^k\,H}{k!}\right](z) = \frac{1}{k!}\,\mathcal{L}[t^k\,H](z-a) = \frac{1}{(z-a)^{k+1}}, \qquad \mathrm{Re}(z) > \mathrm{Re}(a). \tag{4.4.3}$$

Proposition 4.4.6 (Time delay)**.** *Let f be an $L-$transformable causal signal. For $t_0 > 0$, we define $\mathcal{T}_{-t_0} f(t) = f(t-t_0)$. Then $\mathcal{T}_{-t_0} f$ is $L-$transformable with abscissa of convergence $\sigma_{\mathcal{T}_{-t_0} f} = \sigma_f$ and we have*

$$\mathcal{L}[\mathcal{T}_{-t_0} f](z) = e^{-z\,t_0}\,\mathcal{L}[f](z), \qquad \text{for } \mathrm{Re}(z) > \sigma_f.$$

Proof. Observe that the function $\mathcal{T}_{-t_0} f$ is still a causal signal, indeed it vanishes for $t < t_0$. We have

$$\mathcal{L}[\mathcal{T}_{-h_0} f_\lambda](z) = \int_{t_0}^{\infty} e^{-z\,t}\,f(t-t_0)\,dt = \int_0^{+\infty} e^{-z\,s - z\,t_0}\,f(s)\,ds = e^{-z\,t_0}\,\mathcal{L}[f](z),$$

as desired. □

Lemma 4.4.7 ($L-$transformability of periodic signals)**.** *Let $f : \mathbb{R} \to \mathbb{C}$ be a* positively periodic *causal signal, i.e. there exists $T > 0$ such that*

$$f(t+T) = f(t), \qquad \text{for every } t \geq 0.$$

Then we have

$$f \text{ is } L-\text{transformable} \qquad \Longleftrightarrow \qquad f \in L^1([0,T]).$$

Proof. We first observe that if f is $L-$transformable, then $f \in L^1([0,T])$ by Lemma 4.1.7.

We now suppose $f \in L^1([0,T])$ and prove that f is $L-$transformable. We take $\alpha > 0$ and observe that for every $k \in \mathbb{N}$ we have

$$\int_{k\,T}^{(k+1)\,T} e^{-\alpha\,t}\,|f(t)|\,dt \leq e^{-k\,T\,\alpha} \int_{k\,T}^{(k+1)\,T} |f(t)|\,dt = e^{-k\,T\,\alpha} \int_0^T |f(t)|\,dt,$$

where we used the monotonicity of $t \mapsto e^{-\alpha\,t}$ and the periodicity of f. We thus have

$$\begin{aligned}\int_0^{+\infty} e^{-\alpha\,t}\,|f(t)|\,dt &= \lim_{N\to\infty} \sum_{k=0}^{N} \int_{k\,T}^{(k+1)\,T} e^{-\alpha\,t}\,|f(t)|\,dt \\ &\leq \lim_{N\to\infty} \sum_{k=0}^{N} e^{-\alpha\,k\,T} \int_0^T |f(t)|\,dt \\ &= \|f\|_{L^1([0,T])} \lim_{N\to\infty} \sum_{k=0}^{N} e^{-\alpha\,k\,T}.\end{aligned}$$

The last series is a geometric one, with argument $0 < e^{-\alpha\,T} < 1$. Thus it converges and we have

$$e^{-\alpha\,t}\,f \in L^1(\mathbb{R}_+), \qquad \text{for every } \alpha > 0, \tag{4.4.4}$$

which shows that f is $L-$transformable. □

Proposition 4.4.8 (Periodic signals)**.** *Let f be an $L-$transformable causal signal. Let us suppose that f is positively periodic. Then*

$$\sigma_f = 0, \tag{4.4.5}$$

and we have

$$\mathcal{L}[f](z) = \frac{1}{1 - e^{-T\,z}} \int_0^T e^{-z\,t}\, f(t)\, dt, \qquad \textit{for } \mathrm{Re}(z) > 0. \tag{4.4.6}$$

Proof. Of course, the proof has some similarities with that of the corresponding result for the $Z-$transform, see Proposition 2.2.8.

In order to prove (4.4.5), we first observe that we already know that $\sigma_f \le 0$, thanks to (4.4.4). We are left to prove that

$$e^{-\alpha\,t}\, f \notin L^1(\mathbb{R}_+),$$

for every $\alpha < 0$. The proof runs similarly as above, it is sufficient to observe that for $\alpha < 0$

$$\int_{k\,T}^{(k+1)\,T} e^{-\alpha\,t}\, |f(t)|\, dt \ge e^{-k\,T\,\alpha} \int_{k\,T}^{(k+1)\,T} |f(t)|\, dt = e^{-k\,T\,\alpha} \int_0^T |f(t)|\, dt,$$

then by summing with respect to $k \in \mathbb{N}$, we now get

$$\int_0^{+\infty} e^{-\alpha\,t}\, |f(t)|\, dt \ge \lim_{N\to\infty} \sum_{k=0}^{N} e^{-\alpha\,k\,T} \int_0^T |f(t)|\, dt,$$

and the latter diverges to $+\infty$, since the argument of the geometric series is now bigger than 1 (indeed, $e^{-\alpha\,T} > 1$ because $\alpha < 0$).

We now come to formula (4.4.6). For z such that $\mathrm{Re}(z) > 0$, we write

$$\begin{aligned}\mathcal{L}[f](z) &= \sum_{k\in\mathbb{N}} \int_{k\,T}^{(k+1)\,T} e^{-z\,t}\, f(t)\, dt \\ &= \sum_{k\in\mathbb{N}} \int_0^T e^{-z\,k\,T}\, e^{-z\,s}\, f(s + k\,T)\, ds,\end{aligned}$$

where we used the change of variable $t = s + k\,T$. We now use the hypothesis of periodicity on f, thus we get $f(s + k\,T) = f(s)$ and

$$\mathcal{L}[f](z) = \sum_{k\in\mathbb{N}} e^{-z\,k\,T} \int_0^T e^{-z\,s}\, f(s)\, ds.$$

The integral does not depend on k and the sum is just a geometric series, with argument $e^{-z\,T}$. This series is convergent provided $|e^{-z\,T}| < 1$, i.e.

$$e^{-\mathrm{Re}(z)\,T} < 1,$$

which is true, since we took $\mathrm{Re}(z) > 0$. By observing that

$$\sum_{k\in\mathbb{N}} (e^{-z\,T})^k = \frac{1}{1 - e^{-z\,T}},$$

we get the conclusion. □

Remark 4.4.9 (Critical axis and periodic signals)**.** By recalling the properties of the complex exponential, we have that

$$1 - e^{-T\,z} = 0 \qquad \Longleftrightarrow \qquad -T\,z = 2\,\pi\,k\,i,\ \ k \in \mathbb{Z} \qquad \Longleftrightarrow \qquad z = \frac{2\,\pi\,k}{T}\, i,\ \ k \in \mathbb{Z}$$

Thus, from formula (4.4.6), we obtain that the Laplace transform of a $T-$periodic causal signal can be extended to a holomorphic function defined on

$$\mathbb{C} \setminus \{z_k \ : \ k \in \mathbb{Z}\}, \qquad \text{where } z_k = \frac{2\,\pi\,k}{T}\,i.$$

Each point z_k lies on the imaginary axis and represents an isolated singularity for the function

$$F(z) = \frac{1}{1-e^{-T\,z}} \int_0^T e^{-z\,t}\,f(t)\,dt, \qquad z \notin \{z_k \,:\, k \in \mathbb{Z}\},$$

which is the extension of $\mathcal{L}[f]$ to the whole $\mathbb{C} \setminus \{z_k \ : \ k \in \mathbb{Z}\}$.

It is not difficult to see that every z_k is indeed either a removable singularity or a simple pole. Indeed, by observing that $1 = e^{-z_k\,T}$, we have

$$\begin{aligned}
\lim_{z\to z_k} F(z)\,(z-z_k) &= \lim_{z\to z_k} \frac{z-z_k}{1-e^{-z\,T}} \int_0^T e^{-z\,t}\,f(t)\,dt \\
&= \left(\frac{1}{T}\int_0^T e^{-\frac{2\,\pi\,k}{T}\,i\,t}\,f(t)\,dt\right) \lim_{z\to z_k} \frac{(z-z_k)\,T}{e^{-z_k\,T}-e^{-z\,T}} \\
&= \left(\frac{1}{T}\int_0^T e^{-\frac{2\,\pi\,k}{T}\,i\,t}\,f(t)\,dt\right) \lim_{z\to z_k} \frac{(z-z_k)\,T}{e^{-z_k\,T}\,(1-e^{-(z-z_k)\,T})} \\
&= \frac{1}{T}\int_0^T e^{-\frac{2\,\pi\,k}{T}\,i\,t}\,f(t)\,dt.
\end{aligned}$$

Remark 4.4.10 (Laplace transform VS. Fourier series)**.** The formula found in the previous observation gives a remarkable link between the Laplace transform of a periodic signal f and the coefficients of its Fourier series expansion. Indeed, by observing that the latter are given by (see Appendix C)

$$\widehat{f}(k) = \frac{1}{T}\int_0^T e^{-\frac{2\,\pi\,k}{T}\,i\,t}\,f(t)\,dt, \quad \text{for } k \in \mathbb{Z},$$

we have just shown that (with a slight abuse of notation)

$$\lim_{z\to z_k} \mathcal{L}[f](z)\,(z-z_k) = \widehat{f}(k), \qquad \text{where } z_k = \frac{2\,\pi\,k}{T}\,i, \qquad k \in \mathbb{Z}.$$

By recalling that if z_0 is a simple pole for a function F, it holds (see Proposition 1.10.11 with $m = 1$)

$$\mathrm{res}(F, z_0) = \lim_{z\to z_0} (z-z_0)\,F(z),$$

we can rewrite the previous link between the Laplace transform and the Fourier coefficients as follows

$$\mathrm{res}(\mathcal{L}[f], z_k) = \widehat{f}(k), \qquad \text{where } z_k = \frac{2\,\pi\,k}{T}\,i, \qquad k \in \mathbb{Z}. \tag{4.4.7}$$

The following result is analogous to the formula for "time delay" for the $Z-$transform, i.e. Proposition 2.2.2.

Proposition 4.4.11 (Laplace transform of the derivative)**.** *Let f be an $L-$transformable causal signal, which is continuous on $\mathbb{R}_+$. Let us assume that f' is piecewise continuous, with f' having only jump discontinuities at $\{x_0, \dots, x_N, \dots\} \subset \mathbb{R}_+$ and*[1]

$$|x_i - x_j| \geq \delta > 0, \qquad \textit{for every } i \neq j.$$

[1] The hypothesis assures that the discontinuity points are well detached and do not accumulate somewhere.

Let us suppose that f' is $L-$transformable. Then we have the formula

$$\mathcal{L}[f'](z) = z\left[\mathcal{L}[f](z) - \frac{f(0)}{z}\right], \qquad \textit{for } \mathrm{Re}(z) > \max\{\sigma_f, \sigma_{f'}\}. \tag{4.4.8}$$

Here $f(0)$ has to be intended as $\lim_{t\to 0^+} f(t)$.

Proof. We will perform the proof under the stronger assumption that f' is continuous on $\mathbb{R}_+$, the general case is left as an exercise to the reader.

We idea of the proof is very simple: it is based on the integration by parts formula. However, since we are integrating on the unbounded set $\mathbb{R}_+$, some care is needed. We take $M > 0$, then an integration by parts gives

(4.4.9)
$$\begin{aligned}\int_0^M e^{-z\,t}\, f'(t)\, dt &= \Big[e^{-z\,t}\, f(t)\Big]_0^M + z\int_0^M e^{-z\,t}\, f(t)\, dt\\ &= e^{-z\,M}\, f(M) - f(0) + z\int_0^M e^{-z\,t}\, f(t)\, dt, \qquad \text{for } \mathrm{Re}(z) > \max\{\sigma_f, \sigma_{f'}\}.\end{aligned}$$

By $L-$transformability of f and f', we know that both limits

$$\lim_{M\to+\infty} \int_0^M e^{-z\,t}\, f'(t)\, dt \qquad \text{and} \qquad \lim_{M\to+\infty} \int_0^M e^{-z\,t}\, f(t)\, dt$$

exist. Then from (4.4.9) we obtain that the limit

$$\lim_{M\to+\infty} e^{-z\,M}\, f(M),$$

exists as well. Since $t \mapsto e^{-z\,t}\, f(t)$ is $L^1(\mathbb{R}_+)$ by assumption, this limit must be 0: it is a consequence of Lemma 3.3.12 with $g(t) = e^{-z\,t}\, f(t)$. By taking the limit as M goes to $+\infty$ in (4.4.9), we obtain

$$\int_0^{+\infty} e^{-z\,t}\, f'(t)\, dt = -f(0) + z\int_0^{+\infty} e^{-z\,t}\, f(t)\, dt, \qquad \text{for } \mathrm{Re}(z) > \max\{\sigma_f, \sigma_{f'}\}$$

By recalling the definition of Laplace transform, this gives the desired conclusion. □

The previous result can be iterated, provided f is sufficiently regular.

Corollary 4.4.12. *Let f be an $L-$transformable causal signal of class $C^{n-1}(\mathbb{R}_+)$ for some $n \in \mathbb{N}\setminus\{0\}$. Let us suppose that the derivative $f^{(n-1)}$ satisfies the hypotheses of Proposition 4.4.11. Then we have*

$$\mathcal{L}[f^{(n)}](z) = z^n\left[\mathcal{L}[f](z) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{z^{k+1}}\right], \qquad \textit{for } \mathrm{Re}(z) > \max\{\sigma_f, \sigma_{f'}, \dots, \sigma_{f^{(n)}}\}.$$

We recall from Chapter 3 that if $f, g : \mathbb{R} \to \mathbb{C}$ are causal, then their convolution (provided this is well-defined) can be written

$$f * g(t) = \int_0^t f(s)\, g(t-s)\, ds = \int_0^t f(t-s)\, g(s)\, ds. \tag{4.4.10}$$

It is easy to see that $f * g$ is still causal, i.e. it vanishes for negative t.

Proposition 4.4.13 (Laplace transform of the convolution)**.** *Let f, g two $L-$transformable causal signals. Then $f*g$ is $L-$transformable with abscissa of convergence $\sigma_{f*g} \le \max\{\sigma_f, \sigma_g\}$. Moreover, we have*

$$\mathcal{L}[f*g](z) = \mathcal{L}[f](z)\,\mathcal{L}[g](z), \qquad \text{for } \mathrm{Re}(z) > \max\{\sigma_f, \sigma_g\}. \tag{4.4.11}$$

Proof. We first show that $f*g$ is $L-$transformable. Indeed, let us take

$$\alpha > \max\{\sigma_f, \sigma_g\},$$

then we know that

$$F(t) = e^{-\alpha\, t}\, f(t) \in L^1(\mathbb{R}) \qquad \text{and} \qquad G(t) = e^{-\alpha\, t}\, g(t) \in L^1(\mathbb{R}),$$

by definition of $L-$transformability. From Proposition 3.5.4, we know that the convolution $F*G$ is well-defined and is in $L^1(\mathbb{R})$. We thus get

$$\begin{aligned} F*G(t) &= \int_0^{+\infty} e^{-\alpha\, s}\, f(s)\, e^{-\alpha\,(t-s)}\, g(t-s)\, ds \\ &= \int_0^{+\infty} f(s)\, e^{-\alpha\, t}\, g(t-s)\, ds \\ &= e^{-\alpha\, t}\, f*g(t), \end{aligned}$$

which shows that $e^{-\alpha\, t}\, f*g(t) \in L^1(\mathbb{R})$. Thus $f*g$ is $L-$transformable and

$$\sigma_{f*g} \le \alpha, \qquad \text{for every } \alpha > \max\{\sigma_f, \sigma_g\}.$$

In order to prove (4.4.11), we write

$$\begin{aligned} \mathcal{L}[f*g](z) &= \int_0^{+\infty} e^{-z\, t}\, f*g(t)\, dt = \int_0^{+\infty} e^{-z\, t} \left(\int_0^t f(s)\, g(t-s)\, ds \right) dt \\ &= \int_0^{+\infty} \left(\int_0^t e^{-z\, s}\, f(s)\, e^{-z\,(t-s)}\, g(t-s)\, ds \right) dt, \end{aligned}$$

where we used (4.4.10). Observe that for every z such that $\mathrm{Re}(z) > \max\{\sigma_f, \sigma_g\}$, the function

$$(s,t) \mapsto \left| e^{-z\, s}\, f(s)\, e^{-z\,(t-s)}\, g(t-s)\, ds \right| = e^{-\mathrm{Re}(z)\, s}\, |f(s)|\, e^{-\mathrm{Re}(z)\,(t-s)}\, |g(t-s)|,$$

satisfies the hypothesis of Tonelli's Theorem. Thus we obtain that

$$(s,t) \mapsto e^{-z\, s}\, f(s)\, e^{-z\,(t-s)}\, g(t-s),$$

is summable and thus by Fubini's Theorem we can exchange the order of integration. We thus obtain (by further using the change of variable $t - s = \tau$)

$$\begin{aligned} \mathcal{L}[f*g](z) &= \int_0^{+\infty} e^{-z\, s}\, f(s) \left(\int_s^{+\infty} e^{-z\,(t-s)}\, g(t-s)\, dt \right) ds \\ &= \int_0^{+\infty} e^{-z\, s}\, f(s) \left(\int_0^{+\infty} e^{-z\, \tau}\, g(\tau)\, d\tau \right) ds \\ &= \mathcal{L}[f](z)\, \mathcal{L}[g](z), \end{aligned}$$

as desired. □

5. Inversion formula

We have the following inversion formula.

Theorem 4.5.1. *Let f be an $L-$transformable piecewise C^1 causal signal. Let us assume that f and f' have only jump discontinuities at $\{x_0, \dots, x_k, \dots\} \subset [0, +\infty)$, with*

$$|x_k - x_j| \geq \delta > 0, \qquad \textit{for every } k \neq j.$$

We normalize f so that

$$f(x_k) = \frac{1}{2}\left[\lim_{x \to x_k^+} f(x) + \lim_{x \to x_k^-} f(x)\right], \qquad k \in \mathbb{N}.$$

Then for every $t \in \mathbb{R}$ we have

$$\begin{aligned} f(t) &= \frac{1}{2\,\pi\,i} \lim_{L \to +\infty} \int_{\alpha - i\,L}^{\alpha + i\,L} \mathcal{L}[f](z)\, e^{z\,t}\, dz \\ &= \frac{1}{2\,\pi} \lim_{L \to +\infty} \int_{-L}^{L} \mathcal{L}[f](\alpha + i\,y)\, e^{\alpha\,t}\, e^{i\,y\,t}\, dy, \end{aligned} \tag{4.5.1}$$

where α is any real number such that $\alpha > \sigma_f$.

Proof. We will prove this formula in Section 5 of the next chapter, as a consequence of the Inversion Formula for the Fourier Transform. □

Corollary 4.5.2 (Injectivity of the Laplace transform)**.** *Let f and g be two piecewise C^1 causal signals, which are $L-$transformable. Let us assume that they both satisfy the hypotheses of Theorem 4.5.1. If*

$$\mathcal{L}[f](z) = \mathcal{L}[g](z), \qquad \textit{for every } z \in B_R(z_0) \subset \Big\{ w \in \mathbb{C} \,:\, \mathrm{Re}(w) > \max\{\sigma_f, \sigma_g\} \Big\},$$

then we have

$$f(t) = g(t), \qquad \textit{for every } t \in \mathbb{R}.$$

Proof. We first observe that, by using that the Laplace transform is a holomorphic function, from the *unique continuation principle* (see Corollary 1.8.7) we get that

$$\mathcal{L}[f](z) = \mathcal{L}[g](z), \qquad \text{for every } z \text{ such that } \mathrm{Re}(z) > \max\{\sigma_f, \sigma_g\}.$$

We now take $\alpha > \max\{\sigma_f, \sigma_g\}$, then by using the previous information and formula (4.5.1) for f and g, we get

$$\begin{aligned} f(t) &= \frac{1}{2\,\pi\,i} \lim_{L \to +\infty} \int_{-\alpha - i\,L}^{\alpha + i\,L} \mathcal{L}[f](z)\, e^{z\,t}\, dz \\ &= \frac{1}{2\,\pi\,i} \lim_{L \to +\infty} \int_{\alpha - i\,L}^{\alpha + i\,L} \mathcal{L}[g](z)\, e^{z\,t}\, dz = g(t). \end{aligned}$$

This concludes the proof. □

The following result is very useful in order to avoid the use of the inversion formula in some particular situations. It assures that every rational function is indeed the Laplace transform of a suitable causal signal.

Proposition 4.5.3 (Inversion of a rational function)**.** *Let $P, Q : \mathbb{C} \to \mathbb{C}$ be two polynomials such that*

$$n = \deg(P) < \deg(Q) = m. \tag{4.5.2}$$

Let us call $z_1, \dots, z_k$ the roots of Q, each one having multiplicity $m_1, \dots, m_k$ so that

$$m_1 + \dots + m_k = m,$$

Let us consider the function of the complex variable z

$$F(z) = \frac{P(z)}{Q(z)}, \qquad \text{for } z \in \mathbb{C} \setminus \{z_1, \dots, z_k\}.$$

Then there exists an $L-$transformable causal signal $f : \mathbb{R} \to \mathbb{C}$ such that

$$F(z) = \mathcal{L}[f](z), \qquad \text{for } \mathrm{Re}(z) > \max\{\mathrm{Re}(z_1), \dots, \mathrm{Re}(z_k)\}.$$

More precisely, the signal f is given by

$$f(t) = \sum_{j=1}^{k} e^{z_j\, t} \left(\sum_{h=1}^{m_j} \frac{a_{j,h}}{(h-1)!}\, t^{h-1} \right) H(t), \qquad t \in \mathbb{R}, \tag{4.5.3}$$

where $a_{j,h} \in \mathbb{C}$ are the coefficients of the partial fraction decomposition of F, which are given by

$$a_{j,h} = \mathrm{res}\left((z - z_j)^{h-1} \frac{P(z)}{Q(z)}, z_j \right),$$

see Theorem 1.11.7.

Proof. From Theorem 1.11.7, we already know that we have the partial fraction decomposition

$$F(z) = \frac{P(z)}{Q(z)} = \sum_{j=1}^{k} \left(\sum_{h=1}^{m_j} \frac{a_{j,h}}{(z - z_j)^h} \right), \tag{4.5.4}$$

for suitable coefficients $a_{j,h} \in \mathbb{C}$. We know recall that for every $k \in \mathbb{N}$ and every $a \in \mathbb{C}$ (see Example 4.3.7)

$$\frac{1}{(z-a)^{k+1}} = \mathcal{L}\left[\frac{t^k\, e^{a\,t}\, H}{k!} \right](z), \qquad \text{for } \mathrm{Re}(z) > \mathrm{Re}(a).$$

By using this formula with $a = z_i$ and $k = h - 1$, we thus get for every $j \in \{1, \dots, k\}$ and $h \in \{1, \dots, m_j\}$

$$\frac{a_{j,h}}{(z - z_j)^h} = \mathcal{L}\left[a_{j,h}\, \frac{t^{h-1}\, e^{z_j\, t}\, H}{(h-1)!} \right](z), \qquad \text{for } \mathrm{Re}(z) > \mathrm{Re}(z_j). \tag{4.5.5}$$

By using (4.5.5) in (4.5.4), we thus obtain for $\mathrm{Re}(z) > \max\{\mathrm{Re}(z_1), \dots, \mathrm{Re}(z_k)\}$

$$\begin{aligned} \frac{P(z)}{Q(z)} = \sum_{j=1}^{k} \left(\sum_{h=1}^{m_j} \frac{a_{j,h}}{(z - z_j)^h} \right) &= \sum_{j=1}^{k} \left(\sum_{h=1}^{m_j} \mathcal{L}\left[a_{j,h} \frac{t^{h-1}\, e^{z_j\, t}\, H}{(h-1)!} \right](z) \right) \\ &= \mathcal{L}\left[\sum_{j=1}^{k} e^{z_j\, t} \left(\sum_{h=1}^{m_j} a_{j,h} \frac{t^{h-1}}{(h-1)!} \right) H \right](z), \end{aligned}$$

and the latter is the Laplace transform of the signal defined by (4.5.3). This concludes the proof. $\square$

Remark 4.5.4. The signal (4.5.3) considerably simplifies if all the zeros $z_1, \ldots, z_k$ of the denominator Q are simple. In this case $k = m$ and $m_j = 1$ for every $j = 1, \ldots, m$, thus we obtain

$$f(t) = \sum_{i=1}^{m} a_i \, e^{z_i \, t} \, H(t), \qquad t \in \mathbb{R}.$$

The $a_i \in \mathbb{C}$ are still the coefficients of the partial fraction decomposition of $F = P/Q$.

Remark 4.5.5 (Some words on formula (4.5.3))**.** We can resume Proposition 4.5.3 by saying that:

every rational function $F(z) = P(z)/Q(z)$ verifying (4.5.2) and with k distinct poles is the Laplace transform of the sum of k causal signals, each one of the form

$$e^{(\textit{zero of } Q)\, t} \times \textit{polynomial of degree ``(order of the zero)}-1''.$$

In particular, if all the poles are simple, this is just the sum of k exponential signals.

6. Solving linear ordinary differential equations

One of the main applications of the Laplace transform is to the solution of initial value problems for linear ordinary differential equations, with constant coefficients.

For $n \in \mathbb{N} \setminus \{0\}$, we fix the *coefficients* $\beta_0, \ldots, \beta_n \in \mathbb{C}$ (with $\beta_n \neq 0$) and the *initial conditions* $y_0, \ldots, y_{n-1} \in \mathbb{C}$. We also fix an $L-$transformable causal signal f. Then we want to find a causal signal y of class $C^n(\mathbb{R}_+)$ such that

$$(4.6.1) \qquad \left\{ \begin{array}{rcl} \beta_n \, y^{(n)}(t) + \beta_{n-1} \, y^{(n-1)}(t) + \cdots + \beta_1 \, y'(t) + \beta_0 \, y(t) & = & f(t), \\ y(0) & = & y_0, \\ y'(0) & = & y_1, \\ \cdots & \cdots & \\ y^{(n-1)}(0) & = & y_{n-1}. \end{array} \right.$$

- **Preliminary discussion.** It is useful to separate the difficulties in the problem (4.6.1): in other words, we consider the two problems

$$(4.6.2) \qquad \left\{ \begin{array}{rcl} \beta_n \, y^{(n)}(t) + \beta_{n-1} \, y^{(n-1)}(t) + \cdots + \beta_1 \, y'(t) + \beta_0 \, y(t) & = & 0, \\ y(0) & = & y_0, \\ y'(0) & = & y_1, \\ \cdots & \cdots & \\ y^{(n-1)}(0) & = & y_{n-1}, \end{array} \right.$$

and

$$(4.6.3) \qquad \left\{ \begin{array}{rcl} \beta_n \, y^{(n)}(t) + \beta_{n-1} \, y^{(n-1)}(t) + \cdots + \beta_1 \, y'(t) + \beta_0 \, y(t) & = & f(t), \\ y(0) & = & 0, \\ y'(0) & = & 0, \\ \cdots & \cdots & \\ y^{(n-1)}(0) & = & 0, \end{array} \right.$$

Thanks to the linearity of the derivative, it is easy to verify that if y_{hom} solves (4.6.2) and y_f solves (4.6.3), then

$$y = y_{\text{hom}} + y_f$$

is a solution of the original problem (4.6.1). In order to solve (4.6.2) and (4.6.3), let us introduce the *characteristic polynomial*

$$P_{\mathrm{car}}(z) = \beta_n z^n + \beta_{n-1} z^{n-1} + \cdots + \beta_1 z + \beta_0 = \sum_{i=0}^{n} \beta_i z^i, \qquad z \in \mathbb{C}.$$

We denote by $z_1, \ldots, z_k$ its roots, each one having multiplicity $m_1, \ldots, m_k$, so that $m_1 + \cdots + m_k = n$.

- **Solution of problem (4.6.2).** We call y_{hom} the solution of this problem. By taking the Laplace transform, using its linearity and Corollary 4.4.12, the problem (4.6.2) becomes

$$P_{\mathrm{car}}(z)\, \mathcal{L}[y_{\mathrm{hom}}](z) - \sum_{i=1}^{n} \beta_i \left(\sum_{\ell=0}^{i-1} y_\ell\, z^{i-1-\ell} \right) = 0.$$

Thus we easily get the Laplace transform of y_{hom}, this is given by

$$\mathcal{L}[y_{\mathrm{hom}}](z) = \frac{\displaystyle\sum_{i=1}^{n} \beta_i \left(\sum_{\ell=0}^{i-1} y_\ell\, z^{i-1-\ell} \right)}{P_{\mathrm{car}}(z)}. \tag{4.6.4}$$

Though the expression on the right-hand side seems ugly, we can observe that **this is the ratio of two polynomials**. Moreover, the degree of the numerator is (at most) $n-1$ (just take $k = 0$ and $i = n$), while P_{car} has degree n, thus (4.5.2) is verified. We can thus apply Proposition 4.5.3 in order to find y_{hom}. We get

$$y_{\mathrm{hom}}(t) = \sum_{j=1}^{k} e^{z_j t} \left(\sum_{h=1}^{m_j} \frac{b_{j,h}}{(h-1)!}\, t^{h-1} \right) H(t), \qquad t \in \mathbb{R}$$

where $b_{j,h} \in \mathbb{C}$ are the coefficients of the partial fraction decomposition of the rational function in (4.6.4), that is

$$b_{j,h} = \mathrm{res}\left((z - z_j)^{h-1}\, \frac{\displaystyle\sum_{i=1}^{n} \beta_i \left(\sum_{\ell=0}^{i-1} y_\ell\, z^{i-1-\ell} \right)}{P_{\mathrm{car}}(z)}, z_j \right).$$

- **Solution of problem (4.6.3).** We call y_f a such a solution. As before, we take the Laplace transform, use its linearity and Corollary 4.4.12, thus problem (4.6.3) now becomes

$$P_{\mathrm{car}}(z)\, \mathcal{L}[y_f](z) = \mathcal{L}[f](z),$$

where we used that in (4.6.3) all the initial conditions are 0. The Laplace transform of y_f is easily found to be

$$\mathcal{L}[y_f](z) = \frac{\mathcal{L}[f](z)}{P_{\mathrm{car}}(z)} = \frac{1}{P_{\mathrm{car}}(z)}\, \mathcal{L}[f](z).$$

The function of one complex variable

$$F(z) = \frac{1}{P_{\mathrm{car}}(z)},$$

is called *transfer function*[2] of the system. We observe that the transfer function is a rational function, thus by Proposition 4.5.3 again we know that there exists a causal signal Y, such that

$$\mathcal{L}[Y](z) = \frac{1}{P_{\text{car}}(z)}.$$

More precisely, by formula (4.5.3), we have

$$Y(t) = \sum_{j=1}^{k} e^{z_j\, t} \left(\sum_{h=1}^{m_j} \frac{a_{j,h}}{(h-1)!}\, t^{h-1} \right) H(t), \qquad t \in \mathbb{R},$$

with

$$a_{j,h} = \operatorname{res}\left(\frac{(z-z_j)^{h-1}}{P_{\text{car}}(z)}, z_j \right).$$

The causal signal Y is called *impulse response*[3] of the system. We thus have obtained

$$\mathcal{L}[y_f](z) = \frac{1}{P_{\text{car}}(z)}\, \mathcal{L}[f](z) = \mathcal{L}[Y](z)\, \mathcal{L}[f](z) = \mathcal{L}[Y * f](z),$$

where we used Proposition 4.4.13 in the last equality. Since y_f and $Y * f$ have the same Laplace transform, if they are regular enough we can apply Corollary 4.5.2 and finally obtain

$$y_f(t) = Y * f(t) = \int_0^t f(t-s)\, Y(s)\, ds. \tag{4.6.5}$$

- **Conclusion.** The solution of the original problem (4.6.1) is thus given by

$$\begin{aligned} y(t) &= y_{\text{hom}}(t) + y_f(t) \\ &= \sum_{j=1}^{k} e^{z_j\, t} \left(\sum_{h=1}^{m_j} \frac{b_{j,h}}{(h-1)!}\, t^{h-1} \right) H(t) + Y * f(t). \end{aligned}$$

7. Solving linear integral equations

The Laplace transform is also a useful tool to solve *integral equations*. Without any attempt to offer a rigorous or complete treatment of the subject, let us present some ideas and computations.

Let us consider the *Volterra integral equation of the second kind*

$$y(t) = f(t) + \int_0^t \mathcal{K}(t,s)\, y(s)\, ds, \qquad t \geq 0, \tag{4.7.1}$$

where:

- y is the unknown, which we consider as a causal signal ;
- $\mathcal{K}$ is a given function, called *kernel* of the equation;
- f is a given causal signal, called *source*.

[2] "*Funzione di trasferimento*" in italian.
[3] "*Risposta impulsiva*" in italian.

We observe that if the kernel has the following form

$$\mathcal{K}(t,s) = K(t-s),$$

for some function $K : \mathbb{R} \to \mathbb{R}$ which is causal (i.e. $K(t) = 0$ for $t \leq 0$), then by considering y as a causal signal as well we get

$$\int_0^t \mathcal{K}(t,s)\, y(s)\, ds = \int_0^t K(t-s)\, y(s)\, ds = K * y(t),$$

and thus (4.7.1) rewrites

$$y(t) = f(t) + K * y(t).$$

Let us suppose that both f and K are $L-$transformable. By passing to the Laplace transform, using its linearity and Proposition 4.4.13, we thus get

$$\mathcal{L}[y](z) = \mathcal{L}[f](z) + \mathcal{L}[K](z)\, \mathcal{L}[y](z).$$

By supposing that the kernel K is such that

$$\mathcal{L}[K](z) \neq 1, \qquad \text{for } \operatorname{Re}(z) > \sigma_K,$$

then we can determine the Laplace transform of y, which is given by

$$\mathcal{L}[y](z) = \frac{\mathcal{L}[f](z)}{1 - \mathcal{L}[K](z)}, \text{ for } \operatorname{Re}(z) > \max\{\sigma_f, \sigma_K\}.$$

If we are now able the to compute the inverse transform of the right-hand side above, we can then find a solution to (4.7.1). We refer to the exercises below for some examples.

Remark 4.7.1 (Integro-differential equations)**.** It should be clear that we can still use the Laplace transform to solve a combination of the last two types of equations, i. e. ordinary differential equations with constant coefficients and Volterra equations of the second kind. For example, we could use the Laplace transform to solve an *integro-differential equation* of the type

$$\begin{cases} y'(t) + a\, y(t) &= f(t) + \displaystyle\int_0^t K(t-s)\, y(s)\, ds, \qquad \text{for } t \geq 0, \\ y(0) &= y_0, \end{cases}$$

with $a, y_0 \in \mathbb{C}$ given. We do not insist on this point.

8. The bilateral Laplace transform and the Mellin transform

The Laplace transform can be defined also for general functions $f : \mathbb{R} \to \mathbb{C}$, not necessarily causal. However, some care is needed.

Definition 4.8.1. Let $f : \mathbb{R} \to \mathbb{C}$ be a measurable function. We say that f is $L-$*transformable* if there exist $\alpha < \beta \in \mathbb{R}$ such that

$$e^{-\alpha t} f \in L^1(\mathbb{R}_+) \qquad \text{and} \qquad e^{-\beta t} f \in L^1(\mathbb{R}_-),$$

i.e.

$$\int_0^{+\infty} e^{-\alpha t}\, |f(t)|\, dt < +\infty \qquad \text{and} \qquad \int_{-\infty}^0 e^{-\beta t}\, |f(t)|\, dt < +\infty.$$

In this case, we define its *bilateral Laplace transform* by

$$\mathcal{B}[f](z) := \int_{-\infty}^{+\infty} e^{-z t}\, f(t)\, dt, \qquad z \in \mathbb{C} \text{ such that } \alpha \leq \operatorname{Re}(z) \leq \beta.$$

Remark 4.8.2. We observe that the definition is well-posed. Indeed, for every $z \in \mathbb{C}$ such that $\alpha \leq \mathrm{Re}(z) \leq \beta$, we have

$$|e^{-z\,t}\, f(t)| = |e^{-\mathrm{Re}(z)\,t}\, e^{-i\,\mathrm{Im}(z)\,t}\, f(t)| = e^{-\mathrm{Re}(z)\,t}\, |f(t)| \leq \begin{cases} e^{-\alpha\,t}\, |f(t)|, & \text{for } t \geq 0, \\ e^{-\beta\,t}\, |f(t)|, & \text{for } t < 0. \end{cases}$$

By observing that the last function is in $L^1(\mathbb{R})$ by hypothesis, we then get $e^{-z\,t}\, f(t) \in L^1(\mathbb{R})$ and thus $\mathcal{B}[f]$ is well-defined.

Definition 4.8.3. Let $f : \mathbb{R} \to \mathbb{C}$ be an $L-$transformable signal, we define

$$\sigma_f = \inf\{\alpha \in \mathbb{R} \,:\, e^{-\alpha\,t}\, f(t) \in L^1(\mathbb{R}_+)\},$$

and

$$\Sigma_f = \sup\{\beta \in \mathbb{R} \,:\, e^{-\beta\,t}\, f(t) \in L^1(\mathbb{R}_-)\}.$$

Then its Laplace transform is a well-defined function on the strip

$$\{z \in \mathbb{C} \,:\, \sigma_f < \mathrm{Re}(z) < \Sigma_f\}.$$

The number Σ_f is called *upper abscissa of convergence.*

By proceeding as in the proof of Lemma 4.1.7, one can easily get the following

Lemma 4.8.4 (A necessary condition for transformability)**.** *Let f be a $L-$transformable signal. Then for every $T > 0$ we have $f \in L^1([-T, T])$.*

Without any attempt of completeness, we give a sufficient condition for $L-$transformability.

Proposition 4.8.5. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ be such that for some $C, T > 0$ and $\beta > 0$ we have*

$$|f(t)| \leq C\, e^{-\beta\,|t|}, \qquad \textit{for a. e. } |t| \geq T.$$

Then f is $L-$transformable, with

$$\sigma_f \leq -\beta \qquad \textit{and} \qquad \Sigma_f \geq \beta.$$

Proof. The proof is the same of Proposition 4.2.2. The fact that $f \in L^1_{\mathrm{loc}}(\mathbb{R})$, implies that

$$\int_{-T}^{T} e^{-\alpha\,t}\, |f(t)|\, dt < +\infty,$$

for every $\alpha \in \mathbb{R}$. In order to check the summability on $\mathbb{R} \setminus [-T, T]$, we use the assumption on f. Then for every $\alpha > -\beta$ we have

$$\int_{T}^{+\infty} e^{-\alpha\,t} |f(t)|\, dt \leq C \int_{T}^{+\infty} e^{-\alpha\,t}\, e^{-\beta\,t}\, dt = C \int_{T}^{+\infty} e^{-(\beta+\alpha)\,t}\, dt = \frac{C\, e^{-(\beta+\alpha)\,T}}{\beta + \alpha} < +\infty$$

and for every $\alpha < \beta$

$$\int_{-\infty}^{-T} e^{-\alpha\,t}\, |f(t)|\, dt \leq C \int_{-\infty}^{-T} e^{-\alpha\,t}\, e^{\beta\,t}\, dt = C \int_{-\infty}^{-T} e^{(\beta-\alpha)\,t}\, dt = \frac{C\, e^{-(\beta-\alpha)\,T}}{\beta - \alpha} < +\infty.$$

This concludes the proof. □

As in the case of causal signals, from the previous result we immediately get the following

Corollary 4.8.6 (Compactly supported signals)**.** *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ be a compactly supported signal. Then f is $L-$transformable, with $\sigma_f = -\infty$ and $\Sigma_f = +\infty$, i. e. its bilateral Laplace transform is defined on the whole $\mathbb{C}$.*

The following result is analogous to Theorem 4.3.1.

Theorem 4.8.7. *Let f be a $L-$transformable signal, with abscissae of convergence $\sigma_f < \Sigma_f$. Then for every $\sigma_0 > \sigma_f$ and $\Sigma_0 < \Sigma_f$ its bilateral Laplace transform $\mathcal{B}[f]$ is bounded and continuous on the strip $\sigma_0 \leq \mathrm{Re}(z) \leq \Sigma_0$. Moreover, we have*

$$\lim_{|\mathrm{Im}(z)|\to+\infty} \mathcal{B}[f](z) = 0, \qquad \textit{for } \sigma_f < \mathrm{Re}(z) < \Sigma_f. \tag{4.8.1}$$

Remark 4.8.8. We recall that in the case of causal signals, by (4.3.1) we also have

$$\lim_{\mathrm{Re}(z)\to+\infty} \mathcal{B}[f](z) = \lim_{\mathrm{Re}(z)\to+\infty} \mathcal{L}[f](z) = 0.$$

However, if f is not causal, then this property in general fails. See Example 4.8.10 below for a counterexample.

Finally, with a proof similar to that of Theorem 4.3.4, one can prove

Theorem 4.8.9. *Let f be an $L-$transformable signal. Its bilateral Laplace transform $\mathcal{B}[f]$ is a holomorphic function on the strip*

$$\{z \in \mathbb{C} \,:\, \sigma_f < \mathrm{Re}(z) < \Sigma_f\}.$$

Moreover, the function $t \mapsto t\, f(t)$ is still $L-$transformable with the same abscissae of convergence and we have

$$\frac{d}{dz}\mathcal{B}[f](z) = -\mathcal{B}[t\, f](z), \qquad \textit{for } \sigma_f < \mathrm{Re}(z) < \Sigma_f. \tag{4.8.2}$$

Example 4.8.10 (Bilateral Laplace transform of the rectangle)**.** The rectangular function $f(t) = \mathrm{rect}(t)$ is $L-$transformable, with

$$\sigma_f = -\infty \qquad \text{and} \qquad \Sigma_f = +\infty.$$

Indeed, for every $\alpha \in \mathbb{R}$ and every $\beta \in \mathbb{R}$ we have

$$\int_0^{+\infty} e^{-\alpha\, t}\, |\mathrm{rect}(t)|\, dt = \int_0^{1/2} e^{-\alpha\, t}\, dt < +\infty$$

and

$$\int_{-\infty}^{0} e^{-\beta\, t}\, |\mathrm{rect}(t)|\, dt = \int_{-1/2}^{0} e^{-\beta\, t}\, dt < +\infty.$$

Its bilateral Laplace transform is the entire function given by

$$\mathcal{B}[\mathrm{rect}](z) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-z\, t}\, dt = \left[-\frac{e^{-z\, t}}{z}\right]_{-1/2}^{1/2} = \frac{e^{\frac{z}{2}} - e^{-\frac{z}{2}}}{z}.$$

Observe that $z = 0$ is a removable singularity, thus the function is truly holomorphic on the whole $\mathbb{C}$.

Example 4.8.11. The function $f(t) = e^{-|t|}$ is $L-$transformable, with

$$\sigma_f = -1 \qquad \text{and} \qquad \Sigma_f = 1.$$

Indeed, for every $\alpha > -1$ we have

$$\int_0^{+\infty} e^{-\alpha\, t}\, f(t)\, dt = \int_0^{+\infty} e^{-\alpha\, t}\, e^{-t}\, dt = \frac{1}{\alpha + 1} < +\infty,$$

and for every $\beta < 1$
$$\int_{-\infty}^{0} e^{-\beta\,t}\, f(t)\, dt = \int_{-\infty}^{0} e^{-\beta\,t}\, e^{t}\, dt = \frac{1}{1-\beta} < +\infty.$$
For every $z \in \mathbb{C}$ with $-1 < \mathrm{Re}(z) < 1$, the bilateral Laplace transform is given by
$$\begin{aligned}\mathcal{B}[f](z) &= \int_{0}^{+\infty} e^{-z\,t}\, e^{-t}\, dt + \int_{-\infty}^{0} e^{-z\,t}\, e^{t}\, dt \\ &= \left[-\frac{e^{-(z+1)\,t}}{z+1}\right]_{0}^{+\infty} + \left[-\frac{e^{-\,(z-1)\,t}}{z-1}\right]_{-\infty}^{0} \\ &= \frac{1}{z+1} - \frac{1}{z-1} = -\frac{2}{z^2-1}\end{aligned}$$

Once we defined the bilateral Laplace transform, we can define the so-called *Mellin transform*. This is again well-defined for causal signals.

Definition 4.8.12. Let $f : \mathbb{R} \to \mathbb{C}$ be a causal signal. We say that f is $M-$*transformable* if the function
$$g(t) = f(e^{-t}), \qquad t \in \mathbb{R},$$
is $L-$transformable, i.e. (see Definition 4.8.3) if there exist $\alpha < \beta \in \mathbb{R}$ such that
$$\int_{0}^{+\infty} e^{-\alpha\,t}\, |f(e^{-t})|\, dt < +\infty \qquad \text{and} \qquad \int_{-\infty}^{0} e^{-\beta\,t}\, |f(e^{-t})|\, dt < +\infty.$$
In this case, we define its *Mellin transform* by
$$\mathcal{M}[f](z) := \mathcal{B}[g](z) = \int_{-\infty}^{+\infty} e^{-z\,t}\, f(e^{-t})\, dt, \qquad z \in \mathbb{C} \text{ such that } \alpha \le \mathrm{Re}(z) \le \beta.$$

Remark 4.8.13 (Another form of the Mellin transform)**.** We observe that if f is $M-$transformable, by making the change of variable
$$e^{-t} = x \qquad \text{i.e. } t = -\log x,$$
we can also write
$$\begin{aligned}\mathcal{M}[f](z) = \int_{-\infty}^{+\infty} e^{-z\,t}\, f(e^{-t})\, dt &= \int_{0}^{+\infty} e^{z\,\log x}\, f(x)\, \frac{dx}{x} \\ &= \int_{0}^{+\infty} x^{z}\, f(x)\, \frac{dx}{x} \\ &= \int_{0}^{+\infty} x^{z-1}\, f(x)\, dx, \qquad z \in \mathbb{C} \text{ such that } \alpha \le \mathrm{Re}(z) \le \beta.\end{aligned}$$

Example 4.8.14. The causal signal
$$f(t) = \mathrm{rect}(t - 1/2) = \begin{cases} 1, & \text{if } 0 \le t \le 1, \\ 0, & \text{otherwise,} \end{cases}$$
is $M-$transformable. Indeed, we observe that
$$g(t) = f(e^{-t}) = \mathrm{rect}\left(e^{-t} - \frac{1}{2}\right) = \begin{cases} 1, & \text{if } t \ge 0, \\ 0, & \text{if } t < 0. \end{cases}$$
In other words, we have
$$g(t) = H(t),$$

i.e. it coincides with the Heaviside function. By definition, we thus obtain

$$\mathcal{M}[\mathrm{rect}(\cdot - 1/2)](z) = \mathcal{B}[H](z) = \mathcal{L}[H](z) = \frac{1}{z},$$

where we used that the bilateral Laplace transform coincides with the Laplace transform for a causal signal and Example 4.1.5. By using the alternative expression for the Mellin transform

$$\mathcal{M}[f](z) = \int_0^{+\infty} x^{z-1}\, f(x)\, dx,$$

the computations above imply that

$$\int_0^1 x^{z-1}\, dx = \frac{1}{z}, \qquad \text{for } \mathrm{Re}(z) > 0.$$

Example 4.8.15 (The Gamma function)**.** We consider the causal signal

$$f(t) = e^{-t}\, H(t).$$

This function is $M-$transformable, since the function

$$g(t) = f(e^{-t}) = e^{-e^{-t}},$$

is $L-$transformable, with

$$\sigma_g = 0 \qquad \text{and} \qquad \Sigma_g = +\infty.$$

Let us prove this assertion: for every $\alpha > 0$, we have

$$\int_0^{+\infty} e^{-\alpha\, t}\, g(t)\, dt = \int_0^{+\infty} e^{-\alpha\, t}\, e^{-e^{-t}}\, dt \le \int_0^{+\infty} e^{-\alpha\, t}\, dt = \frac{1}{\alpha} < +\infty.$$

This shows also that $\sigma_g = 0$. On the other hand, for every $\beta > 0$, we have

$$\int_{-\infty}^0 e^{-\beta\, t}\, g(t)\, dt = \int_{-\infty}^0 e^{-(\beta\, t + e^{-t})}\, dt < +\infty,$$

thanks to the fact that

$$e^{-(\beta\, t + e^{-t})} = o(e^{-t^2}) \qquad \text{for } t \to -\infty,$$

and the last function is summable on $(-\infty, 0]$. This is true for every $\beta > 0$, thus this also shows that $\Sigma_g = +\infty$. We can then define the Mellin transform of f, by

$$\mathcal{M}[f](z) = \int_0^{+\infty} t^{z-1}\, e^{-t}\, dt, \qquad \text{for } z \in \mathbb{C} \text{ with } \mathrm{Re}(z) > 0.$$

This function of the complex variable z is called *Gamma function*. We observe that the restriction of $\mathcal{M}[f]$ to $\mathbb{N} \setminus \{0\}$ has the following properties:

- $\mathcal{M}[f](1) = \displaystyle\int_0^{+\infty} e^{-t}\, dt = 1$
- the recursive formula for $n \ge 1$

$$\begin{aligned} \mathcal{M}[f](n+1) &= \int_0^{+\infty} t^n\, e^{-t}\, dt \\ &= \left[-t^n\, e^{-t}\right]_0^{+\infty} + n \int_0^{+\infty} t^{n-1}\, e^{-t}\, dt \\ &= n \int_0^{+\infty} t^{n-1}\, e^{-t}\, dt = n\, \mathcal{M}[f](n). \end{aligned}$$

The previous relations imply that

$$\mathcal{M}[f](1) = 1, \quad \mathcal{M}[f](2) = 1, \quad \mathcal{M}[f](3) = 2, \quad \mathcal{M}[f](4) = 2 \cdot 3 = 6$$

and so on....in other words we obtained the following remarkable relation

$$\mathcal{M}[f](n) = (n-1)!, \qquad \text{for every } n \in \mathbb{N}.$$

For this reason, the Gamma function can be seen as an extension of the factorial function.

9. Exercises

Exercise 4.9.1. *Let $\omega > 0$, show that the Laplace transform of the causal signal $\cos(\omega\, t)\, H(t)$ is given by*

$$\mathcal{L}[\cos(\omega\, t)\, H](z) = \frac{z}{z^2+\omega^2}, \qquad \mathrm{Re}(z) > 0.$$

Solution. We recall that

$$\cos(\omega\, t) = \frac{e^{i\,\omega\, t} + e^{-i\,\omega\, t}}{2},$$

thus from (4.4.2) with $a = i\,\omega$ we get

$$\begin{aligned}\mathcal{L}[\cos(\omega\, t)\, H](z) &= \frac{1}{2}\Big\{\mathcal{L}[e^{i\,\omega\, t}\, H](z) + \mathcal{L}[e^{-i\,\omega\, t}\, H](z)\Big\}\\ &= \frac{1}{2}\left\{\frac{1}{z-i\,\omega} + \frac{1}{z+i\,\omega}\right\} = \frac{z}{z^2+\omega^2}, \qquad \mathrm{Re}(z) > 0,\end{aligned}$$

as desired. □

Exercise 4.9.2. *Let $\omega > 0$, show that the Laplace transform of the causal signal $\sin(\omega\, t)\, H(t)$ is given by*

$$\mathcal{L}[\sin(\omega\, t)\, H](z) = \frac{\omega}{z^2+\omega^2}, \qquad \mathrm{Re}(z) > 0. \tag{4.9.1}$$

Solution. This is very similar to the previous one. We recall that

$$\sin(\omega\, t) = \frac{e^{i\,\omega\, t} - e^{-i\,\omega\, t}}{2\,i},$$

thus from (4.4.2) with $a = i\,\omega$ we get

$$\begin{aligned}\mathcal{L}[\sin(\omega\, t)\, H](z) &= \frac{1}{2\,i}\Big\{\mathcal{L}[e^{i\,\omega\, t}\, H](z) - \mathcal{L}[e^{-i\,\omega\, t}\, H](z)\Big\}\\ &= \frac{1}{2\,i}\left\{\frac{1}{z-i\,\omega} - \frac{1}{z+i\,\omega}\right\} = \frac{\omega}{z^2+\omega^2}, \qquad \mathrm{Re}(z) > 0,\end{aligned}$$

as desired. □

Exercise 4.9.3 (A positively periodic signal)**.** *Let us consider the following positively periodic causal signal (*sawtooth wave[4]*)*

$$SW(t) = \sum_{k=0}^{\infty} (t-k)\,\Big[H(t-k) - H(t-k-1)\Big],$$

see Figure 1. Compute its Laplace transform.

[4]"*Onda a dente di sega*" in italian.

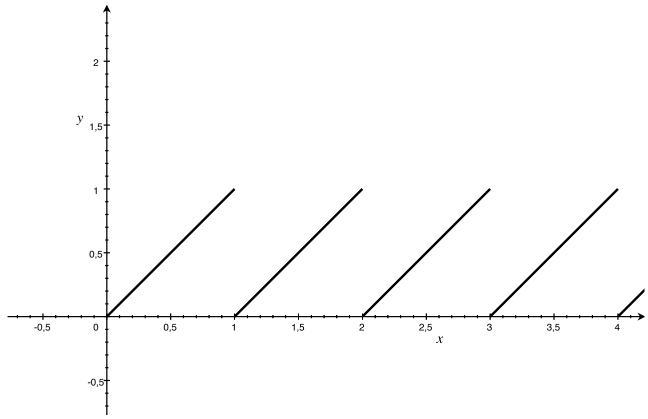

Figure 1. The graph of the *sawtooth wave*

Solution. Observe that the period of this signal is $T = 1$. Also notice that $SW \in L^1([0,1])$, thus by appealing to Lemma 4.4.7, we obtain that SW is $L-$transformable. Moreover, $\sigma_{SW} = 0$ thanks to Proposition 4.4.8.

From formula (4.4.6), we get

$$\mathcal{L}[SW](z) = \frac{1}{1-e^{-z}} \int_0^1 e^{-z\,t}\, t\, dt, \qquad \text{for } \mathrm{Re}(z) > 0.$$

We compute the last integral: by using an integration by parts

$$\begin{aligned} \int_0^1 e^{-z\,t}\, t\, dt &= \left[-\frac{e^{-z\,t}}{z}\, t \right]_0^1 + \int_0^1 \frac{e^{-z\,t}}{z}\, dt \\ &= -\frac{e^{-z}}{z} + \left[-\frac{e^{-z\,t}}{z^2} \right]_0^1 \\ &= -\frac{e^{-z}}{z} + \frac{1}{z^2} - \frac{e^{-z}}{z^2}, \qquad \text{for } \mathrm{Re}(z) > 0. \end{aligned}$$

After some elementary manipulations, we thus get

$$\mathcal{L}[f](z) = \frac{1}{z^2} - \frac{1}{z\,(e^z - 1)} = \frac{e^z - z - 1}{z^2}\, \frac{1}{e^z - 1}, \qquad \text{for } \mathrm{Re}(z) > 0.$$

This concludes the exercise. □

Exercise 4.9.4. *Solve the following intial value problem for the linear ordinary differential equation of second order with constant coefficients*

$$\begin{cases} y''(t) + 4\, y'(t) + 3\, y(t) &= 0, \qquad t \geq 0 \\ y(0) &= 3, \\ y'(0) &= 1. \end{cases}$$

Solution. Observe that this is a homogeneous equation. Let us introduce the characteristic polynomial

$$P_{\mathrm{car}}(z) = z^2 + 4\,z + 3.$$

By passing to the Laplace transform, from formula (4.6.4) we find the Laplace transform of the solution

$$\mathcal{L}[y](z) = \frac{a_1\, y(0) + a_2\, (y(0)\, z + y'(0))}{P_{\mathrm{car}}(z)} = \frac{3\, z + 13}{z^2 + 4\, z + 3}.$$

We now compute the partial fraction decomposition of the last rational function: observe that

$$z \mapsto \frac{3\, z + 13}{z^2 + 4\, z + 3},$$

has two simple poles, in correspondence of

$$z_1 = -3 \qquad \text{and} \qquad z_2 = -1.$$

We thus seek for two coefficients $A, B \in \mathbb{C}$ such that

$$\frac{3\, z + 13}{z^2 + 4\, z + 3} = \frac{A}{z + 3} + \frac{B}{z + 1}.$$

By recalling Corollary 1.11.8, we have

$$A = \mathrm{res}\left(\frac{3\, z + 13}{z^2 + 4\, z + 3}, -3\right) = -2 \qquad \text{and} \qquad B = \mathrm{res}\left(\frac{3\, z + 13}{z^2 + 4\, z + 3}, -1\right) = 5.$$

Thus we have

$$\mathcal{L}[y](z) = -\frac{2}{z + 3} + \frac{5}{z + 1}, \qquad \mathrm{Re}(z) > -1.$$

This finally gives

$$y(t) = -2\, e^{-3\, t} + 5\, e^{-t}, \qquad t \geq 0,$$

thanks to Proposition 4.5.3. □

Exercise 4.9.5. *Solve the following initial value problem for the linear ordinary differential equation of second order with constant coefficients*

$$\begin{cases} y''(t) + 2\, y'(t) + 5\, y(t) &= 0, \qquad t \geq 0 \\ y(0) &= 2, \\ y'(0) &= -4. \end{cases}$$

Solution. Again, this is a homogeneous equation. Let us introduce the characteristic polynomial

$$P_{\mathrm{car}}(z) = z^2 + 2\, z + 5.$$

By passing to the Laplace transform, from formula (4.6.4) we find the Laplace transform of the solution

$$\mathcal{L}[y](z) = \frac{a_1\, y(0) + a_2\, (y(0)\, z + y'(0))}{P_{\mathrm{car}}(z)} = \frac{2\, z}{z^2 + 2\, z + 5}.$$

We have to compute the partial fraction decomposition of the last rational function. The function

$$F(z) = \frac{2\, z}{z^2 + 2\, z + 5},$$

has two simple poles, in correspondence of

$$z_1 = -1 - 2\, i \qquad \text{and} \qquad z_2 = -1 + 2\, i.$$

We thus seek for two coefficients $A, B \in \mathbb{C}$ such that

$$F(z) = \frac{2\, z}{z^2 + 2\, z + 5} = \frac{A}{z + 1 + 2\, i} + \frac{B}{z + 1 - 2\, i}.$$

By using Corollary 1.11.8, we get

$$A = \mathrm{res}(F, z_1) = \frac{z_1}{z_1 + 1} = \frac{1 + 2\,i}{2\,i} = 1 - \frac{i}{2},$$

and

$$B = \mathrm{res}(F, z_2) = \frac{z_2}{z_2 + 1} = \frac{-1 + 2\,i}{2\,i} = 1 + \frac{i}{2}.$$

Finally, we obtain

$$\mathcal{L}[y](z) = \left(1 - \frac{i}{2}\right)\, \frac{1}{z + 1 + 2\,i} + \left(1 + \frac{i}{2}\right)\, \frac{1}{z + 1 - 2\,i},$$

which gives

$$y(t) = \left(1 - \frac{i}{2}\right)\, e^{-t}\, e^{-2\,i\,t} + \left(1 + \frac{i}{2}\right)\, e^{-t}\, e^{2\,i\,t}, \qquad t \geq 0,$$

thanks to Proposition 4.5.3. Observe that we can rewrite the solution in a different fashion as

$$\begin{aligned} y(t) &= e^{-t}\, \left[e^{-2\,i\,t} + e^{2\,i\,t} + i\, \frac{e^{2\,i\,t} - e^{-2\,i\,t}}{2}\right] \\ &= e^{-t}\, \Big[2\, \cos(2\,t) - \sin(2\,t)\Big], \qquad t \geq 0, \end{aligned}$$

concluding the exercise. □

Exercise 4.9.6. *Solve the following initial value problem for the linear ordinary differential equation of second order with constant coefficients*

$$\begin{cases} y''(t) + y(t) &= R(t), \qquad t \geq 0 \\ y(0) &= 0, \\ y'(0) &= 0. \end{cases}$$

where as above $t \mapsto R(t) = t\,H(t)$ is the unitary ramp function.

Solution. As always, let us introduce the characteristic polynomial

$$P_{\mathrm{car}}(z) = z^2 + 1.$$

Observe that this is a non-homogeneous equation (due to the presence of the source term R), but with homogeneous initial conditions. Thus we already know from formula (4.6.5) that the solution can be written in the form

$$y(t) = Y * R(t), \qquad t \geq 0,$$

where Y is the causal signal such that

$$\mathcal{L}[Y](z) = \frac{1}{P_{\mathrm{car}}(z)} = \frac{1}{z^2 + 1}, \qquad \text{for } \mathrm{Re}(z) > 0.$$

We recognize that the right-hand side is the Laplace transform of the causal signal $t \mapsto \sin t\, H(t)$, thanks to Exercise 4.9.2. Thus we obtain $Y(t) = \sin t\, H(t)$ and

$$\begin{aligned} y(t) = Y * R(t) = \int_0^t (t - s)\, \sin s\, ds &= t - \int_0^t \cos(t - s)\, ds \\ &= t - \sin t, \qquad \text{for } t \geq 0, \end{aligned}$$

thus concluding the exercise. □

Exercise 4.9.7. *Solve the following intial value problem for the linear ordinary differential equation of second order with constant coefficients*

$$\begin{cases} y''(t) + y(t) &=& R(t), \qquad t \geq 0 \\ y(0) &=& 1, \\ y'(0) &=& -1. \end{cases}$$

where as above $t \mapsto R(t) = t\,H(t)$ *is the unitary ramp function.*

Solution. As always, let us introduce the characteristic polynomial

$$P_{\text{car}}(z) = z^2 + 1.$$

Observe that now we are in the general situation faced in Section 6. We thus decompose the problem in the two problems

$$\begin{cases} y''(t) + y(t) &=& 0, \qquad t \geq 0 \\ y(0) &=& 1, \\ y'(0) &=& -1, \end{cases}$$

and

$$\begin{cases} y''(t) + y(t) &=& R(t), \qquad t \geq 0 \\ y(0) &=& 0, \\ y'(0) &=& 0. \end{cases}$$

The seeked solution of the initial problem will be the sum of y_{hom} (solving the first one) and y_f (solving the second one). We observe that y_f has already been computed in the previous exercise. In order to find the solution y_{hom}, we cans use the Laplace transform and obtain from (4.6.4)

$$\mathcal{L}[y_{\text{hom}}](z) = \frac{a_1\,y(0) + a_2\,(y(0)\,z + y'(0))}{P_{\text{car}}(z)} = \frac{z-1}{z^2+1}, \qquad \text{for } \operatorname{Re}(z) > 0.$$

We could now proceed to compute the partial fraction decomposition of the right-hand side. Otherwise, we can recognize directly that

$$\frac{z-1}{z^2+1} = \frac{z}{z^2+1} - \frac{1}{z^2+1} = \mathcal{L}[\cos t\,H](z) - \mathcal{L}[\sin t\,H](z),$$

thanks to Execises 4.9.1 and 4.9.2. We thus get

$$y_{\text{hom}}(t) = \cos t - \sin t, \qquad \text{for } t \geq 0,$$

and finally

$$y(t) = y_{\text{hom}} + y_f(t) = \cos t - 2\,\sin t + t, \qquad \text{for } t \geq 0,$$

thus concluding the exercise. □

Exercise 4.9.8. *Solve the following initial value problem for the linear ordinary differential equation of third order with constant coefficients*

$$\begin{cases} y'''(t) - 3\,y'(t) + 2\,y(t) &=& e^t, \qquad t \geq 0 \\ y(0) &=& 0, \\ y'(0) &=& 0, \\ y''(0) &=& 0. \end{cases}$$

Solution. The characteristic polynomial is given by

$$P_{\rm car}(z) = z^3 - 3\,z + 2.$$

This is a non-homogeneous equation (due to the presence of the source term e^t), but with homogeneous initial conditions. By formula (4.6.5), we know that the solution can be written as

$$y(t) = Y * e^t, \qquad t \geq 0,$$

where Y is the *impulse response*, i.e. the causal signal such that

$$\mathcal{L}[Y](z) = \frac{1}{P_{\rm car}(z)} = \frac{1}{z^3 - 3\,z + 2}.$$

We observe that

$$z^3 - 3\,z + 2 = (z-1)^2\,(z+2),$$

thus we have

$$\mathcal{L}[Y](z) = \frac{1}{(z-1)^2\,(z+2)}.$$

In order to find Y, we need to perfom a partial fraction decomposition, i.e. we need to find $A, B, C \in \mathbb{C}$ such that

$$\mathcal{L}[Y](z) = \frac{1}{(z-1)^2\,(z+2)} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z+2}.$$

Observe that we have a multiple pole, thus we need to use the general formula of Theorem 1.11.7. We thus get

$$A = \mathrm{res}\left(\frac{1}{(z-1)^2\,(z+2)}, 1\right) = \lim_{z\to 1} \frac{d}{dz}\frac{1}{z+2} = -\frac{1}{9},$$

$$B = \mathrm{res}\left((z-1)\,\frac{1}{(z-1)^2\,(z+2)}, 1\right) = \frac{1}{3},$$

and

$$C = \mathrm{res}\left(\frac{1}{(z-1)^2\,(z+2)}, -2\right) = \frac{1}{9}.$$

We used Proposition 1.10.11 in order to compute the residues above. We thus obtained

$$\mathcal{L}[Y](z) = -\frac{1}{9}\,\frac{1}{z-1} + \frac{1}{3}\,\frac{1}{(z-1)^2} + \frac{1}{9}\,\frac{1}{z+2}.$$

This implies that

$$Y(t) = \left(-\frac{1}{9}\,e^t + \frac{1}{3}\,t\,e^t + \frac{1}{9}\,e^{-2\,t}\right)\,H(t).$$

Finally, we solution is given by

$$y(t) = Y * e^t = -\frac{1}{9}\int_0^t e^s\,e^{t-s}\,ds + \frac{1}{3}\int_0^t s\,e^s\,e^{t-s}\,ds + \frac{1}{9}\int_0^t e^{-2\,s}\,e^{t-s}\,ds.$$

The previous integrals can be easily computed, we leave the details to the reader. $\square$

10. Advanced exercises

Exercise 4.10.1. *Let $a \in \mathbb{C}$ be such that $\mathrm{Re}(a) > 0$. Compute the bilateral Laplace transform of the signal $f : \mathbb{R} \to \mathbb{C}$ defined by*

$$f(t) = \begin{cases} e^{-a\,t}, & \text{for } t \geq 0, \\ e^{a\,t}, & \text{for } t < 0. \end{cases}$$

Solution. We first observe that f is $L-$transformable, since

$$e^{-\alpha\,t}\, f \in L^1(\mathbb{R}_+), \qquad \text{for every } \alpha > -\mathrm{Re}(a),$$

and

$$e^{-\beta\,t}\, f \in L^1(\mathbb{R}_+), \qquad \text{for every } \beta < \mathrm{Re}(a).$$

This also shows that

$$\sigma_f = -\mathrm{Re}(a) \qquad \text{and} \qquad \Sigma_f = \mathrm{Re}(a).$$

We now compute the bilateral Laplace transform. For every $z \in \mathbb{C}$ with $-\mathrm{Re}(a) < \mathrm{Re}(z) < \mathrm{Re}(a)$, we have

$$\begin{aligned} \mathcal{B}[f](z) = \int_{\mathbb{R}} e^{-z\,t}\, f(t)\, dt &= \int_0^{+\infty} e^{-(z+a)\,t}\, dt + \int_{-\infty}^0 e^{-(z-a)\,t}\, dt \\ &= \lim_{M \to +\infty} \left[-\frac{e^{-(z+a)\,t}}{z+a} \right]_0^M + \lim_{M \to -\infty} \left[-\frac{e^{-(z-a)\,t}}{z-a} \right]_M^0 \\ &= \frac{1}{z+a} - \frac{1}{z-a} = \frac{2\,a}{a^2 - z^2}. \end{aligned}$$

Observe that we used that

$$\lim_{M \to +\infty} -\frac{e^{-(z+a)\,M}}{z+a} = 0, \qquad \text{for } \mathrm{Re}(z) > -\mathrm{Re}(a),$$

and

$$\lim_{M \to -\infty} -\frac{e^{-(z-a)\,M}}{z-a} = 0, \qquad \text{for } \mathrm{Re}(z) < \mathrm{Re}(a).$$

This concludes the exercise. □

Exercise 4.10.2. *Show that the causal signal*

$$f(t) = (\mathrm{sinc}\, t)\, H(t),$$

is $L-$transformable and compute its Laplace transform.

Solution. We have already seen that $\mathrm{sinc} \in L^p(\mathbb{R})$ for every $1 < p \leq \infty$, see Example 3.3.15. Thus f is $L-$transformable by Proposition 4.2.5 and we have

$$\sigma_f \leq 0.$$

In order to compute $\mathcal{L}[f]$, we introduce the causal signal

$$g(t) = \frac{1}{\pi}\, \sin(\pi\, t)\, H(t).$$

Then by recalling the definition of the cardinal sinus, we get

$$g(t) = t\, f(t).$$

By Theorem 4.3.4, we have that g is $L-$transformable, as well. Moreover, $\sigma_g = \sigma_f$ and

$$\mathcal{L}[g](z) = \mathcal{L}[t\, f](z) = -\frac{d}{dz}\mathcal{L}[f](z), \qquad \text{for } \mathrm{Re}(z) > \sigma_f.$$

On the other hand, we can use Exercise 4.9.2 to compute $\mathcal{L}[g]$: by using formula (4.9.1) with $\omega = \pi$, we find

$$\mathcal{L}[g](z) = \frac{1}{\pi}\,\frac{\pi}{z^2+\pi^2} = \frac{1}{z^2+\pi^2}, \qquad \text{for } \mathrm{Re}(z) > \sigma_g = 0.$$

This shows that $\sigma_f = \sigma_g = 0$ and that

$$\frac{d}{dz}\mathcal{L}[f](z) = -\frac{1}{z^2+\pi^2}, \qquad \text{for } \mathrm{Re}(z) > 0.$$

We now introduce the function

$$F(z) = -\frac{1}{\pi}\,\mathrm{Arctan}\left(\frac{z}{\pi}\right),$$

where Arctan is the function of Exercise 1.13.2, and observe that

$$F'(z) = -\frac{1}{z^2+\pi^2}.$$

This shows that $\mathcal{L}[f] - F$ has derivative constantly equals to 0, on the connected set $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$. Thus there exists a constant $C \in \mathbb{C}$ such that

$$\mathcal{L}[f](z) = F(z) + C = -\frac{1}{\pi}\,\mathrm{Arctan}\left(\frac{z}{\pi}\right) + C.$$

It is only left to compute the constant C: we take the limit as $\mathrm{Re}(z) \to +\infty$ in the previous identity. By using Theorem 4.3.1, we obtain

$$0 = \lim_{\mathrm{Re}(z)\to+\infty} -\frac{1}{\pi}\,\mathrm{Arctan}\left(\frac{z}{\pi}\right) + C, \qquad \text{that is} \qquad C = \lim_{\mathrm{Re}(z)\to+\infty} \frac{1}{\pi}\mathrm{Arctan}\left(\frac{z}{\pi}\right) = \frac{1}{2}.$$

In the last computation we used that Arctan has the explicit expression

$$\mathrm{Arctan}(w) = \frac{1}{2}\,\mathrm{Arg}\left(\frac{1+i\,w}{1-i\,w}\right) + i\,\log\sqrt{\left|\frac{1-i\,w}{1+i\,w}\right|}, \qquad \text{for } w \in \mathbb{C}\setminus\{-i,\, i\}.$$

In conclusion, we get

$$\mathcal{L}[f](z) = F(z) + C = \frac{1}{2} - \frac{1}{\pi}\,\mathrm{Arctan}\left(\frac{z}{\pi}\right).$$

This concludes the exercise. □

Exercise 4.10.3. *Compute the Fourier coefficients $\{\widehat{f}(k)\}_{k\in\mathbb{Z}}$ of the sawtooh wave.*

Solution. We already observed that the sawtooth wave is $T-$positively periodic, with $T = 1$. We set

$$F(z) = \mathcal{L}[f](z) = \frac{e^z - z - 1}{z^2\,(e^z - 1)},$$

then we want to use formula (4.4.7) with $T = 1$, to infer that

$$\widehat{f}(k) = \mathrm{res}(F, z_k), \qquad \text{where } z_k = 2\,\pi\,k\,i, \qquad k \in \mathbb{Z}$$

For $k = 0$, we have $z_0 = 0$ and thus

$$\widehat{f}(0) = \mathrm{res}(F, 0) = \lim_{z\to0} z\,\frac{e^z - z - 1}{z^2\,(e^z - 1)} = \lim_{z\to0}\frac{e^z - z - 1}{z\,(e^z - 1)} = \lim_{z\to0}\frac{\dfrac{z^2}{2} + o(z^2)}{z^2 + o(z^2)} = \frac{1}{2},$$

where we used the second order Taylor expansion of the exponential.

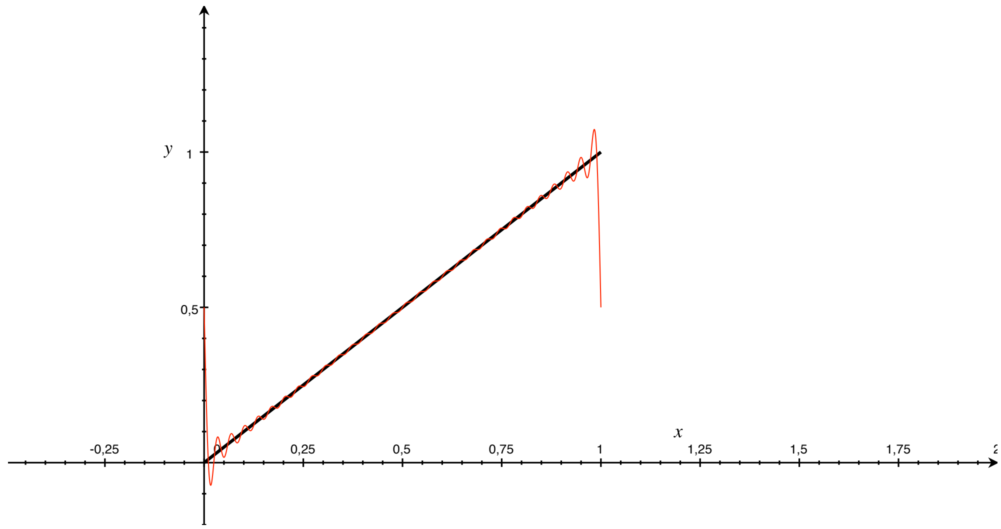

Figure 2. The sawtooth wave (in black) and the first 30 terms of its Fourier series (in red).

For $k \neq 0$, we can observe that the function F can be written as

$$F(z) = \frac{h(z)}{g(z)}, \qquad \text{with } h(z) = \frac{e^z - z - 1}{z^2} \text{ and } \ g(z) = \frac{1}{e^z - 1}.$$

We can then compute the residue at z_k by using Corollary 1.10.13 and get

$$\widehat{f}(k) = \mathrm{res}(F, z_k) = \frac{h(z_k)}{g'(z_k)} = \frac{e^{z_k} - z_k - 1}{z_k^2 \, e^{z_k}}.$$

If we now recall that $e^{z_k} = 1$, we finally obtain

$$\widehat{f}(k) = \frac{1 - z_k - 1}{z_k^2 \cdot 1} = -\frac{1}{z_k} = -\frac{1}{2\,\pi\,k\,i} = \frac{i}{2\,\pi\,k}, \qquad k \in \mathbb{Z} \setminus \{0\}.$$

This concludes the exercise. □

Exercise 4.10.4. *Find a causal signal y solving the* Volterra integral equation of the second kind

$$y(t) = f(t) + \int_0^t K(t-s)\, y(s)\, ds, \qquad t \geq 0,$$

where

$$f(t) = t\, e^{-t}\, H(t), \qquad K(t) = e^{-t}\, H(t).$$

Solution. The equation can be rewritten as

$$y(t) = f(t) + K * y(t), \qquad t \geq 0,$$

and f and K are both $L-$transformable, with $\sigma_f = \sigma_K = -1$, thanks to (4.4.3). By taking the Laplace transform, the previous equation becomes

$$\mathcal{L}[y](z) = \mathcal{L}[f](z) + \mathcal{L}[K](z)\, \mathcal{L}[y](z), \qquad \text{for } \mathrm{Re}(z) > -1.$$

This gives

$$\Big(1 - \mathcal{L}[K](z)\Big)\, \mathcal{L}[y](z) = \mathcal{L}[f](z), \qquad \text{for } \mathrm{Re}(z) > -1.$$

We now observe that

$$\begin{aligned}\mathcal{L}[f](z) = \mathcal{L}[t\,e^{-t}\,H](z) &= -\frac{d}{dz}\,\mathcal{L}[e^{-t}\,H] \\ &= -\frac{d}{dz}\frac{1}{z+1} = \frac{1}{(z+1)^2}, \qquad \text{for } \mathrm{Re}(z) > -1,\end{aligned}$$

where we used Theorem 4.3.4 in the first equality and Proposition 4.4.3 in the second one. Similarly, we compute

$$\mathcal{L}[K](z) = \mathcal{L}[e^{-t}\,H](z) = \frac{1}{z+1}, \qquad \text{for } \mathrm{Re}(z) > -1,$$

and observe that $\mathcal{L}[K](z) \neq 1$ for $\mathrm{Re}(z) > 0$. By confining our analysis to this half-plane, we thus obtain

$$\begin{aligned}\mathcal{L}[y](z) = \frac{\mathcal{L}[f](z)}{\Big(1 - \mathcal{L}[K](z)\Big)} &= \frac{1}{(z+1)^2}\,\frac{1}{1 - \dfrac{1}{z+1}} \\ &= \frac{1}{z\,(z+1)}, \qquad \mathrm{Re}(z) > 0.\end{aligned}$$

We can use the partial fractional decomposition of the last term: in this case, this is particularly simple since we have

$$\frac{1}{z\,(z+1)} = \frac{1}{z} - \frac{1}{z+1}, \qquad z \neq 0 \text{ and } z \neq -1.$$

The last two terms are two known Laplace transforms, indeed

$$\frac{1}{z} = \mathcal{L}[H](z) \qquad \text{and} \qquad \frac{1}{z+1} = \mathcal{L}[e^{t}\,H](z).$$

Thus we get the solution

$$y(t) = H(t) - e^{t}\,H(t), \qquad t \in \mathbb{R}.$$

concluding the exercise. We can easily verify that this causal signal is indeed a solution of the initial Volterra equation (*check it!*). □

Chapter 5

The Fourier Transform

1. Fourier transform of L^1 functions

Definition 5.1.1. Let $f \in L^1(\mathbb{R})$ be a complex-valued function, we define its *Fourier transform* by

$$\mathcal{F}[f](\omega) = \int_{\mathbb{R}} e^{-i\,t\,\omega}\, f(t)\, dt, \qquad \omega \in \mathbb{R}.$$

The definition is well-posed, since for every $\omega \in \mathbb{R}$ we have

$$\left|e^{-i\,t\,\omega}\, f(t)\right| = \left|e^{-i\,t\,\omega}\right|\, |f(t)| = |f(t)| \in L^1(\mathbb{R}),$$

where we used that

$$|e^{-i\,t\,\omega}| = 1,$$

thanks to (1.5.3).

Remark 5.1.2 (Relation with the Laplace transform)**.** To every function $f \in L^1(\mathbb{R})$ we can associate two $L-$transformable causal signals: these are given by

$$f_{\rightarrow}(t) := f(t)\, H(t), \qquad t \in \mathbb{R} \qquad (\textit{forward signal}),$$

and

$$f_{\leftarrow}(t) := f(-t)\, H(t), \qquad t \in \mathbb{R} \qquad (\textit{backward signal}).$$

Observe that by construction we have

$$f(t) = f_{\rightarrow}(t) + f_{\leftarrow}(-t), \qquad \text{for every } t \in \mathbb{R}.$$

They are both in $L^1(\mathbb{R}_+)$, thus by Lemma 4.2.5 they are $L-$transformable with $\sigma_{f_\rightarrow}, \sigma_{f_\leftarrow} \leq 0$. Moreover, the L^1 hypothesis entails that their Laplace transforms

$$z \mapsto \mathcal{L}[f_{\rightarrow}](z) \qquad \text{and} \qquad z \mapsto \mathcal{L}[f_{\leftarrow}](z),$$

can be extended up to the imaginary axis $\{z \in \mathbb{C} : \mathrm{Re}(z) = 0\}$, still by Proposition 4.2.5. We can thus consider

$$\mathcal{L}[f_{\rightarrow}](i\,\omega) = \int_0^{+\infty} e^{-i\,t\,\omega}\, f(t)\, dt, \qquad \omega \in \mathbb{R},$$

and

$$\mathcal{L}[f_{\leftarrow}](i\,\omega) = \int_0^{+\infty} e^{-i\,t\,\omega}\, f(-t)\, dt, \qquad \omega \in \mathbb{R}.$$

Finally, we observe that with a simple change of variables

$$\begin{aligned}\mathcal{L}[f_{\rightarrow}](i\,\omega) + \mathcal{L}[f_{\leftarrow}](-i\,\omega) &= \int_0^{+\infty} e^{-i\,t\,\omega}\, f(t)\, dt + \int_0^{+\infty} e^{i\,t\,\omega}\, f(-t)\, dt \\ &= \int_0^{+\infty} e^{-i\,t\,\omega}\, f(t)\, dt + \int_{-\infty}^{0} e^{-i\,s\,\omega}\, f(s)\, ds = \mathcal{F}[f](\omega).\end{aligned}$$

This gives the relation between the two transforms.

Remark 5.1.3 (Relation with the bilateral Laplace transform)**.** If $f \in L^1(\mathbb{R})$ is $L-$trasformable with bilateral Laplace transform

$$\mathcal{B}[f](z) = \int_{-\infty}^{+\infty} e^{-z\,t}\, f(t)\, dt \qquad \text{and} \qquad \sigma_f < 0 < \Sigma_f,$$

then the relation with the Fourier transform is more direct. Indeed, we just have

$$\mathcal{F}[f](\omega) = \mathcal{B}[f](i\,\omega),$$

i.e. $\mathcal{F}[f]$ coincides with the restriction of the bilateral Laplace transform to the imaginary axis.

Example 5.1.4 (Fourier Transform of the rectangular function)**.** We recall that the rectangular function is defined by

$$\operatorname{rect}(t) = \begin{cases} 1, & \text{if } t \in \left[-\dfrac{1}{2}, \dfrac{1}{2}\right], \\ 0, & \text{otherwise.}\end{cases}$$

This is an $L^1(\mathbb{R})$ function, thus we can define its Fourier transform. Observe that rect is $L-$transformable, as seen in Example 4.8.10, with abscissae of convergence given by

$$\sigma_{\text{rect}} = -\infty \qquad \text{and} \qquad \Sigma_{\text{rect}} = +\infty.$$

Its bilateral Laplace transform is given by the entire function

$$\mathcal{B}[\text{rect}](z) = \frac{e^{\frac{z}{2}} - e^{-\frac{z}{2}}}{z}.$$

By taking the restriction of this function to $z = i\,\omega$, we get

$$\mathcal{F}[\text{rect}](\omega) = \mathcal{B}[\text{rect}](i\,\omega) = \frac{e^{\frac{i\,\omega}{2}} - e^{-\frac{i\,\omega}{2}}}{i\,\omega}.$$

By recalling that

$$\frac{e^{i\,\vartheta} - e^{-i\,\vartheta}}{2\,i} = \sin\vartheta, \qquad \vartheta \in \mathbb{R},$$

we get

$$\mathcal{F}[\text{rect}](\omega) = \frac{\sin\left(\dfrac{\omega}{2}\right)}{\dfrac{\omega}{2}}.$$

By recalling the definition of the *cardinal sine function*

$$\operatorname{sinc}\omega = \begin{cases} \dfrac{\sin(\pi\,\omega)}{\pi\,\omega}, & \text{if } \omega \neq 0, \\ 1, & \text{if } \omega = 0,\end{cases}$$

we can also rewrite the previous formula as

$$\mathcal{F}[\mathrm{rect}](\omega) = \mathrm{sinc}\left(\frac{\omega}{2\,\pi}\right). \tag{5.1.1}$$

Example 5.1.5. The Fourier transform of the function $f(t) = e^{-|t|}$ is given by

$$\mathcal{F}[f](\omega) = \frac{2}{1+\omega^2}.$$

Indeed, we recall that the function $f(t) = e^{-|t|}$ is $L-$transformable and its bilateral Laplace transform is given by

$$\mathcal{B}[f](z) = -\frac{2}{z^2-1}, \qquad \text{for } -1 < \mathrm{Re}(z) < 1,$$

see Example 4.8.11. Thus by Remark 5.1.3 we get

$$\mathcal{F}[f](\omega) = \mathcal{B}[f](i\,\omega) = -\frac{2}{-\omega^2-1} = \frac{2}{\omega^2+1}.$$

2. Properties of the Fourier transform

Theorem 5.2.1. *Let $f \in L^1(\mathbb{R})$ be a complex-valued function. Then its Fourier transform $\mathcal{F}[f]$ is a bounded and continuous function on $\mathbb{R}$. Moreover, we have*

$$\Big\|\mathcal{F}[f]\Big\|_{L^\infty(\mathbb{R})} \le \|f\|_{L^1(\mathbb{R})}, \tag{5.2.1}$$

and

$$\lim_{|\omega|\to\infty} \Big|\mathcal{F}[f](\omega)\Big| = 0 \qquad (\textit{Riemann-Lebesgue Lemma}). \tag{5.2.2}$$

Proof. We start by proving (5.2.1). By definition and properties of the Lebesgue integral, we have

$$|\mathcal{F}[f](\omega)| = \left|\int_{\mathbb{R}} e^{-i\,t\,\omega}\, f(t)\,dt\right| \le \int_{\mathbb{R}} \left|e^{-i\,t\,\omega}\, f(t)\right|\,dt = \int_{\mathbb{R}} |f(t)|\,dt$$

where we used again that $|e^{-i\,t\,\omega}| = 1$ for every $t, \omega \in \mathbb{R}$. Since the previous estimate is valid for every $\omega \in \mathbb{R}$, we thus obtain

$$\Big\|\mathcal{F}[f]\Big\|_{L^\infty(\mathbb{R})} \le \int_{\mathbb{R}} |f(t)|\,dt,$$

as desired.

Let us now prove that $\omega \mapsto \mathcal{F}[f](\omega)$ is a continuous function. We fix $\omega_0 \in \mathbb{R}$, we need to prove that

$$\lim_{\omega\to\omega_0} \Big|\mathcal{F}[f](\omega) - \mathcal{F}[f](\omega_0)\Big| = 0.$$

By definition and elementary properties of the Lebesgue integral, we have

$$\begin{aligned}
\Big|\mathcal{F}[f](\omega) - \mathcal{F}[f](\omega_0)\Big| &= \left|\int_{\mathbb{R}} e^{-i\,t\,\omega}\, f(t)\,dt - \int_{\mathbb{R}} e^{-i\,t\,\omega_0}\, f(t)\,dt\right| \\
&= \left|\int_{\mathbb{R}} \left(e^{-i\,t\,\omega} - e^{-i\,t\,\omega_0}\right)\, f(t)\,dt\right| \\
&\le \int_{\mathbb{R}} \left|e^{-i\,t\,\omega} - e^{-i\,t\,\omega_0}\right|\, |f(t)|\,dt.
\end{aligned}$$

We now observe that

$$\lim_{\omega\to\omega_0}\left|e^{-i\,t\,\omega}-e^{-i\,x\,\omega_0}\right|=0,$$

thus in order to conclude we want to pass the limit under the integral sign. By noticing that

$$\left|e^{-i\,t\,\omega}-e^{-i\,x\,\omega_0}\right|\,|f(t)|\le\left(\left|e^{-i\,t\,\omega}\right|+\left|e^{-i\,x\,\omega_0}\right|\right)\,|f(t)|=2\,|f(t)|\in L^1(\mathbb{R}),$$

we can invoke the Lebesgue Dominated Convergence Theorem and conclude.

We now prove (5.2.2). We notice that the proof is exactly the same as in the case of Laplace transform, see Theorem 4.3.1. We start by observing that for $\omega\in\mathbb{R}$ we have

$$\mathcal{F}[f](\omega)=-\int_{\mathbb{R}}e^{-i\,t\,\omega-i\,\pi}\,f(t)\,dt,$$

since $e^{-i\,\pi}=-1$. Then if $\omega\neq 0$ we have

$$\begin{aligned}\mathcal{F}[f](\omega)=-\int_{\mathbb{R}}e^{-i\,t\,\omega-i\,\pi}\,f(t)\,dt&=-\int_{\mathbb{R}}e^{-i\,\left(t+\frac{\pi}{\omega}\right)\,\omega}\,f(t)\,dt\\&=-\int_{\mathbb{R}}e^{-i\,y\,\omega}\,f\left(y-\frac{\pi}{\omega}\right)\,dy.\end{aligned}$$

On the other hand, by definition of Fourier transform

$$\mathcal{F}[f](\omega)=\int_{\mathbb{R}}e^{-i\,t\,\omega}\,f(t)\,dt.$$

By summing up the two previous identities and dividing by 2, we thus obtain

$$\mathcal{F}[f](\omega)=\frac{1}{2}\int_{\mathbb{R}}e^{-i\,y\,\omega}\,\left[f(y)-f\left(y-\frac{\pi}{\omega}\right)\right]\,dy.$$

By taking the modulus, we get

$$\left|\mathcal{F}[f](\omega)\right|\le\frac{1}{2}\int_{\mathbb{R}}\left|f(y)-f\left(y-\frac{\pi}{\omega}\right)\right|\,dy=\frac{1}{2}\,\|f-\mathcal{T}_{-\frac{\pi}{\omega}}f\|_{L^1(\mathbb{R})},$$

where we used the usual notation $\mathcal{T}_h f(t)=f(t+h)$ for translations. We conclude by using the continuity of the L^1 norm with respect to translations (see Proposition 3.4.5) and the fact that π/ω converges to 0, as $|\omega|$ goes to $+\infty$. □

Remark 5.2.2. From Theorem 5.2.1 we can infer that if $f\in L^1(\mathbb{R})$, then $\mathcal{F}[f](\omega)$ is indeed *uniformly continuous* on $\mathbb{R}$. We do not insist on this point.

Proposition 5.2.3 (Higher regularity of the transform)**.** *Let $f\in L^1(\mathbb{R})$ be such that the function $t\mapsto t\,f(t)$ is in $L^1(\mathbb{R})$ as well. Then the Fourier transform $\mathcal{F}[f]$ is a $C^1(\mathbb{R})$ function and we have*

$$\frac{d}{d\omega}\mathcal{F}[f](\omega)=-i\,\mathcal{F}[t\,f](\omega).\tag{5.2.3}$$

Moreover we have

$$\frac{d}{d\omega}\mathcal{F}[f]\in L^\infty(\mathbb{R})\qquad\text{and}\qquad\lim_{|\omega|\to+\infty}\left|\frac{d}{d\omega}\mathcal{F}[f](\omega)\right|=0.$$

Proof. We first observe that since $t\mapsto t\,f(t)$ is in $L^1(\mathbb{R})$, we already know from Theorem 5.2.1 that $\mathcal{F}[t\,f]$ is continuous, bounded and converges to 0 as $|\omega|\to+\infty$. Thus it is sufficient to show that $\mathcal{F}[f]$ is derivable and that formula (5.2.3) holds.

We start by observing that

$$\lim_{h\to 0}\frac{\mathcal{F}[f](\omega+h)-\mathcal{F}[f](\omega)}{h}=\lim_{h\to 0}\int_{\mathbb{R}}\frac{e^{-i\,t\,h}-1}{h}\,e^{-i\,t\,\omega}\,f(t)\,dt, \tag{5.2.4}$$

and since

$$\lim_{h\to 0}\frac{e^{-i\,t\,h}-1}{h}=-i\,t,$$

in order to conclude we just need the take the limit under the integral sign. We observe that

$$\begin{aligned}|e^{-i\,t\,h}-1| &= \sqrt{(\cos(t\,h)-1)^2+\sin^2(t\,h)}\\ &= \sqrt{2-2\,\cos(t\,h)}\\ &= \sqrt{2}\,\sqrt{1\,-\cos(t\,h)}=2\left|\sin\left(\frac{t\,h}{2}\right)\right|,\end{aligned}$$

thus we get

$$\begin{aligned}\left|\frac{e^{-i\,t\,h}-1}{h}\right| &= 2\left|\sin\left(\frac{t\,h}{2}\right)\right|\,\frac{1}{|h|}\\ &= |t|\,\left|\sin\left(\frac{t\,h}{2}\right)\right|\,\frac{1}{\left|\dfrac{t\,h}{2}\right|}\le |t|,\end{aligned}$$

where we used the well-known trigonometric facts

$$\sin^2\left(\frac{\alpha}{2}\right)=1-\cos\alpha \qquad \text{and} \qquad \left|\frac{\sin\alpha}{\alpha}\right|\le 1.$$

Thus for the function under the integral sign in (5.2.4), we have

$$\left|\frac{e^{-i\,t\,h}-1}{h}\,e^{-i\,t\,\omega}\,f(t)\right|\le |t\,f(t)|, \qquad t\in\mathbb{R}.$$

The last function is in L^1 by assumption and does not depend on h, thus we can apply the Lebesgue Dominated Convergence Theorem and get

$$\lim_{h\to 0}\frac{\mathcal{F}[f](\omega+h)-\mathcal{F}[f](\omega)}{h}=-i\int_{\mathbb{R}}t\,e^{-i\,t\,\omega}\,f(t)\,dt,$$

as desired. □

Corollary 5.2.4. *Let $f\in L^1(\mathbb{R})$ be such that the function $t\mapsto t^n\,f(t)$ is in $L^1(\mathbb{R})$ as well, for some $n\in\mathbb{N}\setminus\{0\}$. Then the Fourier transform $\mathcal{F}[f]$ is a $C^n(\mathbb{R})$ function and we have*

$$\frac{d^k}{d\omega^k}\mathcal{F}[f](\omega)=(-i)^k\,\mathcal{F}[t^k\,f](\omega), \qquad k=1,\dots,n.$$

Moreover for every $k=1,\dots,n$ we have

$$\frac{d^k}{d\omega^k}\mathcal{F}[f]\in L^\infty(\mathbb{R}) \qquad \textit{and} \qquad \lim_{|\omega|\to+\infty}\left|\frac{d^k}{d\omega^k}\mathcal{F}[f](\omega)\right|=0.$$

Remark 5.2.5. The hypothesis "$t\mapsto t^n\,f(t)$ is in $L^1(\mathbb{R})$" holds true if f is such that

$$\lim_{|t|\to+\infty}|t^\alpha\,f(t)|<+\infty, \qquad \text{for some } \alpha>n+1.$$

Thus from the previous result we could say that "*the faster the signal decays at infinity, the more regular its Fourier transform is*".

Proposition 5.2.6. *Let $f \in L^1(\mathbb{R})$, then:*

- *if f is real-valued and even, then $\mathcal{F}[f]$ is real-valued and even;*
- *if f is real-valued and odd, then $\mathcal{F}[f]$ is purely imaginary and odd.*

Proof. Let us suppose that f is real-valued and even, i.e. $f(-t) = f(t)$ for every $t \in \mathbb{R}$. Then with a simple change of variable we get

$$\begin{aligned}\operatorname{Im}(\mathcal{F}[f](\omega)) = -\int_{\mathbb{R}} \sin(t\,\omega)\, f(t)\, dt &= -\int_0^{+\infty} \sin(t\,\omega)\, f(t)\, dt - \int_{-\infty}^0 \sin(t\,\omega)\, f(t)\, dt \\ &= -\int_0^{+\infty} \sin(t\,\omega)\, f(t)\, dt - \int_0^{+\infty} \sin(-s\,\omega)\, f(-s)\, ds \\ &= -\int_0^{+\infty} \sin(t\,\omega)\, f(t)\, dt + \int_0^{+\infty} \sin(s\,\omega)\, f(s)\, ds = 0,\end{aligned}$$

where we used the fact that f is even, while the sinus is odd. This shows that $\mathcal{F}[f]$ is real-valued. Moreover, it is an even function since

$$\mathcal{F}[f](-\omega) = \int_{\mathbb{R}} e^{i\,t\,\omega}\, f(t)\, dt = \int_{\mathbb{R}} e^{-i\,s\,\omega}\, f(-s)\, ds = \int_{\mathbb{R}} e^{-i\,s\,\omega}\, f(s)\, ds.$$

In the case f is real-valued and odd the proof is similar, it is left to the reader as an exercise. □

3. Remarkable formulas

The following properties of the Fourier transform are analogous to the ones for the Laplace transform.

Proposition 5.3.1 (Linearity). *Let $f, g \in L^1(\mathbb{R})$, then for every $c_1, c_2 \in \mathbb{C}$ the function $c_1\, f + c_2\, g$ is in L^1 and*

$$\mathcal{F}[c_1\, f + c_2\, g] = c_1\, \mathcal{F}[f] + c_2\, \mathcal{F}[g]. \tag{5.3.1}$$

Proof. This is an easy consequence of the linearity of the Lebesgue integral. □

Proposition 5.3.2 (Dilations). *Let $f \in L^1(\mathbb{R})$ and $\lambda > 0$. We set $f_\lambda(x) = f(\lambda\, x)$, then we get*

$$\mathcal{F}[f_\lambda](\omega) = \frac{1}{\lambda}\, \mathcal{F}[f]\left(\frac{\omega}{\lambda}\right).$$

Proof. It is sufficient to use the definition and a change of variables, we have

$$\mathcal{F}[f_\lambda](\omega) = \int_{\mathbb{R}} f(\lambda\, t)\, e^{-i\,t\,\omega}\, dt = \frac{1}{\lambda} \int_{\mathbb{R}} f(y)\, e^{-i\,y\,\frac{\omega}{\lambda}}\, dy.$$

This concludes the proof. □

Proposition 5.3.3 (Translations). *Let $f \in L^1(\mathbb{R})$ and $h \in \mathbb{R}$. We set $\mathcal{T}_h f(t) = f(t+h)$, then we get*

$$\mathcal{F}[\mathcal{T}_h f](\omega) = e^{i\,h\,\omega}\, \mathcal{F}[f](\omega).$$

Proof. We use the change of variable $x + h = y$, so to get

$$\mathcal{F}[\mathcal{T}_h f](\omega) = \int_{\mathbb{R}} f(t+h)\, e^{-i\,t\,\omega}\, dt = \int_{\mathbb{R}} f(y)\, e^{-i\,(y-h)\,\omega}\, dy,$$

which concludes the proof. □

It may be useful to state explicitely a formula for the composition of dilations and translations.

Corollary 5.3.4 (Dilations & translations)**.** *Let $f \in L^1(\mathbb{R})$, $\lambda > 0$ and $h \in \mathbb{R}$. Then for the function $t \mapsto f_{\lambda,h}(t) = f(\lambda\,t + h)$ we have*

$$\mathcal{F}[f_{\lambda,h}](\omega) = \frac{e^{i\,\frac{h}{\lambda}\,\omega}}{\lambda}\,\mathcal{F}[f]\left(\frac{\omega}{\lambda}\right).$$

Proof. It is sufficient to observe that

$$f(\lambda\,t + h) = f\left(\lambda\,\left(t + \frac{h}{\lambda}\right)\right) = \left(\mathcal{T}_{\frac{h}{\lambda}}\,f\right)_{\lambda}(t),$$

thus from the previous formulas we get

$$\mathcal{F}[f_{\lambda,h}](\omega) = \frac{1}{\lambda}\,\mathcal{F}[\mathcal{T}_{\frac{h}{\lambda}}\,f]\left(\frac{\omega}{\lambda}\right) = \frac{e^{i\,\frac{h}{\lambda}\,\omega}}{\lambda}\,\mathcal{F}[f]\left(\frac{\omega}{\lambda}\right).$$

Alternatively, we can prove the formula directly, by using a simple change of variables

$$\begin{aligned}\mathcal{F}[f_{\lambda,h}](\omega) = \int_{\mathbb{R}} e^{-i\,t\,\omega}\,f(\lambda\,t + h)\,dt &= \int_{\mathbb{R}} e^{-i\,\frac{s-h}{\lambda}\,\omega}\,f(s)\,\frac{ds}{\lambda}\\ &= \frac{e^{i\,\frac{h}{\lambda}\,\omega}}{\lambda}\,\int_{\mathbb{R}} e^{-i\,s\,\frac{\omega}{\lambda}}\,f(s)\,ds,\end{aligned}$$

which gives the desired formula. □

Proposition 5.3.5 (Phase multiplication)**.** *Let $f \in L^1(\mathbb{R})$ and $\omega_0 \in \mathbb{R}$. Then the function $t \mapsto e^{i\,t\,\omega_0}\,f(t)$ is in $L^1(\mathbb{R})$ and*

$$\mathcal{F}[e^{i\,t\,\omega_0}\,f](\omega) = \mathcal{F}[f](\omega - \omega_0).$$

Proof. By using the definition we have

$$\mathcal{F}[e^{i\,t\,\omega_0}\,f](\omega) = \int_{\mathbb{R}} e^{-i\,t\,\omega}\,e^{i\,t\,\omega_0}\,f(t)\,dt = \int_{\mathbb{R}} e^{-i\,t\,(\omega-\omega_0)}\,f(t)\,dt,$$

which proves the formula. □

Proposition 5.3.6 (Fourier transform of the derivative)**.** *Let $f \in L^1(\mathbb{R}) \cap C^0(\mathbb{R})$ be a continuous summable function. Let us assume that f' is piecewise continuous, with f' having only jump discontinuities at $\{x_0, \dots, x_N, \dots\} \subset \mathbb{R}$ and*

$$|x_j - x_k| \geq \delta > 0, \qquad \textit{for every } j \neq k.$$

Let us suppose that f' is in $L^1(\mathbb{R})$. Then we have the formula

$$\mathcal{F}[f'](\omega) = i\,\omega\,\mathcal{F}[f](\omega). \tag{5.3.2}$$

In particular, we get

$$\lim_{|\omega|\to\infty} \Big|\omega\,\mathcal{F}[f](\omega)\Big| = 0. \tag{5.3.3}$$

Proof. We first observe that (5.3.3) is a plain consequence of (5.3.2) and the Riemann-Lebesgue Lemma applied to f', which is in $L^1(\mathbb{R})$ by assumption. Indeed, we would have

$$\lim_{|\omega|\to\infty} \Big|\omega\,\mathcal{F}[f](\omega)\Big| = \lim_{|\omega|\to\infty} \Big|\mathcal{F}[f'](\omega)\Big| = 0.$$

Let us prove formula (5.3.2) under the stronger assumption that f' *is continuous on* $\mathbb{R}$. The general case can be handled as we did in the analogous case for the Laplace transform, we leave the details as an interesting exercise for the reader.

In this case, we observe that by basic calculus

$$f(t) - f(0) = \int_0^t f'(s)\, ds,$$

and since we are assuming $f' \in L^1(\mathbb{R})$, then the limit

$$\lim_{t\to+\infty} \int_0^t f'(s)\, ds,$$

exists and is finite. By using the identity above, this implies that the limit

$$\lim_{t\to+\infty} f(t),$$

exists and is finite. Since $f \in L^1(\mathbb{R})$, this limit is 0 by Lemma 3.3.12. With a similar argument, we also get

$$\lim_{t\to-\infty} f(t) = 0.$$

We now observe that by using an integration by parts

$$\begin{aligned}\mathcal{F}[f'](\omega) = \lim_{L\to\infty} \int_{-L}^{L} e^{-i\,t\,\omega}\, f'(t)\, dt &= \lim_{L\to+\infty} \left[e^{-i\,L\,\omega}\, f(L) - e^{i\,L\,\omega}\, f(-L))\right] \\ &\quad + i\,\omega \lim_{L\to+\infty} \int_{-L}^{L} e^{-i\,t\,\omega}\, f(t)\, dt \\ &= \lim_{L\to+\infty} \left[e^{-i\,L\,\omega}\, f(L) - e^{i\,L\,\omega}\, f(-L))\right] \\ &\quad + i\,\omega\, \mathcal{F}[f](\omega),\end{aligned}$$

and this gives the desired conclusion, since we proved above that

$$\lim_{L\to+\infty} \left[e^{-i\,L\,\omega}\, f(L) - e^{i\,L\,\omega}\, f(-L)\right] = 0.$$

This conclues the proof in the case that f' is continuous. $\square$

Corollary 5.3.7. *Let $f \in L^1(\mathbb{R}) \cap C^{(n-1)}(\mathbb{R})$ be such that $f', \dots, f^{(n-1)} \in L^1(\mathbb{R})$. Let us suppose that $f^{(n-1)}$ verifies the hypothesis of Proposition 5.3.6. Then we have the formula for $k = 1, \dots, n$*

$$\mathcal{F}[f^{(k)}](\omega) = (i\,\omega)^k\, \mathcal{F}[f](\omega), \qquad \omega \in \mathbb{R}. \tag{5.3.4}$$

In particular, we get

$$\lim_{|\omega|\to\infty} \Big| |\omega|^n\, \mathcal{F}[f](\omega) \Big| = 0. \tag{5.3.5}$$

Remark 5.3.8. The previous result can be summarized by saying that "*the more regular the signal f is, the faster the Fourier transform decays at infinity*". This is the converse of Remark 5.2.5.

Proposition 5.3.9 (Fourier transform of a convolution)**.** *Let $f, g \in L^1(\mathbb{R})$, then we have*

$$\mathcal{F}[f * g](\omega) = \mathcal{F}[f](\omega)\, \mathcal{F}[g](\omega).$$

Proof. We already know that $f * g \in L^1(\mathbb{R})$, thus we can compute the Fourier transform. We have

$$\mathcal{F}[f * g](\omega) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(t-y)\, g(y)\, dy \right) e^{-i\,t\,\omega}\, dt. \tag{5.3.6}$$

We now observe that for every $\omega \in \mathbb{R}$ the function

$$(t, y) \mapsto f(t-y)\, g(y)\, e^{-i\,t\,\omega},$$

is summable over $\mathbb{R} \times \mathbb{R}$, since

$$|f(t-y)\, g(y)\, e^{-i\,t\,\omega}| \le |f(t-y)|\, |g(y)|,$$

and the function $(t, y) \mapsto |f(t-y)|\, |g(y)|$ is positive and such that:

- for a. e. $y \in \mathbb{R}$ the function $t \mapsto |f(t-y)|\, |g(y)|$ is summable on $\mathbb{R}$;
- the function $y \mapsto \int_{\mathbb{R}} |f(t-y)|\, |g(y)|\, dt$ is summable on $\mathbb{R}$.

By applying Tonelli's Theorem, we thus obtain summability of

$$(t, y) \mapsto |f(t-y)|\, |g(y)|$$

and this in turn implies summability of $(t, y) \mapsto f(t-y)\, g(y)\, e^{-i\,t\,\omega}$.

We can thus apply Fubini's Theorem in (5.3.6) and exchange the order of integration, so to get (with a simple change of variable in the second identity)

$$\begin{aligned}
\mathcal{F}[f * g](\omega) &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(t-y)\, e^{-i\,(t-y)\,\omega}\, dt \right) g(y)\, e^{-i\,y\,\omega}\, dy \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(t)\, e^{-i\,t\,\omega}\, dt \right) g(y)\, e^{-i\,y\,\omega}\, dy \\
&= \mathcal{F}[f](\omega) \int_{\mathbb{R}} g(y)\, e^{-i\,y\,\omega}\, dy = \mathcal{F}[f](\omega)\, \mathcal{F}[g](\omega),
\end{aligned}$$

as desired. □

Example 5.3.10 (Fourier transform of the triangular function)**.** We recall that the triangular function is defined by

$$\mathrm{tri}(t) = \begin{cases} 0, & \text{if } |t| \ge 1, \\ 1 - |t|, & \text{if } |t| < 1. \end{cases}$$

We have already seen in Example 3.5.8 that

$$\mathrm{tri}(t) = \mathrm{rect} * \mathrm{rect}(t).$$

By Proposition 5.3.9 and Example 5.1.4, we thus get

$$\mathcal{F}[\mathrm{tri}](\omega) = \mathcal{F}[\mathrm{rect} * \mathrm{rect}](\omega) = (\mathcal{F}[\mathrm{rect}](\omega))^2 = \left(\mathrm{sinc}\left(\frac{\omega}{2\,\pi} \right) \right)^2.$$

By recalling the definition of cardinal sine function, we can also write

$$\mathcal{F}[\mathrm{tri}](\omega) = \frac{4}{\omega^2}\, \sin^2\left(\frac{\omega}{2} \right).$$

Observe that this is an L^1 function.

4. Inversion formula

At first, we need the following variant of the Riemann-Lebesgue Lemma above. The idea of the proof is the same already exploited in Theorems 4.3.1 and 5.2.1

Lemma 5.4.1. *Let $F \in L^1([a,b])$, then we have*

$$\lim_{L\to+\infty}\int_a^b F(y)\,\sin(L\,y)\,dy=\lim_{L\to+\infty}\int_a^b F(y)\,\cos(L\,y)\,dy=0.$$

Proof. The proof is the same as that of (5.2.2). Let us focus on the first limit, the proof for the second one being exactly the same. By elementary properties of the trigonometric functions, we have

$$\int_a^b F(y)\,\sin(L\,y)\,dy=-\int_a^b F(y)\,\sin(L\,y-\pi)\,dy=-\int_{a-\frac{\pi}{L}}^{b-\frac{\pi}{L}} F\left(s+\frac{\pi}{L}\right)\,\sin(L\,s)\,ds.$$

Thus we can infer

$$\begin{aligned}\int_a^b F(y)\,\sin(L\,y)\,dy&=\frac{1}{2}\left[\int_a^b F(y)\,\sin(L\,y)\,dy-\int_{a-\frac{\pi}{L}}^{b-\frac{\pi}{L}} F\left(y+\frac{\pi}{L}\right)\,\sin(L\,y)\,dy\right]\\&=\frac{1}{2}\int_a^{b-\frac{\pi}{L}}\left[F(y)-F\left(y+\frac{\pi}{L}\right)\right]\,\sin(L\,y)\,dy\\&+\frac{1}{2}\int_{b-\frac{\pi}{L}}^{b} F(y)\,\sin(L\,y)\,dy-\frac{1}{2}\int_{a-\frac{\pi}{L}}^{a} F\left(y+\frac{\pi}{L}\right)\,\sin(L\,y)\,dy,\end{aligned}$$

and the 3 integrals converges to 0. For the first one we have to use the continuity of translations with respect to L^p norms (see Proposition 3.4.5), since

$$\left|\int_a^{b-\frac{\pi}{L}}\left[F(y)-F\left(y+\frac{\pi}{L}\right)\right]\,\sin(L\,y)\,dy\right|\le\int_a^{b-\frac{\pi}{L}}\left|F(y)-\mathcal{T}_{\frac{\pi}{L}}\,F\,(y)\right|\,dy,$$

where we used the usual notation $\mathcal{T}_h F(t)=F(t+h)$. The other two integrals converge to 0 because they are the integral of a summable function on an interval which is "squeezing" (i.e. the width of the interval tends to 0). □

Theorem 5.4.2 (Inversion for piecewise C^1 signals)**.** *Let $f \in L^1(\mathbb{R})$ be a piecewise C^1 function. Let us assume that f and f' have only jump discontinuities at $\{t_0,\dots,t_k,\dots\}$, with*

$$|t_k-t_j|\ge\delta>0,\qquad \textit{for every } k\ne j.$$

We normalize it so that

$$f(t_j)=\frac{1}{2}\left[\lim_{x\to t_j^+}f(t)+\lim_{x\to t_j^-}f(t)\right],\qquad j=1,2,\dots.$$

Then we have

$$f(t)=\frac{1}{2\,\pi}\lim_{L\to+\infty}\int_{-L}^{L}\mathcal{F}[f](\omega)\,e^{i\,t\,\omega}\,d\omega. \tag{5.4.1}$$

In the case $\mathcal{F}[f]\in L^1(\mathbb{R})$, the formula can be written directly as

$$f(t)=\frac{1}{2\,\pi}\int_{\mathbb{R}}\mathcal{F}[f](\omega)\,e^{i\,t\,\omega}\,d\omega.$$

Proof. Let us fix $t \in \mathbb{R}$, we set

$$f(t^+) = \lim_{s \to t^+} f(s) \qquad \text{and} \qquad f(t^-) = \lim_{s \to t^-} f(s).$$

For every $L > 0$, we observe that by Fubini's Theorem for every $t \in \mathbb{R}$ we have[1]

$$\begin{aligned}\int_{-L}^{L} \mathcal{F}[f](\omega)\, e^{i\,t\,\omega}\, d\omega &= \int_{-L}^{L} \left(\int_{\mathbb{R}} f(x)\, e^{-i\,x\,\omega}\, dx\right) e^{i\,t\,\omega}\, d\omega \\ &= \int_{\mathbb{R}} f(x) \left(\int_{-L}^{L} e^{-i\,(x-t)\,\omega}\, d\omega\right) dx \\ &= \int_{\mathbb{R}} f(x)\, \mathcal{F}\Big[1_{[-L,L]}\Big](x-t)\, dx \\ &= \int_{\mathbb{R}} f(t-y)\, \mathcal{F}\Big[1_{[-L,L]}\Big](-y)\, dy.\end{aligned}$$

In the last identity we used a change of variable. We now observe that (see Remark 5.8.2)

$$\mathcal{F}\Big[1_{[-L,L]}\Big](-y) = 2\,L\,\operatorname{sinc}\left(-\frac{L}{\pi}\,y\right) = 2\,L\,\operatorname{sinc}\left(\frac{L}{\pi}\,y\right), \qquad y \in \mathbb{R},$$

where we also used that sinc is an even function. Thus we have obtained for every $t \in \mathbb{R}$

$$\frac{1}{2\,\pi} \int_{-L}^{L} \mathcal{F}[f](\omega)\, e^{i\,t\,\omega}\, d\omega = \frac{L}{\pi} \int_{\mathbb{R}} f(t-y)\, \operatorname{sinc}\left(\frac{L}{\pi}\,y\right)\, dy. \tag{5.4.2}$$

In order to conclude the proof, it is sufficient to show that

$$\lim_{L \to +\infty} \frac{L}{\pi} \int_0^{+\infty} f(t-y)\, \operatorname{sinc}\left(\frac{L}{\pi}\,y\right)\, dy = \frac{1}{2}\, f(t^-), \tag{5.4.3}$$

and

$$\lim_{L \to +\infty} \frac{L}{\pi} \int_{-\infty}^{0} f(t-y)\, \operatorname{sinc}\left(\frac{L}{\pi}\,y\right)\, dy = \frac{1}{2}\, f(t^+).$$

If we are able to prove this, then (5.4.1) will follow from (5.4.2). We focus on proving (5.4.3), the other formula is then obtained exactly in the same manner.

We recall that (see Exercise 3.7.4)

$$\int_{\mathbb{R}} \operatorname{sinc} s\, ds = \lim_{R \to +\infty} \int_{-R}^{R} \operatorname{sinc} s\, ds = 1,$$

and $y \mapsto \operatorname{sinc} y$ is an even function, thus

$$\int_0^{+\infty} \operatorname{sinc} s\, ds = \int_{-\infty}^{0} \operatorname{sinc} s\, ds = \frac{1}{2}.$$

With a change of variable, we also have

$$\frac{1}{2} = \int_0^{+\infty} \operatorname{sinc} s\, ds = \frac{L}{\pi} \int_0^{+\infty} \operatorname{sinc}\left(\frac{L}{\pi}\,y\right)\, dy.$$

---

[1] The summability of the function $(x,\omega) \mapsto f(x)\, e^{-i\,(x-t)\,\omega}$ (needed to apply Fubini's Theorem) can be inferred from Tonelli's Theorem: indeed, observe that the positive function $(x,\omega) \mapsto |f(x)\, e^{-i\,(x-t)\,\omega}| = |f(x)|$ defined on $\mathbb{R} \times [-L,L]$ is such that

- for a. e. $\omega \in [-L,L]$, the function $x \mapsto |f(x)|$ is summable on $\mathbb{R}$ (because $f \in L^1(\mathbb{R})$)
- the function $\omega \mapsto \int_{\mathbb{R}} |f(x)|\, dx$ is summable on the bounded interval $[-L,L]$ (indeed, this is a constant function).

By Tonelli's Theorem this entails that $(x,\omega) \mapsto |f(x)\, e^{-i\,(x-t)\,\omega}|$ is summable and thus the same can be said for $(x,\omega) \mapsto f(x)\, e^{-i\,(x-t)\,\omega}$, as desired.

Thus we can write

$$\begin{aligned}\frac{L}{\pi}\int_0^{+\infty} f(t-y)\,\mathrm{sinc}\left(\frac{L}{\pi}\,y\right)\,dy-\frac{1}{2}\,f(t^-)\\ =\frac{L}{\pi}\int_0^{+\infty}[f(t-y)-f(t^-)]\,\mathrm{sinc}\left(\frac{L}{\pi}\,y\right)\,dy.\end{aligned}$$

In order to prove (5.4.3), it is sufficient to prove that the last integral converges to 0, as L goes to $+\infty$.

We now take $T>1$ and write

$$\begin{aligned}&\frac{L}{\pi}\int_0^{+\infty}\Big[f(t-y)-f(t^-)\Big]\,\mathrm{sinc}\left(\frac{L}{\pi}\,y\right)\,dy\\ &=\frac{L}{\pi}\int_0^{T}\Big[f(t-y)-f(t^-)\Big]\,\mathrm{sinc}\left(\frac{L}{\pi}\,y\right)\,dy\\ &+\frac{L}{\pi}\int_T^{+\infty}\Big[f(t-y)-f(t^-)\Big]\,\mathrm{sinc}\left(\frac{L}{\pi}\,y\right)\,dy=:\mathcal{I}_1+\mathcal{I}_2.\end{aligned}\tag{5.4.4}$$

We have to handle carefully the two integrals $\mathcal{I}_1$ and $\mathcal{I}_2$, due to the fact that $\mathrm{sinc}\not\in L^1(\mathbb{R})$ but we only have $\mathrm{sinc}\in L^1_{\mathrm{loc}}(\mathbb{R})$.

Estimate of the first integral $\mathcal{I}_1$. We start with the first one: we have

$$\mathcal{I}_1=\frac{L}{\pi}\int_0^{T}[f(t-y)-f(t^-)]\,\mathrm{sinc}\left(\frac{L}{\pi}\,y\right)\,dy=\frac{1}{\pi}\int_0^{T}\frac{f(t-y)-f(t^-)}{y}\,\sin(L\,y)\,dy,$$

and observe that for every $t\in\mathbb{R}$ the function

$$F(y)=\frac{f\,(t-y)-f(t^-)}{y},$$

is in $L^1([0,T])$. Indeed, since $f\in L^1(\mathbb{R})$ we have that $F\in L^1([\varepsilon,T])$, for every $\varepsilon>0$. Moreover, when $y\in[0,\varepsilon]$, we can use the mean-value Theorem[2] to infer existence of $\xi_y\in[t-y,t]$ such that

$$|F(y)|=\left|\frac{f\,(t-y)-f(t^-)}{y}\right|=|f'(\xi_y)|\le\max_{\xi\in[t-\varepsilon,t]}|f'(\xi)|,$$

i.e. the function F is in $L^\infty([0,\varepsilon])\subset L^1([0,\varepsilon])$. By using Lemma 5.4.1 we thus get

$$\lim_{L\to+\infty}\mathcal{I}_1=\lim_{L\to+\infty}\frac{1}{\pi}\int_0^{T}\frac{f(t-y)-f(t^-)}{y}\,\sin(L\,y)\,dy=0.$$

Up to now, from (5.4.4) we obtained that

$$\begin{aligned}\lim_{L\to+\infty}&\left|\frac{L}{\pi}\int_0^{+\infty}f(t-y)\,\mathrm{sinc}\left(\frac{L}{\pi}\,y\right)\,dy-\frac{1}{2}\,f(t^-)\right|\\ &\le\lim_{L\to+\infty}|\mathcal{I}_1|+\lim_{L\to+\infty}|\mathcal{I}_2|=\lim_{L\to+\infty}|\mathcal{I}_2|.\end{aligned}\tag{5.4.5}$$

[2] *Teorema di Lagrange*, in italian.

Estimate of the second integral $\mathcal{I}_2$. In order to conclude, we need to show that the last limit is 0. We have

$$
\begin{aligned}
|\mathcal{I}_2| &= \left|\frac{L}{\pi}\int_T^{+\infty}[f(t-y)-f(t^-)]\,\mathrm{sinc}\left(\frac{L}{\pi}\,y\right)\,dy\right| \\
&\le \frac{L}{\pi}\int_T^{+\infty}|f(t-y)|\,\left|\mathrm{sinc}\left(\frac{L}{\pi}\,y\right)\right|\,dy + \left|\int_{\frac{L}{\pi}T}^{+\infty} f(t^-)\,\mathrm{sinc}\,s\,ds\right| \\
&\le \frac{1}{\pi}\int_T^{+\infty}|f(t-y)|\,\,dy + |f(t^-)|\,\left|\int_{\frac{L}{\pi}T}^{+\infty}\mathrm{sinc}\,s\,ds\right|,
\end{aligned}
$$

where in the last inequality we used that

$$
\frac{L}{\pi}\,\left|\mathrm{sinc}\left(\frac{L}{\pi}\,y\right)\right| = \frac{1}{\pi}\,\frac{|\sin(L\,y)|}{|y|} \le \frac{1}{\pi}, \qquad \text{for } y \ge T > 1.
$$

By using an integration by parts, for every $L \ge \pi$ we get

$$
\begin{aligned}
\left|\int_{\frac{L}{\pi}T}^{+\infty}\mathrm{sinc}\,s\,ds\right| = \left|\int_{\frac{L}{\pi}T}^{+\infty}\frac{\sin(\pi\,s)}{\pi\,s}\,ds\right| &= \left|\frac{\cos(L\,T)}{\pi\,L\,T} - \int_{\frac{L}{\pi}T}^{+\infty}\frac{\cos(\pi\,s)}{\pi^2\,s^2}\,ds\right| \\
&\le \left|\frac{\cos(L\,T)}{\pi\,L\,T}\right| + \left|\int_{\frac{L}{\pi}T}^{+\infty}\frac{\cos(\pi\,s)}{\pi^2\,s^2}\,ds\right| \\
&\le \frac{1}{\pi\,L\,T} + \frac{1}{\pi^2}\int_{\frac{L}{\pi}T}^{+\infty}\frac{1}{s^2}\,ds \\
&= \frac{1}{\pi\,L\,T} + \frac{1}{\pi\,L\,T} \le \frac{2}{\pi\,L}.
\end{aligned}
$$

This implies

$$
\lim_{L\to+\infty}\left|\int_{\frac{L}{\pi}T}^{+\infty}\mathrm{sinc}\,s\,ds\right| = 0,
$$

and thus

$$
\lim_{L\to+\infty}|\mathcal{I}_2| \le \frac{1}{\pi}\int_T^{+\infty}|f(t-y)|\,\,dy.
$$

Conclusion. By resuming everything, from (5.4.5) we obtained

$$
\lim_{L\to+\infty}\left|\frac{L}{\pi}\int_0^{+\infty}f(t-y)\,\mathrm{sinc}\left(\frac{L}{\pi}\,y\right)\,dy - \frac{1}{2}\,f(t^-)\right| \le \frac{1}{\pi}\int_T^{+\infty}|f(t-y)|\,\,dy,
$$

which is valid for every $T > 1$. In order to conclude, we just have to observe that since

$$
\int_{\mathbb{R}}|f(t-y)|\,dy < +\infty,
$$

then we have

$$
\lim_{T\to+\infty}\int_T^{+\infty}|f(t-y)|\,\,dy = 0.
$$

This finally gives

$$
\lim_{L\to+\infty}\left|\frac{L}{\pi}\int_0^{+\infty}f(t-y)\,\mathrm{sinc}\left(\frac{L}{\pi}\,y\right)\,dy - \frac{1}{2}\,f(t^-)\right| = 0,
$$

as desired. □

Remark 5.4.3. Observe that (5.4.2) can also be written as

$$\frac{1}{2\pi}\int_{-L}^{L}\mathcal{F}[f](\omega)\,e^{i\,t\,\omega}\,d\omega = f*\mathcal{H}_L(t),\qquad \text{where}\quad \mathcal{H}_L(y)=\frac{L}{\pi}\,\mathrm{sinc}\left(\frac{L}{\pi}\,y\right).$$

Then in the previous proof we have shown that for every $t_0\in\mathbb{R}$ we have

$$\lim_{L\to+\infty} f*\mathcal{H}_L(t_0)=f(t_0),$$

provided that f verifies the hypotheses of Theorem 5.4.2 and is continuous at t_0.

Corollary 5.4.4 (Duality formula)**.** *Let $f\in L^1(\mathbb{R})$ verify the hypotheses of Theorem 5.4.2. Let us suppose that $\mathcal{F}[f]\in L^1(\mathbb{R})$, then*

$$\mathcal{F}\Big[\mathcal{F}[f]\Big](\omega)=2\,\pi\,f(-\omega),\qquad \omega\in\mathbb{R}. \tag{5.4.6}$$

Proof. From the inversion formula we know that

$$\int_{\mathbb{R}}\mathcal{F}[f](\omega)\,e^{i\,t\,\omega}\,d\omega=2\,\pi\,f(t).$$

By changing the name of the variables we get

$$\int_{\mathbb{R}}\mathcal{F}[f](t)\,e^{i\,t\,\omega}\,dt=2\,\pi\,f(\omega).$$

It is now sufficient to observe that the left-hand side coincides with the Fourier transform of $\mathcal{F}[f]$, evaluated at $-\omega$. This shows that

$$\mathcal{F}\Big[\mathcal{F}[f]\Big](-\omega)=2\,\pi\,f(\omega),$$

and thus concludes the proof. □

Remark 5.4.5. When $\mathcal{F}[f]\notin L^1(\mathbb{R})$ formula (5.4.6) still holds in the following form

$$\lim_{L\to+\infty}\int_{-L}^{L}\mathcal{F}[f](t)\,e^{-i\,t\,\omega}\,dt=2\,\pi\,f(-\omega),\qquad \omega\in\mathbb{R}.$$

Let us record the following more general result.

Theorem 5.4.6 (Inversion for L^1 signals)**.** *Let $f\in L^1(\mathbb{R})$ be a function such that $\mathcal{F}[f]\in L^1(\mathbb{R})$ as well. Then we have*

$$f(t)=\frac{1}{2\,\pi}\int_{\mathbb{R}}\mathcal{F}[f](\omega)\,e^{i\,t\,\omega}\,d\omega,\qquad \text{for a. e. } t\in\mathbb{R}.$$

Proof. Let us take $g\in C^\infty_0(\mathbb{R})$ a function such that $\int_{\mathbb{R}}g(t)\,dt=1$. For every $n\in\mathbb{N}\setminus\{0\}$, we define the sequence

$$g_n(t)=n\,g(n\,t).$$

By Theorem 3.5.13, we know that

$$\lim_{n\to\infty}\|g_n*f-f\|_{L^1(\mathbb{R})}=0, \tag{5.4.7}$$

and $g_n*f\in C^\infty(\mathbb{R})\cap L^1(\mathbb{R})$. Observe that thanks to Proposition 5.3.9, we have

$$\mathcal{F}[g_n*f](\omega)=\mathcal{F}[g_n](\omega)\,\mathcal{F}[f](\omega).$$

Since by assumption $\mathcal{F}[f]\in L^1(\mathbb{R})$ and thanks to Proposition 5.2.1 we have $\mathcal{F}[g]\in L^\infty(\mathbb{R})$, from the previous identity we get

$$\mathcal{F}[g_n*f]\in L^1(\mathbb{R}),$$

thanks to Hölder's inequality (see Proposition 3.3.5). By smoothness of $g_n * f$ and integrability of its Fourier transform, we thus obtain from Theorem 5.4.2

$$g_n * f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}[g_n * f](\omega)\, e^{i\,t\,\omega}\, d\omega, \qquad t \in \mathbb{R}.$$

We now observe that from (5.4.7) we can suppose to have[3]

$$\lim_{n\to\infty} g_n * f(t) = f(t), \qquad \text{for a. e. } t \in \mathbb{R},$$

which implies

$$f(t) = \frac{1}{2\pi} \lim_{n\to\infty} \int_{\mathbb{R}} \mathcal{F}[g_n * f](\omega)\, e^{i\,t\,\omega}\, d\omega, \qquad \text{for a. e. } t \in \mathbb{R}.$$

In order to conclude, we need to take the limit under the integral sign. Let us start by observing that

$$\mathcal{F}[g_n * f](\omega) = \mathcal{F}[g_n](\omega)\, \mathcal{F}[f](\omega) = \mathcal{F}[g]\left(\frac{\omega}{n}\right) \mathcal{F}[f](\omega),$$

where we used Proposition 5.3.2 and the definition of g_n. This shows that

$$\lim_{n\to\infty} \mathcal{F}[g_n * f](\omega)\, e^{i\,t\,\omega} = \mathcal{F}[g](0)\, \mathcal{F}[f](\omega)\, e^{i\,t\,\omega}, \qquad \text{for a. e. } \omega \in \mathbb{R}.$$

Moreover, we can easily produce a uniform L^1 domination of this function: indeed

$$\left|\mathcal{F}[g_n * f](\omega)\, e^{i\,t\,\omega}\right| = \left|\mathcal{F}[g]\left(\frac{\omega}{n}\right) \mathcal{F}[f](\omega)\right| \le \|\mathcal{F}[g]\|_{L^\infty(\mathbb{R})}\, |\mathcal{F}[f](\omega)|,$$

and the last function is L^1 (by assumption) and independent of n. We can thus apply Lebesgue Dominated Convergence Theorem and obtain

$$f(t) = \frac{\mathcal{F}[g](0)}{2\pi} \int_{\mathbb{R}} \mathcal{F}[f](\omega)\, e^{i\,t\,\omega}\, d\omega, \qquad \text{for a. e. } t \in \mathbb{R}.$$

In order to conclude, it is only left to observe that

$$\mathcal{F}[g](0) = \int_{\mathbb{R}} g(t)\, dt = 1,$$

since g has been chosen at the beginning with this property. This concludes the proof. □

5. Back to the Laplace transform

We already seen in Remark 5.1.2 how the Fourier transform of a signal $f \in L^1(\mathbb{R})$ is linked to the Laplace transform of two causal signals obtained from f. Conversely, if $f : \mathbb{R} \to \mathbb{C}$ is an $L-$transformable causal signal and

$$\sigma_f = \inf\left\{\alpha \in \mathbb{R} \,:\, e^{-\alpha\,t} f \in L^1(\mathbb{R})\right\} < +\infty,$$

for $\alpha > \sigma_f$ we can define the new causal signal

$$g(t) = e^{-\alpha\,t} f(t),$$

[3]We use the following remarkable fact: if $\{f_n\}_{n\in\mathbb{N}} \subset L^p(\mathbb{R})$ converges in L^p norm to a function $f \in L^p(\mathbb{R})$, then there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} f_{n_k}(t) = f(t), \qquad \text{for a. e. } t \in \mathbb{R}.$$

and take its Fourier transform. Indeed, observe that this is in $L^1(\mathbb{R})$ by construction. We thus get the relation for $\omega \in \mathbb{R}$ and $\alpha > \sigma_f$

$$\mathcal{F}[g](\omega) = \int_0^{+\infty} e^{-i\,t\,\omega}\, e^{-\alpha\,t}\, f(t)\,dt = \int_0^{\infty} e^{-(\alpha+i\,\omega)\,t}\, f(t)\,dt = \mathcal{L}[f](\alpha + i\,\omega). \tag{5.5.1}$$

We can now exploit this relation in order to prove the Inversion Formula for the Laplace transform.

Proof of Theorem 4.5.1. With the notation above, we further assume that f verifies the hypotheses of Theorem 4.5.1. Then the signal $g(t) = e^{-\alpha\,t}\, f(t)$ verifies them, as well, and from the Inversion Formula for the Fourier transform of Theorem 5.4.2 we get

$$g(t) = \frac{1}{2\,\pi} \lim_{L\to+\infty} \int_{-L}^{L} e^{i\,\omega\,t}\, \mathcal{F}[g](\omega)\,d\omega.$$

By using this in (5.5.1) and recalling the definition of g, we get

$$e^{-\alpha\,t}\, f(t) = \frac{1}{2\,\pi} \lim_{L\to+\infty} \int_{-L}^{L} e^{i\,\omega\,t}\, \mathcal{L}[f](\alpha + i\,\omega)\,d\omega,$$

that is, multiplying both sides for $e^{\alpha\,t}$

$$\begin{aligned} f(t) &= \frac{1}{2\,\pi} \lim_{L\to+\infty} \int_{-L}^{L} e^{(\alpha+i\,\omega)\,t}\, \mathcal{L}[f](\alpha + i\,\omega)\,d\omega \\ &= \frac{1}{2\,\pi\,i} \lim_{L\to+\infty} \int_{\alpha-i\,L}^{\alpha+i\,L} e^{z\,t}\, \mathcal{L}[f](z)\,dz. \end{aligned}$$

This proves Theorem 4.5.1. □

Theorem 5.5.1 (Uncertainty principle)**.** *Let $f \in L^1(\mathbb{R})$ be a such that there exist $C, T > 0$ and $\alpha > 0$ for which*

$$|f(t)| \le C\, e^{-\alpha\,|t|}, \qquad \textit{for a. e. } |t| \ge T.$$

Let us suppose that f does not identically vanish. Then the set

$$\{\omega \in \mathbb{R} \,:\, \mathcal{F}[f](\omega) = 0\},$$

is either empty or made of isolated points.

Proof. By using Proposition 4.8.5 and Theorem 4.8.9, the function f is $L-$transformable and its bilateral Laplace transform $\mathcal{B}[f]$ is holomorphic in the strip

$$\{z \in \mathbb{C} \,:\, -\alpha < \mathrm{Re}(z) < \alpha\},$$

thanks to the growth assumption on f. This strip contains the imaginary axis in its interior, thus we have

$$\mathcal{F}[f](\omega) = \mathcal{B}[f](i\,\omega), \qquad \omega \in \mathbb{R}.$$

In order to conclude, we can now use the properties of the zeros of holomorphic functions, see Proposition 1.8.8. □

By observing that a compactly supported function satisfies the hypothesis of the previous theorem (recall Lemma 3.7.6), we get the following

Corollary 5.5.2. *Let $f \in L^1(\mathbb{R})$ be a compactly supported function. Then its Fourier transform $\mathcal{F}[f]$ can not be compactly supported, unless f identically vanishes.*

6. The Schwartz class and the Fourier transform in L^2

We recall that $C^\infty(\mathbb{R})$ is the set of functions $\varphi : \mathbb{R} \to \mathbb{C}$ which are differentiable infinitely many times. The *Schwartz class* $\mathcal{S}$ is an important subset of $C^\infty(\mathbb{R})$, which plays a major rôle in the theory of the Fourier transform.

Definition 5.6.1. Let $\varphi \in C^\infty(\mathbb{R})$, we say that φ belongs to the Schwartz class $\mathcal{S}$ if for every $m, k \in \mathbb{N}$ we have

$$[\varphi]_{m,k} := \sup_{t\in\mathbb{R}} \left| t^m \, \varphi^{(k)}(t) \right| < +\infty. \tag{5.6.1}$$

In other words, a function from $\mathcal{S}$ is such that it and all its derivatives decay to 0 at infinity faster than any polynomial. It is easy to verify that $\mathcal{S}$ is a vector space over $\mathbb{C}$: if $\alpha, \beta \in \mathbb{C}$ and $\varphi, \psi \in \mathcal{S}$, then we have

$$\alpha\,\varphi + \beta\,\psi \in \mathcal{S},$$

as well.

The following simple result will be quite useful in order to verify that a function belongs to $\mathcal{S}$.

Lemma 5.6.2. *Let $\varphi \in C^\infty(\mathbb{R})$, then we have*

$$\varphi \in \mathcal{S} \qquad \Longleftrightarrow \qquad \lim_{|t|\to+\infty} \left| t^n \, \varphi^{(k)}(t) \right| = 0, \qquad \textit{for every } n, k \in \mathbb{N}.$$

Proof. Let us suppose that $\varphi \in \mathcal{S}$, we want to show that

$$\lim_{|t|\to+\infty} \left| t^n \, \varphi^{(k)}(t) \right| = 0, \qquad \text{for every } n, k \in \mathbb{N}.$$

For every $|t| \geq 1$ and every $n, k \in \mathbb{N}$, we have

$$\left| t^n \, \varphi^{(k)}(t) \right| = \frac{1}{|t|} \left| t^{n+1} \, \varphi^{(k)}(t) \right| \leq \frac{1}{|t|} \sup_{t\in\mathbb{R}} \left| t^{n+1} \, \varphi^{(k)}(t) \right| = \frac{1}{|t|} \, [\varphi]_{n+1,k}.$$

By taking the limit as $|t|$ goes to $+\infty$, we get

$$\lim_{|t|\to+\infty} \left| t^n \, \varphi^{(k)}(t) \right| \leq [\varphi]_{n+1,k} \lim_{|t|\to+\infty} \frac{1}{|t|} = 0,$$

as desired.

We now prove the converse implication. We suppose that

$$\lim_{|t|\to+\infty} \left| t^n \, \varphi^{(k)}(t) \right| = 0, \qquad \text{for every } n, k \in \mathbb{N},$$

we need to show that

$$[\varphi]_{m,k} = \sup_{t\in\mathbb{R}} \left| t^m \, \varphi^{(k)}(t) \right| < +\infty.$$

By hypothesis and using the definition of limit, for every $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that

$$\left| t^n \, \varphi^{(k)}(t) \right| < \varepsilon, \qquad \text{for every } |t| > M_\varepsilon.$$

On the other hand, the function $t \mapsto |t^n \, \varphi^{(k)}(t)|$ is continuous on $\mathbb{R}$, thus by the Weiestrass' Theorem we have

$$\sup_{t\in[-M_\varepsilon, M_\varepsilon]} |t^n \, \varphi^{(k)}(t)| = \max_{t\in[-M_\varepsilon, M_\varepsilon]} |t^n \, \varphi^{(k)}(t)| < +\infty.$$

In conclusion, we get

$$[\varphi]_{m,k} = \sup_{t\in\mathbb{R}} \left|t^m\, \varphi^{(k)}(t)\right| \le \sup_{|t|>M_\varepsilon} \left|t^m\, \varphi^{(k)}(t)\right| + \sup_{t\in[-M_\varepsilon, M_\varepsilon]} |t^n\, \varphi^{(k)}(t)|$$
$$< \varepsilon + \max_{t\in[-M_\varepsilon, M_\varepsilon]} |t^n\, \varphi^{(k)}(t)| < +\infty,$$

as desired. □

Example 5.6.3. We give some examples:

(1) it is not difficult to see that if $\varphi \in C_0^\infty(\mathbb{R})$, then $\varphi \in \mathcal{S}$. Indeed, by assumption, there exists an interval $[a, b] \subset \mathbb{R}$ such that

$$|\varphi^{(k)}(t)| = 0 \qquad \text{for every } t \in \mathbb{R} \setminus [a, b] \text{ and every } k \in \mathbb{N}.$$

This implies that for every $m, k \in \mathbb{N}$, we have

$$[\varphi]_{m,k} = \sup_{t\in\mathbb{R}} \left|t^m\, \varphi^{(k)}(t)\right| = \max_{t\in[a,b]} \left|t^m\, \varphi^{(k)}(t)\right|$$
$$\le \max\{|a|^m, |b|^m\} \max_{t\in[a,b]} |\varphi^{(k)}(t)| < +\infty,$$

thanks to the Weierstrass' Theorem;

(2) as an example of function in $\mathcal{S}$ not having compact support, we can take the standard Gaussian function $\varphi(t) = e^{-t^2}$. In order to verify that it belong to $\mathcal{S}$, it is sufficient to recall that

$$e^{-t^2} = o(|t|^{-n}), \qquad \text{as } |t| \to +\infty, \text{ for every } n \in \mathbb{N}.$$

In other words, we have

$$\lim_{|t|\to+\infty} |t|^n\, e^{-t^2} = 0, \qquad \text{for every } n \in \mathbb{N}.$$

By using Lemma 5.6.2, we can now easily prove that e^{-t^2} belongs to $\mathcal{S}$;

(3) on the other hand, the function

$$\varphi(t) = \frac{1}{1+t^2}, \qquad \text{for } t \in \mathbb{R},$$

is $C^\infty(\mathbb{R})$, but it does not belong to $\mathcal{S}$. Indeed, we have

$$[\varphi]_{3,0} = \sup_{t\in\mathbb{R}} \left|\frac{t^3}{1+t^2}\right| = +\infty.$$

Proposition 5.6.4. *For every $1 \le p \le \infty$, we have $\mathcal{S} \subset L^p(\mathbb{R})$.*

Proof. Let $\varphi \in \mathcal{S}$. We first consider the case $p = \infty$. In this case, it is sufficient to observe that by definition

$$\|\varphi\|_{L^\infty(\mathbb{R})} = \sup_{t\in\mathbb{R}} |\varphi(t)| = [\varphi]_{0,0} < +\infty,$$

thus φ is bounded on $\mathbb{R}$, i.e. $\varphi \in L^\infty(\mathbb{R})$.

We now prove that $\varphi \in L^1(\mathbb{R})$. We have

$$
\begin{aligned}
\int_{\mathbb{R}} |\varphi(t)|\, dt &= \int_{\mathbb{R}} (1+t^2)\, |\varphi(t)|\, \frac{1}{1+t^2}\, dt \\
&= \int_{\mathbb{R}} \Big(|\varphi(t)| + t^2\, |\varphi(t)|\Big)\, \frac{1}{1+t^2}\, dt \\
&\le ([\varphi]_{0,0} + [\varphi]_{2,0}) \int_{\mathbb{R}} \frac{1}{1+t^2}\, dt = \pi\, ([\varphi]_{0,0} + [\varphi]_{2,0}) < +\infty,
\end{aligned}
$$

as desired. Finally, by using that $\varphi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, we can easily prove that $\varphi \in L^p(\mathbb{R})$ for $1 < p < \infty$, as well. Indeed, we have

$$
\int_{\mathbb{R}} |\varphi(t)|^p\, dt = \int_{\mathbb{R}} |\varphi(t)|^{p-1}\, |\varphi(t)|\, dt \le \|\varphi\|_{L^\infty(\mathbb{R})}^{p-1} \int_{\mathbb{R}} |\varphi(t)|\, dt < +\infty.
$$

This concludes the proof. □

Proposition 5.6.5. *Let $\varphi \in \mathcal{S}$, then we have*

$$
t\,\varphi \in \mathcal{S} \qquad \text{and} \qquad \varphi' \in \mathcal{S}.
$$

More generally, for every $n, \ell \in \mathbb{N}$, we have

$$
t^n\,\varphi \in \mathcal{S} \qquad \text{and} \qquad \varphi^{(k)} \in \mathcal{S}.
$$

Proof. We observe that it is sufficient to prove the first part of the statement, the second part just follows by iterating this result.

In order to prove the first fact, we notice that both $t \mapsto t$ and $t \mapsto \varphi(t)$ are C^∞ functions, thus their product is C^∞ as well. We now take $m \in \mathbb{N}$ and $k \in \mathbb{N} \setminus \{0\}$, by recalling that

$$
\frac{d^k}{dt^k}(t\,\varphi(t)) = \sum_{j=0}^{k} \binom{k}{j} \frac{d^j}{dt^j} t\, \varphi^{(k-j)}(t) = t\,\varphi^{(k)}(t) + k\,\varphi^{(k-1)}(t),
$$

we get

$$
\left| t^m\, \frac{d^k}{dt^k}(t\,\varphi(t)) \right| \le |t^{m+1}\, \varphi^{(k)}(t)| + k\, |t^m\, \varphi^{(k-1)}(t)|.
$$

By taking the supremum over $\mathbb{R}$, we obtain

$$
[t\,\varphi]_{m,k} \le [\varphi]_{m+1,k} + k\,[\varphi]_{m,k-1} < +\infty.
$$

As for the case $k = 0$, it is sufficient to observe that

$$
[t\,\varphi]_{m,0} = \sup_{t\in\mathbb{R}} |t^m\, t\,\varphi| = [\varphi]_{m+1,0} < +\infty.
$$

We now prove the second fact. The fact that $\varphi' \in C^\infty(\mathbb{R})$ is a plain consequence of $\varphi \in C^\infty(\mathbb{R})$. Moreover, for every $m, k \in \mathbb{N}$ we have

$$
[\varphi']_{m,k} = \sup_{t\in\mathbb{R}} \left| t^m\, \frac{d^k}{dt^k} \varphi'(t) \right| = \sup_{t\in\mathbb{R}} \left| t^m\, \varphi^{(k+1)}(t) \right| = [\varphi]_{m,k+1} < +\infty.
$$

This concludes the proof. □

The class $\mathcal{S}$ is important for its remarkable properties with respect to the Fourier transform. These are collected in the following result.

Theorem 5.6.6 ("Schwartz meets Fourier")**.** *We have the following facts:*

i) for every $\varphi \in \mathcal{S}$, we have $\mathcal{F}[\varphi] \in \mathcal{S}$ as well;

ii) for every $\varphi \in \mathcal{S}$ there exists $\psi \in \mathcal{S}$ such that

$$\varphi = \mathcal{F}[\psi];$$

iii) for every $\varphi, \psi \in \mathcal{S}$ we have[4]

$$\int_{\mathbb{R}} \varphi(t)\,\psi(t)^*\,dt = \frac{1}{2\,\pi}\int_{\mathbb{R}} \mathcal{F}[\varphi](\omega)\,\mathcal{F}[\psi](\omega)^*\,d\omega, \qquad (\textit{Parseval's formula}); \tag{5.6.2}$$

iv) in particular, for every $\varphi \in \mathcal{S}$ we have

$$\|\varphi\|^2_{L^2(\mathbb{R})} = \frac{1}{2\,\pi}\left\|\mathcal{F}[\varphi]\right\|^2_{L^2(\mathbb{R})}, \qquad (\textit{Plancherel's formula}). \tag{5.6.3}$$

Proof. We prove i). Let $\varphi \in \mathcal{S}$, thanks to Proposition 5.6.5, we have

$$t^n\,\varphi \in \mathcal{S}, \qquad \text{for every } n \in \mathbb{N}.$$

We can thus use Proposition 5.6.4 and infer that

$$t^n\,\varphi \in L^1(\mathbb{R}), \qquad \text{for every } n \in \mathbb{N}. \tag{5.6.4}$$

By using (5.6.4) and Corollary 5.2.4, we get that $\mathcal{F}[\varphi]$ belongs to $C^\infty(\mathbb{R})$. In order to conclude the proof of point i), we need to prove that

$$\Big[\mathcal{F}[\varphi]\Big]_{m,n} < +\infty,$$

for every $n, m \in \mathbb{N}$. Still by Proposition 5.6.5, this time applied to $t^n\,\varphi \in \mathcal{S}$, we also obtain

$$\frac{d^m}{dt^m}\,(t^n\,\varphi) \in \mathcal{S}.$$

By applying Proposition 5.6.4, we obtain

$$\frac{d^m}{dt^m}\,(t^n\,\varphi) \in L^1(\mathbb{R}), \qquad \text{for every } n, m \in \mathbb{N}. \tag{5.6.5}$$

By using the informations (5.6.4) e (5.6.5), we thus get

$$\begin{aligned}
\left|\omega^m\,\frac{d^n}{d\omega^n}\mathcal{F}[\varphi](\omega)\right| &= |\omega^m\,(-i)^n\,\mathcal{F}[t^n\,\varphi](\omega)| \\
&= |\omega^m\,\mathcal{F}[t^n\,\varphi](\omega)| \\
&= |(i\,\omega)^m\,\mathcal{F}[t^n\,\varphi](\omega)| \\
&= \left|\mathcal{F}\left[\frac{d^m}{dt^m}\,(t^n\,\varphi)\right](\omega)\right|,
\end{aligned}$$

where in the first equality we used Corollary 5.2.4 with the choice,

$$f(t) = t^n\,\varphi(t),$$

while in the last equality we used Corollary 5.3.7 with the choice

$$f(t) = \frac{d^m}{dt^m}\,(t^n\,\varphi)\,.$$

The identity above, in conjunction with the Riemann-Lebesgue Lemma, guarantees that

$$\lim_{|\omega|\to+\infty}\left|\omega^m\,\frac{d^n}{d\omega^n}\mathcal{F}[\varphi](\omega)\right| = 0, \qquad \text{for every } n, m \in \mathbb{N}.$$

[4] Recall that for a complex number $z = x + i\,y$, the symbol z^* denotes its complex conjugate, i.e. $z^* = x - i\,y$.

This finally shows that $\mathcal{F}[\varphi] \in \mathcal{S}$, thanks to Lemma 5.6.2.

We now prove point *ii*). Let $\varphi \in \mathcal{S}$, we want to prove that there exists $\psi \in \mathcal{S}$ such that $\mathcal{F}[\psi] = \varphi$. From the duality formula (5.4.6) we get

$$2\,\pi\,\varphi(-\omega) = \mathcal{F}\Big[\mathcal{F}[\varphi]\Big](\omega).$$

We now set

$$\eta(t) = 2\,\pi\,\varphi(-t) \in \mathcal{S},$$

and use again the duality formula, so to get

$$\varphi(\omega) = \frac{1}{2\,\pi}\,\eta(-\omega) = \frac{1}{4\,\pi^2}\,\mathcal{F}\Big[\mathcal{F}[\eta]\Big](\omega).$$

If we set

$$\psi = \frac{1}{4\,\pi^2}\,\mathcal{F}[\eta],$$

we get the desired result, since $\eta \in \mathcal{S}$ and thus $\mathcal{F}[\eta] \in \mathcal{S}$ thanks to the first part of the proof.

Let us prove the identity (5.6.2). By using the definition of Fourier transform and exchanging the order of integration[5], we have

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{F}[\varphi](\omega)\,\mathcal{F}[\psi](\omega)^*\,d\omega &= \int_{\mathbb{R}} \mathcal{F}[\varphi](\omega)\,\left(\int_{\mathbb{R}} e^{-i\,t\,\omega}\psi(t)\,dt\right)^*\,d\omega\\ &= \int_{\mathbb{R}} \mathcal{F}[\varphi](\omega)\,\left(\int_{\mathbb{R}} e^{i\,t\,\omega}\,\psi(t)^*\,dt\right)\,d\omega\\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathcal{F}[\varphi](\omega)\,e^{i\,t\,\omega}\,d\omega\right)\,\psi(t)^*\,dt.\end{aligned}$$

We finally observe that $\varphi \in \mathcal{S}$ verifies the hypotheses of the Inversion Formula Theorem 5.4.2, thus from (5.4.1) we get the desired formula.

Formula (5.6.3) is a direct consequence of (5.6.2), it is sufficient to take $\psi = \varphi$. $\square$

Proposition 5.6.7. *Let $f \in L^1(\mathbb{R})$ and let $\varphi \in \mathcal{S}$. Then we have $f * \varphi \in C^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Moreover, we have*

$$\frac{d^k}{dt^k}(f * \varphi) = f * \frac{d^k}{dt^k}\varphi. \tag{5.6.6}$$

Proof. By Proposition 5.6.4, we have $\varphi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. By using Proposition 3.5.4 with the choices $q = p = 1$, we get

$$f * \varphi \in L^1(\mathbb{R}).$$

Moreover, by using Proposition 3.5.6 with $p = 1$, we also get

$$f * \varphi \in L^\infty(\mathbb{R}) \cap C(\mathbb{R}).$$

[5]Observe that the positive function

$$(t,\omega) \mapsto \Big|\mathcal{F}[f](\omega)\Big|\,|\psi(t)|,$$

is such that

- for a. e. $\omega \in \mathbb{R}$, the function $t \mapsto |\mathcal{F}[f](\omega)|\,|\psi(t)|$ is summable on $\mathbb{R}$ (since $\psi \in \mathcal{S} \subset L^1(\mathbb{R})$);
- the function $\omega \mapsto \int_{\mathbb{R}} |\mathcal{F}[f](\omega)|\,|\psi(t)|\,dt$ is summable on $\mathbb{R}$ (since $\mathcal{F}[\psi] \in \mathcal{S} \subset L^1(\mathbb{R})$).

By Tonelli's Theorem, these entails that the function is in $L^1(\mathbb{R} \times \mathbb{R})$. We can thus apply Fubini's Theorem and exchange the order of integration.

We are left to show that the convolution is C^∞, the proof is similar to that of Proposition 3.5.11. Let $t \in \mathbb{R}$, for every $|h| < 1$ we have

$$\frac{f * \varphi(t+h) - f * \varphi(t)}{h} = \int_{\mathbb{R}} f(y)\, \frac{\varphi(t+h-y) - \varphi(t-y)}{h}\, dy.$$

Thanks to the regularity of φ, we have

$$\lim_{h \to 0} \frac{\varphi(x+h-y) - \varphi(x-y)}{h} = \varphi'(x-y),$$

in order to pass the limit under the integral sign, we need to find a domination with an L^1 function. We have

$$\left| \frac{\varphi(t+h-y) - \varphi(t-y)}{h} \right| = |\varphi'(\xi)| \le \|\varphi'\|_{L^\infty(\mathbb{R})} < +\infty,$$

where ξ in a point belonging to interval $(t-y, t-y+h)$. In the last inequality we used that

$$\|\varphi'\|_{L^\infty(\mathbb{R})} = [\varphi]_{0,1} < +\infty.$$

In conclusion, for every $|h| < 1$ we get

$$\left| f(y)\, \frac{\varphi(t+h-y) - \varphi(t-y)}{h} \right| \le \|\varphi'\|_{L^\infty(\mathbb{R})}\, |f(y)| \in L^1(\mathbb{R}).$$

We can apply Lebesgue Dominated Convergence Theorem (Theorem 3.2.5) and obtain

$$\lim_{h \to 0} \frac{f * \varphi(t+h) - f * \varphi(t)}{h} = \int_{\mathbb{R}} f(y)\, \varphi'(t-y)\, dy = f * \varphi'(t).$$

This shows that $f * \varphi$ is derivable and that formula (5.6.6) holds for $k = 1$. Finally, by observing that φ' still belongs to $\mathcal{S}$ by Proposition 5.6.5, we can iterate the argument and obtain the desired result. □

The Fourier transform of an L^2 function can be defined through an approximation procedure, using the functions in $\mathcal{S}$.

Theorem 5.6.8 (Fourier transform of an L^2 function)**.** *Let $f \in L^2(\mathbb{R})$, then we have:*

(1) *there exists a sequence $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{S}$ such that*

$$\lim_{n \to \infty} \|f_n - f\|_{L^2(\mathbb{R})} = 0; \tag{5.6.7}$$

(2) *there exists $F \in L^2(\mathbb{R})$ such that*

$$\lim_{n \to \infty} \left\| \mathcal{F}[f_n] - F \right\|_{L^2(\mathbb{R})} = 0.$$

The function F is called Fourier transform of f *and denoted by $\mathcal{F}_{L^2}[f]$;*

(3) *the function F does not depend on the particular choice of the sequence $\{f_n\}_{n \in \mathbb{N}}$;*

(4) *if $f, g \in L^2(\mathbb{R})$, we have*

$$\int_{\mathbb{R}} f(t)\, g(t)^*\, dt = \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}_{L^2}[f](\omega)\, \mathcal{F}_{L^2}[g](\omega)^*\, d\omega, \tag{5.6.8}$$

and

$$\int_{\mathbb{R}} |f(t)|^2\, dt = \frac{1}{2\pi} \int_{\mathbb{R}} |\mathcal{F}_{L^2}[f](\omega)|^2\, d\omega. \tag{5.6.9}$$

(5) *finally, if* $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$*, then*

$$\mathcal{F}_{L^2}[f](\omega) = \mathcal{F}[f](\omega) = \int_{\mathbb{R}} e^{-i\,t\,\omega}\, f(t)\, dt.$$

Proof. We construct explicitely the sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ of point (1). We take $\varphi(t) \in C_0^\infty(\mathbb{R})$ such that

$$\int_{\mathbb{R}} \varphi(t)\, dt = 1,$$

and define

$$\varphi_n(t) = n\,\varphi(n\,t), \qquad n \in \mathbb{N}.$$

Then we set

$$f_n(t) = \varphi_n * (f\,1_{[-n,n]})(t) = \int_{-n}^{n} \varphi_n(t-s)\, f(s)\, ds.$$

By using Minkowski's inequality and Young's inequality for convolutions (see Proposition 3.5.4), we have

$$\begin{aligned} \|f_n - f\|_{L^2(\mathbb{R})} &\le \|\varphi_n * (f\,1_{[-n,n]}) - \varphi_n * f\|_{L^2(\mathbb{R})} \\ &\quad + \|\varphi_n * f - f\|_{L^2(\mathbb{R})} \\ &\le \|f\,1_{[-n,n]} - f\|_{L^2(\mathbb{R})}\, \|\varphi_n\|_{L^1(\mathbb{R})} \\ &\quad + \|\varphi_n * f - f\|_{L^2(\mathbb{R})}. \end{aligned} \tag{5.6.10}$$

Observe that

$$\lim_{n\to\infty} \|f\,1_{[-n,n]} - f\|_{L^2(\mathbb{R})} = \lim_{n\to\infty} \left(\int_{|t|>n} |f(t)|^2\, dt \right)^{\frac{1}{2}} = 0,$$

$$\|\varphi_n\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} |\varphi(t)|\, dt, \qquad \text{for every } n \in \mathbb{N},$$

and by Theorem 3.5.13 of Chapter 3

$$\lim_{n\to\infty} \|\varphi_n * f - f\|_{L^2(\mathbb{R})} = 0.$$

By using these informations in (5.6.10), we thus get

$$\lim_{n\to\infty} \|f_n - f\|_{L^2(\mathbb{R})} = 0.$$

It is only left to verify that $f_n \in \mathcal{S}$. The fact that $f_n \in C^\infty(\mathbb{R})$ follows from Theorem 3.5.13 of Chapter 3, by choosing $\varepsilon = 1/n$. Moreover, since both φ_n and $f\,1_{[-n,n]}$ have compact support, by Lemma 3.5.15 their convolution as well has compact support. Thus we get $f_n \in C_0^\infty(\mathbb{R}) \subset \mathcal{S}$.

The proof of point (2) uses Plancherel's formula for $\mathcal{S}$ and *completeness* of the space $L^2(\mathbb{R})$, i.e. Theorem 3.4.2. Indeed, observe that $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ is a Cauchy sequence in $L^2(\mathbb{R})$, since we proved in point (1) that it is converging. On the other hand, by Plancherel's formula (5.6.3), we have for every $n, m \in \mathbb{N}$

$$\|\mathcal{F}[f_n] - \mathcal{F}[f_m]\|^2_{L^2(\mathbb{R})} = \|\mathcal{F}[f_n - f_m]\|^2_{L^2(\mathbb{R})} = 2\,\pi\, \|f_n - f_m\|_{L^2(\mathbb{R})},$$

which shows that also $\{\mathcal{F}[f_n]\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\mathbb{R})$. By using Theorem 3.4.2, there exists $F \in L^2(\mathbb{R})$ such that

$$\lim_{n\to\infty} \|\mathcal{F}[f_n] - F\|_{L^2(\mathbb{R})} = 0,$$

ad desired.

Let us prove point (3). Let us take another sequence $\{h_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ such that

$$\lim_{n\to\infty} \|h_n - f\|_{L^2(\mathbb{R})} = 0.$$

By repeating the argument of point (2), we know that there exists $H \in L^2(\mathbb{R})$ such that

$$\lim_{n\to\infty} \Big\|\mathcal{F}[h_n] - H\Big\|_{L^2(\mathbb{R})} = 0.$$

We need to prove that $F = H$, where F is the function found at point (2). By using Minkowski inequality (Theorem 3.3.7) and Plancherel's formula (5.6.3) for functions in $\mathcal{S}$, we have

$$\begin{aligned}
\|F - H\|_{L^2(\mathbb{R})} &\le \Big\|F - \mathcal{F}[f_n]\Big\|_{L^2(\mathbb{R})} + \Big\|\mathcal{F}[f_n] - \mathcal{F}[h_n]\Big\|_{L^2(\mathbb{R})} + \Big\|\mathcal{F}[h_n] - H\Big\|_{L^2(\mathbb{R})} \\
&= \Big\|F - \mathcal{F}[f_n]\Big\|_{L^2(\mathbb{R})} + \Big\|\mathcal{F}[f_n - h_n]\Big\|_{L^2(\mathbb{R})} + \Big\|\mathcal{F}[h_n] - H\Big\|_{L^2(\mathbb{R})} \\
&= \Big\|F - \mathcal{F}[f_n]\Big\|_{L^2(\mathbb{R})} + \sqrt{2\,\pi}\,\Big\|f_n - h_n\Big\|_{L^2(\mathbb{R})} + \Big\|\mathcal{F}[h_n] - H\Big\|_{L^2(\mathbb{R})} \\
&\le \Big\|F - \mathcal{F}[f_n]\Big\|_{L^2(\mathbb{R})} + \sqrt{2\,\pi}\,\Big\|f_n - f\Big\|_{L^2(\mathbb{R})} \\
&+ \sqrt{2\,\pi}\,\Big\|h_n - f\Big\|_{L^2(\mathbb{R})} + \Big\|\mathcal{F}[h_n] - H\Big\|_{L^2(\mathbb{R})} \stackrel{n\to\infty}{\longrightarrow} 0,
\end{aligned}$$

thus $F = H$ as desired.

We now prove Parseval's formula for functions in $L^2(\mathbb{R})$, i.e. point (4). We start by observing that since

$$f, g \in L^2(\mathbb{R}) \qquad \text{and} \qquad F, G \in L^2(\mathbb{R}),$$

then the two integrals

$$\int_{\mathbb{R}} f(t)\, g(t)^*\, dt \qquad \text{and} \qquad \int_{\mathbb{R}} F(\omega)\, G(\omega)^*\, d\omega,$$

are well-defined, thanks to Hölder inequality. Let $\{g_n\}_{n\in\mathbb{N}}$ be the sequence of point (1) for the function g, then we observe that

$$\begin{aligned}
\left|\int_{\mathbb{R}} f_n(t)\, g_n(t)^*\, dt - \int_{\mathbb{R}} f(t)\, g(t)^*\, dt\right| &= \left|\int_{\mathbb{R}} (f_n(t) - f(t))\, g_n(t)^*\, dt + \int_{\mathbb{R}} f(t)\, (g_n(t)^* - g(t)^*)\, dt\right| \\
&\le \left|\int_{\mathbb{R}} (f_n(t) - f(t))\, g_n(t)^*\, dt\right| \\
&+ \left|\int_{\mathbb{R}} f(t)\, (g_n(t)^* - g(t)^*)\, dt\right| \\
&\le \|f_n - f\|_{L^2(\mathbb{R})}\, \|g_n\|_{L^2(\mathbb{R})} \\
&+ \|f\|_{L^2(\mathbb{R})}\, \|g_n - g\|_{L^2(\mathbb{R})},
\end{aligned}$$

where we used Hölder inequality in the last estimate[6]. By using that

$$\lim_{n\to\infty} \|f_n - f\|_{L^2(\mathbb{R})} = \lim_{n\to\infty} \|g_n - g\|_{L^2(\mathbb{R})} = 0,$$

[6]We also used that if

$$\lim_{n\to\infty} \|g_n - g\|_{L^2(\mathbb{R})} = 0,$$

then

$$\|g_n\|_{L^2(\mathbb{R})},$$

from the previous estimate we conclude

$$\lim_{n\to\infty} \left| \int_{\mathbb{R}} f_n(t)\, g_n(t)^*\, dt - \int_{\mathbb{R}} f(t)\, g(t)^*\, dt \right| = 0,$$

that is

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(t)\, g_n(t)^*\, dt = \int_{\mathbb{R}} f(t)\, g(t)^*\, dt. \tag{5.6.11}$$

Observe that with the very argument, we can also prove that

$$\lim_{n\to\infty} \int_{\mathbb{R}} \mathcal{F}[f_n](\omega)\, \mathcal{F}[g_n](\omega)^*\, dt = \int_{\mathbb{R}} F(\omega)\, G(\omega)^*\, d\omega. \tag{5.6.12}$$

On the other, by Parseval's formula (5.6.2) for $\mathcal{S}$, we know that

$$\int_{\mathbb{R}} f_n(t)\, g_n(t)^*\, dt = \frac{1}{2\,\pi} \int_{\mathbb{R}} \mathcal{F}[f_n](\omega)\, \mathcal{F}[g_n](\omega)^*\, dt.$$

By taking the limit as n goes to ∞ on both sides and using (5.6.11) and (5.6.12), we finally get Parseval's formula for L^2 functions. We can now obtain Plancherel's formula by simply taking $g = f$.

In order to conclude, we need to prove point (5). We observe that if $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, a closer inspection of the proof of point (1) reveals that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is also such that

$$\lim_{n\to\infty} \|f_n - f\|_{L^1(\mathbb{R})} = 0,$$

(*verify this assertion!*). By recalling (5.2.1), we then get

$$\Big\|\mathcal{F}[f_n] - \mathcal{F}[f]\Big\|_{L^\infty(\mathbb{R})} = \Big\|\mathcal{F}[f_n - f]\Big\|_{L^\infty(\mathbb{R})} \le \|f_n - f\|_{L^1(\mathbb{R})},$$

thus we obtain

$$\lim_{n\to\infty} \Big\|\mathcal{F}[f_n] - \mathcal{F}[f]\Big\|_{L^\infty(\mathbb{R})} = 0.$$

This shows that $\mathcal{F}[f_n]$ converges uniformly to $\mathcal{F}[f]$. In particular, for every $M > 0$ we get

$$\lim_{n\to\infty} \Big\|\mathcal{F}[f_n] - \mathcal{F}[f]\Big\|_{L^2([-M,M])} = 0,$$

thanks to Proposition 3.3.10, used with $E = [-M, M]$, $p = 2$ and $q = \infty$. On the other hand, we also know that

$$\lim_{n\to\infty} \Big\|\mathcal{F}[f_n] - F\Big\|_{L^2([-M,M])} = 0,$$

thanks to point (2). By Minkowski inequality, we then get

$$\Big\|\mathcal{F}[f] - F\Big\|_{L^2([-M,M])} \le \Big\|\mathcal{F}[f] - \mathcal{F}[f_n]\Big\|_{L^2([-M,M])} + \Big\|\mathcal{F}[f_n] - F\Big\|_{L^2([-M,M])},$$

and by taking the limit as n goes to ∞

$$\Big\|\mathcal{F}[f] - F\Big\|_{L^2([-M,M])} = 0.$$

---

is uniformly bounded. This easily follows from Minkowski inequality, i.e.

$$\|g_n\|_{L^2(\mathbb{R})} \le \|g_n - g\|_{L^2(\mathbb{R})} + \|g\|_{L^2(\mathbb{R})},$$

and the first term on the right-hand side is converging to 0, while the second one does not depend on n.

This shows that

$$F(\omega) = \mathcal{F}[f](\omega), \qquad \text{for a.e. } \omega \in [-M, M].$$

By arbitrariness of M, we get the desired conclusion. $\square$

7. Band-limited signals and a sampling formula

Definition 5.7.1. Let $f \in L^1(\mathbb{R})$, we say that f is a *band-limited signal* if there exists $M > 0$ such that

$$\mathcal{F}[f](\omega) = 0, \qquad \text{for } |\omega| > M.$$

In other words, a band-limited signal is such that its Fourier transform has compact support. For such a function, we call *band limit* the number

$$\omega_f = \inf\{M > 0 \,:\, \mathcal{F}[f](\omega) = 0, \text{ for } |\omega| > M\}.$$

Lemma 5.7.2 (A necessary condition for being band-limited)**.** *Let $f \in L^1(\mathbb{R})$ be a band-limited signal. Then we have $f \in C^\infty(\mathbb{R})$ with*

$$\frac{d^k f}{dt^k} \in L^\infty(\mathbb{R}) \qquad \textit{and} \qquad \lim_{|t|\to+\infty} \left|\frac{d^k}{dt^k} f(t)\right| = 0, \qquad \textit{for every } k \in \mathbb{N}.$$

Proof. Let us call ω_f the band limit of f, then we first observe that

$$\int_{\mathbb{R}} |\mathcal{F}(\omega)|\, d\omega = \int_{-\omega_f}^{\omega_f} |\mathcal{F}(\omega)|\, d\omega \le \Big\|\mathcal{F}[f]\Big\|_{L^\infty(\mathbb{R})} 2\,\omega_f < +\infty,$$

thanks to (5.2.1). Thus in particular $\mathcal{F}[f] \in L^1(\mathbb{R})$ and by the duality formula (5.4.6) we have

$$f(-\omega) = \frac{1}{2\,\pi}\,\mathcal{F}\Big[\mathcal{F}[f]\Big](\omega), \qquad \omega \in \mathbb{R},$$

which shows that f is the Fourier transform of a compactly supported L^1 function. By Corollary 5.2.4, we get the desired conclusion. $\square$

Remark 5.7.3. The previous conditions are necessary but NOT sufficient. Indeed, $f(t) = e^{-t^2}$ verifies the properties above, but its Fourier transform is not compactly supported (see Exercise 5.8.5).

Example 5.7.4 (An example of band-limited signal)**.** We will construct an example of band-limited signal as follows: recall from Example 5.3.10 that

$$\mathcal{F}[\mathrm{tri}](\omega) = \left(\mathrm{sinc}\left(\frac{\omega}{2\,\pi}\right)\right)^2 \in L^1(\mathbb{R}).$$

We then take $g(x) = \mathcal{F}[\mathrm{tri}](x)$, then by using the duality formula (5.4.6) and the fact that the triangular function is an even function, we obtain

$$\mathcal{F}[g](\omega) = \mathcal{F}\left[\mathcal{F}[\mathrm{tri}]\right](\omega) = 2\,\pi\,\mathrm{tri}(\omega). \tag{5.7.1}$$

This shows that the signal g is band-limited with band limit $\omega_g = 1$, since tri identically vanishes outside $[-1, 1]$.

Starting from this example, we can construct more general band-limited signal as follows: take $f \in L^1(\mathbb{R})$ and define the new signal

$$F(t) = f * g(t) = \int_{\mathbb{R}} f(y)\,\left(\mathrm{sinc}\left(\frac{t-y}{2\,\pi}\right)\right)^2\, dy.$$

Observe that this is a $L^1(\mathbb{R})$ function, as a convolution of two L^1 functions. We thus can take its Fourier transform, by Proposition 5.3.9 we get

$$\mathcal{F}[F](\omega) = \mathcal{F}[f](\omega)\,\mathcal{F}[g](\omega) = 2\,\pi\,\mathcal{F}[f](\omega)\,\mathrm{tri}(\omega),$$

which is again band-limited, with band limit $\omega_F \le \omega_g = 1$.

Remark 5.7.5 (Low-pass filters)**.** In signal processing, the operation of taking the convolution of a signal f with a band-limited signal g corresponds to apply an *(ideal) low-pass filter*. In this case, the band-limited signal g is also called *low-pass filter*.

Remark 5.7.6 (Band-pass filters)**.** More generally, one could be interested in using a filter that admits only frequencies in a given range $[a, b]$. This means that we want to take a convolution $f * g$, with a filter g having the property

$$|\mathcal{F}[g](\omega)| = 0, \qquad \text{for } \omega \notin [a, b].$$

In this case, we say that g is an *(ideal) band-pass filter*. We observe that we can always construct a band-pass filter starting from a low-pass filter: indeed, if h is a band-limited signal with band limit $\omega_h > 0$, by taking

$$g(t) = e^{i\,\frac{b+a}{2}\,t}\,h\left(\frac{b-a}{2\,\omega_h}\,t\right),$$

we obtain

$$\mathcal{F}[g](\omega) = \mathcal{F}\left[h_{\frac{b-a}{2\,\omega_h}}\right]\left(\omega - \frac{b+a}{2}\right) = \frac{2\,\omega_h}{b-a}\,\mathcal{F}[h]\left(\frac{2\,\omega_h}{b-a}\,\omega - \frac{2\,\omega_h}{b-a}\,\frac{b+a}{2}\right).$$

Observe that this Fourier transform is not identically zero if and only if

$$-\omega_h \le \frac{2\,\omega_h}{b-a}\,\omega - \frac{2\,\omega_h}{b-a}\,\frac{b+a}{2} \le \omega_h.$$

With simple algebraic manipulations, this is the same as

$$a \le \omega \le b,$$

thus the Fourier transform of the signal g has compact support, given by the interval $[a, b]$. In other words, the new signal g is a band-pass filter.

The main result of this section is the following sampling formula. The proof exploits the theory of Fourier series expansions in L^2, for which we refer to Appendix C and the books [1, 2].

Theorem 5.7.7 (Shannon-Whittaker formula)**.** *Let $f \in L^1(\mathbb{R})$ be a band-limited signal with band limit $\omega_f > 0$. Then for every $M \ge \omega_f$ we have the following formula*

$$f(t) = \sum_{n\in\mathbb{Z}} f\left(n\,\frac{\pi}{M}\right)\,\mathrm{sinc}\left(\left(t - n\,\frac{\pi}{M}\right)\frac{M}{\pi}\right), \qquad \textit{for } t \in \mathbb{R}. \tag{5.7.2}$$

In other words, the signal f can be completely reconstructed from its regular sampling

$$\left\{f\left(n\,\frac{\pi}{M}\right)\right\}_{n\in\mathbb{Z}}.$$

Proof. We first define the $(2\,M)$−periodic extension on $\mathcal{F}[f]$, i.e.

$$F(\omega) = \sum_{k\in\mathbb{Z}} \mathcal{F}[f](\omega - 2\,k\,M).$$

The Fourier coefficients of F are given by

$$\widehat{F}(n) = \frac{1}{2\,M}\int_{-M}^{M} F(\omega)\, e^{-i\,n\,\frac{\pi}{M}\,\omega}\, d\omega = \frac{1}{2\,M}\int_{-M}^{M} \mathcal{F}[f](\omega)\, e^{-i\,n\,\frac{\pi}{M}\,\omega}\, d\omega. \tag{5.7.3}$$

We already know by Theorem 5.2.1 that $\mathcal{F}[f]$ is bounded. Moreover, by assumption it is compactly supported, thus in particular $\mathcal{F}[f] \in L^1(\mathbb{R})$. Observe that thanks to the choice of M, we have

$$\mathcal{F}[f](\omega) = 0, \qquad \text{for } \omega \in \mathbb{R} \setminus [-M, M].$$

We can use the inversion formula of Theorem 5.4.6 and get

$$2\,\pi\, f(-t) = \int_{-M}^{M} \mathcal{F}[f](\omega)\, e^{-i\,t\,\omega}\, d\omega, \qquad \text{for a. e. } t \in \mathbb{R}. \tag{5.7.4}$$

By joining (5.7.3) and (5.7.4), we get

$$\widehat{F}(n) = \frac{\pi}{M}\, f\left(-n\,\frac{\pi}{M}\right).$$

For every fixed $t \in \mathbb{R}$, we now consider the function

$$g_t(\omega) = e^{i\,t\,\frac{\pi}{M}\,\omega} \cdot 1_{[-M,M]}(\omega),$$

and its $(2\,M)-$periodic extension

$$G_t(\omega) = \sum_{k\in\mathbb{Z}} g_t(\omega - 2\,k\,M),$$

whose Fourier coefficients are given by

$$\begin{aligned}\widehat{G_t}(n) = \frac{1}{2\,M}\int_{-M}^{M} G_t(\omega)\, e^{-i\,n\,\frac{\pi}{M}\,\omega}\, d\omega &= \frac{1}{2\,M}\int_{-M}^{M} e^{i\,(t-n)\,\frac{\pi}{M}\,\omega}\, d\omega \\ &= \frac{1}{2\,M}\,\mathcal{F}\Big[1_{[-M,M]}\Big]\left(\frac{n-t}{M}\,\pi\right) \\ &= \operatorname{sinc}(t-n),\end{aligned}$$

where we used Remark 5.8.2 and the fact that sinc is an even function. We now observe that by *Parseval's formula* for Fourier series (see Theorem C.2.7) we have

$$\int_{-M}^{M} F(\omega)\, G_t(\omega)^*\, d\omega = 2\,M \sum_{n\in\mathbb{Z}} \widehat{F}(n)\,(\widehat{G_t}(n))^* = 2\,\pi \sum_{n\in\mathbb{Z}} f\left(-n\,\frac{\pi}{M}\right) \operatorname{sinc}(t-n).$$

On the other hand, by definition of G_t we have

$$\int_{-M}^{M} F(\omega)\, G_t(\omega)^*\, d\omega = \int_{-M}^{M} \mathcal{F}[f](\omega)\, e^{-i\,t\,\frac{\pi}{M}\omega}\, d\omega = 2\,\pi\, f\left(-t\,\frac{\pi}{M}\right),$$

where we used again the inversion formula. We thus get

$$f\left(-t\,\frac{\pi}{M}\right) = \sum_{n\in\mathbb{Z}} f\left(-n\,\frac{\pi}{M}\right) \operatorname{sinc}(t-n).$$

By changing variable $k = -n$ in the sum, we get

$$f\left(-t\,\frac{\pi}{M}\right) = \sum_{k\in\mathbb{Z}} f\left(k\,\frac{\pi}{M}\right) \operatorname{sinc}(t+k).$$

If we finally change variable $s = -t\,\pi/M$ and observe that $x \mapsto \operatorname{sinc} x$ is an even function, we get the conclusion. □

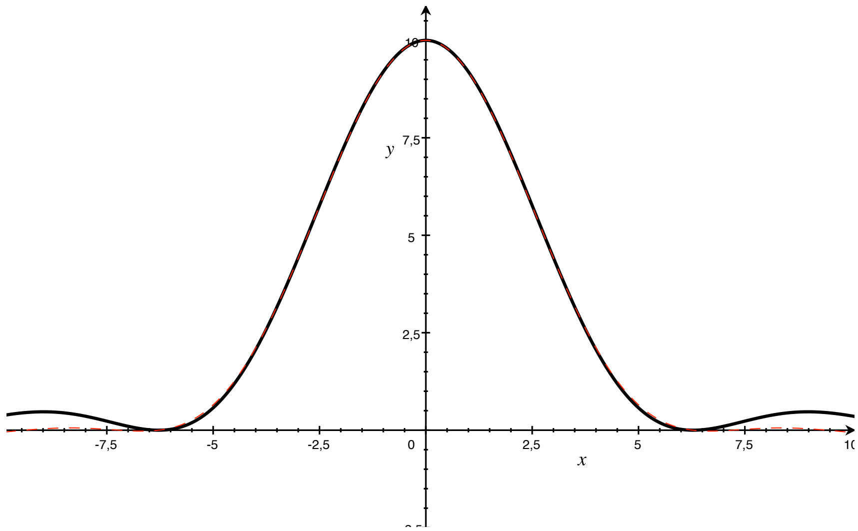

Figure 1. The continuous line is the graph of the band-limited signal

$$f(t) = 10\left(\operatorname{sinc}\left(\frac{t}{2\,\pi}\right)\right)^2.$$

The dotted red line represents the partial sum

$$f(t) = \sum_{n=-2}^{2} f\left(n\,\frac{\pi}{M}\right)\,\operatorname{sinc}\left(\left(t - n\,\frac{\pi}{M}\right)\,\frac{M}{\pi}\right)$$

in (5.7.2).

Remark 5.7.8 (Aliasing)**.** The requirement

$$M \geq \omega_f,$$

is crucial. In other words, if we take a regular sampling

$$\left\{f\left(n\,\frac{\pi}{M}\right)\right\}_{n\in\mathbb{Z}},$$

with $M < \omega_f$, in general it is not possible to reconstruct the signal. Take for example the two band-limited signals

$$f(t) = \left(\operatorname{sinc}\left(\frac{t}{2\,\pi}\right)\right)^2 \qquad \text{and} \qquad g(t) = f(2\,t) = \left(\operatorname{sinc}\left(\frac{t}{\pi}\right)\right)^2.$$

We have seen in Example 5.7.4 that f has band limit $\omega_f = 1$. Moreover, we have

$$\mathcal{F}[g](\omega) = \frac{1}{2}\,\mathcal{F}[f]\left(\frac{\omega}{2}\right),$$

thus g has band limit $\omega_g = 2$. Let us now take

$$M = \frac{1}{2} < \omega_f < \omega_g,$$

i.e. we consider the regular samplings

$$\{f\,(2\,n\,\pi)\}_{n\in\mathbb{Z}} \qquad \text{and} \qquad \{g\,(2\,n\,\pi)\}_{n\in\mathbb{Z}}.$$

We observe that from the definition of cardinal sinus, we have for every $n \in \mathbb{Z}\setminus\{0\}$

$$f\,(2\,n\,\pi) = (\operatorname{sinc} n)^2 = 0 = (\operatorname{sinc}(2\,n))^2 = g(2\,n\,\pi),$$

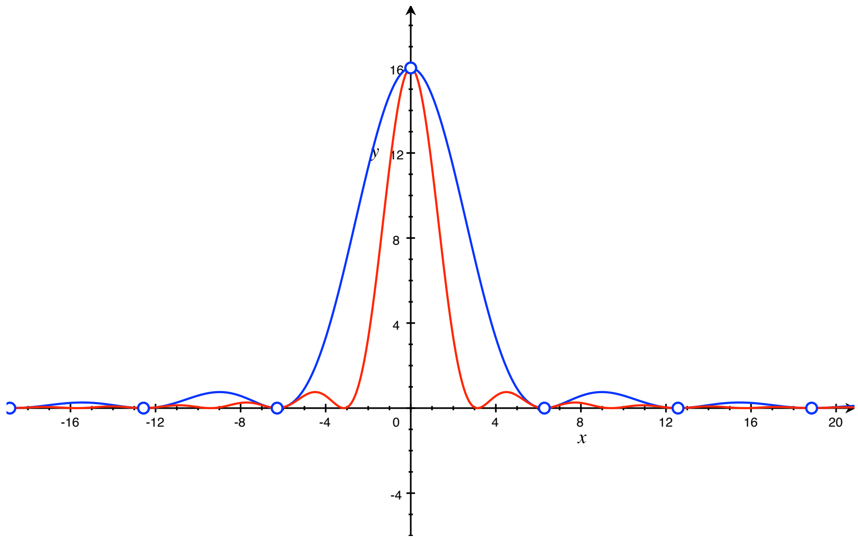

Figure 2. The two band-limited signals are indistinguishable if we consider the sampling corresponding to the dots. In this case the crucial requirement $M \geq \omega_f$ is violated.

and also

$$f(0) = (\mathrm{sinc}(0))^2 = g(0),$$

Then of course the Shannon-Whittaker formula (5.7.2) can not hold now. Observe that we can not distinguish between the two signals f and g, just by looking at their values on the regular sampling grid $\{2\,n\,\pi\}_{n\in\mathbb{Z}}$ (see Figure 2). This phenomenon is called *aliasing* in signal processing.

Remark 5.7.9 (An equivalent form of the Shannon-Whittaker formula)**.** Let $f \in L^1(\mathbb{R})$ be a band-limited signal, with band limit $\omega_f > 0$. In many textbooks on Signal Processing, the Fourier transform is defined by

$$X[f](\omega) = \int_{\mathbb{R}} e^{-2\,\pi\,i\,\omega\,t}\, f(t)\,dt.$$

The relation with our definition is thus given by

$$X[f](\omega) = \mathcal{F}[f](2\,\pi\,\omega). \tag{5.7.5}$$

If we then define the corresponding band limit as

$$\widetilde{\omega}_f = \inf\{M > 0\,:\, X[f](\omega) = 0,\ \text{for } |\omega| > M\},$$

by (5.7.5) we get the relation

$$\widetilde{\omega}_f = \frac{\omega_f}{2\,\pi}.$$

This implies that the sampling step

$$\frac{\pi}{M}, \qquad \text{for } M \geq \omega_f,$$

needed for the validity of (5.7.2) in our notation, can be read as

$$\frac{\pi}{M} = \frac{2\,\pi}{2\,M} = \frac{1}{2\,\widetilde{M}}, \qquad \text{for } \widetilde{M} = \frac{M}{2\,\pi} \geq \frac{\omega_f}{2\,\pi} = \widetilde{\omega}_f.$$

In this way, we end up with the statement of the Shannon-Whittaker formula which is commonly stated in the textbooks, asserting that to reconstruct the signal, the sampling rate should be *at least* the double of the band limit. This threshold rate is called *Nyquist frequency* in Signal Processing.

Remark 5.7.10. We observe that if we set

$$e_n(t) = \sqrt{\frac{M}{\pi}}\ \mathrm{sinc}\left(\left(t - n\,\frac{\pi}{M}\right)\,\frac{M}{\pi}\right),$$

then $\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal family in $L^2(\mathbb{R})$, with respect to the standard scalar product of $L^2(\mathbb{R})$, given by

$$\langle f, g\rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f(t)\,g(t)^*\,dt.$$

Indeed, let us recall that (see Exercise 5.9.5)

$$\mathcal{F}_{L^2}[\mathrm{sinc}](\omega) = 1_{[-\pi,\pi]}(\omega),$$

thus by the translation and dilation properties of the Fourier transform (Corollary 5.3.4) we have

$$\mathcal{F}_{L^2}[e_n](\omega) = \sqrt{\frac{\pi}{M}}\,e^{-i\,\frac{n\,\pi}{M}\omega}\,1_{[-M,M]}(\omega).$$

We now use Parseval's formula for the Fourier transform in $L^2(\mathbb{R})$ (see Theorem 5.6.8), this gives

$$\begin{aligned}\int_{\mathbb{R}} e_n\,e_k^*\,dt = \frac{1}{2\,\pi}\int_{\mathbb{R}} \mathcal{F}_{L^2}[e_n]\,\mathcal{F}_{L^2}[e_k]^*\,d\omega &= \frac{\pi}{M}\,\frac{1}{2\,\pi}\int_{-M}^{M} e^{-i\,\frac{(n-k)\,\pi}{M}\omega}\,d\omega\\ &= \begin{cases} 1, & \text{if } n = k\\ 0, & \text{if } n \neq k.\end{cases}\end{aligned}$$

Thus from the Shannon-Whittaker formula we get in particular that if $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is band-limited, then

$$\int_{\mathbb{R}} |f(t)|^2\,dt = \frac{\pi}{M}\sum_{n\in\mathbb{Z}}\left|f\left(n\,\frac{\pi}{M}\right)\right|^2,$$

i.e. the energy of the signal can be computed from the sampling.

8. Exercises

Exercise 5.8.1. *Let $a < b$, show that the Fourier transform of the generalized rectangular function $f(t) = 1_{[a,b]}(t)$ is given by*

$$\mathcal{F}[f](\omega) = (b-a)\,e^{-\frac{b+a}{2}\,i\,\omega}\ \mathrm{sinc}\left(\frac{b-a}{2\,\pi}\,\omega\right).$$

Solution. Rather then computing it directly, we appeal to Example 5.1.4 and Corollary 5.3.4. Indeed, we already seen (recall formula (3.6.2)) that

$$1_{[a,b]}(t) = \mathrm{rect}\left(\frac{t-a}{b-a} - \frac{1}{2}\right).$$

Thus by setting

$$\lambda = \frac{1}{b-a} \qquad \text{and} \qquad h = -\left(\frac{a}{b-a} + \frac{1}{2}\right),$$

we have

$$1_{[a,b]}(t) = \mathrm{rect}(\lambda\,t + h).$$

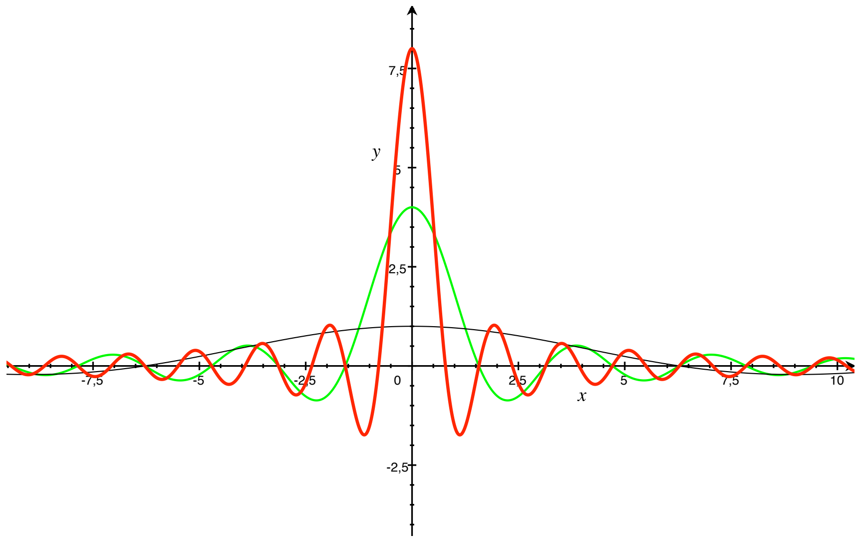

Figure 3. The Fourier transform of the generalized rectangular function $1_{[-L,L]}$, for $L = 1/2$ (black), $L = 2$ (green) and $L = 4$ (red).

By using Corollary 5.3.4 we get

$$\begin{aligned}\mathcal{F}\left[1_{[a,b]}\right](\omega) &= \frac{e^{i\,\frac{h}{\lambda}\,\omega}}{\lambda}\,\mathcal{F}[\mathrm{rect}]\left(\frac{\omega}{\lambda}\right)\\ &= (b-a)\,e^{-a\,i\,\omega-\frac{b-a}{2}\,i\,\omega}\ \mathrm{sinc}\left(\frac{b-a}{2\,\pi}\,\omega\right).\end{aligned}$$

By observing that

$$-a-\frac{b-a}{2} = -\frac{b+a}{2},$$

we get the desired conclusion. □

Remark 5.8.2. In particular, by taking $a = -L$ and $b = L$ in the previous formula, we get

$$\mathcal{F}\left[1_{[-L,L]}\right](\omega) = 2\,L\ \mathrm{sinc}\left(\frac{L}{\pi}\,\omega\right),$$

see Figure 3. We recall that this is essentially the family of functions entering in the proof of the inversion formula, see Remark 5.4.3.

Exercise 5.8.3. *Compute the Fourier transform of the function*

$$g(t) = \frac{1}{1+t^2}.$$

Show that this is given by

$$\mathcal{F}[g](\omega) = \pi\,e^{-|\omega|}.$$

Solution. We could compute the Fourier transform directly, but here we prefer to take advantage of the duality formula (5.4.6). Indeed, if we set $f(t) = e^{-|t|}$, this function satisfies the hypothesis of Theorem 5.4.2. Moreover, from Example 5.1.5 we have

$$\mathcal{F}[f](\omega) = \frac{2}{1+\omega^2},$$

and this function is in $L^1(\mathbb{R})$. From Corollary 5.4.4 we obtain

$$\mathcal{F}\Big[\mathcal{F}[f]\Big](\omega) = 2\,\pi\, f(-\omega) = 2\,\pi\, e^{-|\omega|}.$$

By using the explicit expression for $\mathcal{F}[f]$, this can be rewritten as

$$\int_{\mathbb{R}} \frac{2}{1+t^2}\, e^{-i\,t\,\omega}\, dt = 2\,\pi\, e^{-|\omega|},$$

that is

$$\int_{\mathbb{R}} \frac{1}{1+t^2}\, e^{-i\,t\,\omega}\, dt = \pi\, e^{-|\omega|}.$$

Finally, this proves

$$\mathcal{F}[g](\omega) = \pi\, e^{-|\omega|},$$

thus concluding. □

Exercise 5.8.4. *For every $n \in \mathbb{N}$, compute the Fourier transform of*

$$f(t) = t^n\, e^{-|t|}.$$

Solution. For $n = 0$, we have already computed this transform in Example 5.1.5. For $n \geq 1$, it is sufficient to use Corollary 5.2.4, which gives

$$\mathcal{F}[f](\omega) = \mathcal{F}[t^n\, e^{-|t|}](\omega) = \frac{1}{(-i)^n}\, \frac{d^n}{d\omega^n} \mathcal{F}[e^{-|t|}](\omega) = \frac{1}{(-i)^n}\, \frac{d^n}{d\omega^n} \frac{2}{1+\omega^2}.$$

For example, for $n = 1$ we get

$$\mathcal{F}[t\, e^{-|t|}](\omega) = -\frac{1}{-i}\, \frac{4\,\omega}{(1+\omega^2)^2} = -\frac{4\,i\,\omega}{(1+\omega^2)^2}.$$

This concludes the exercise. □

Exercise 5.8.5. *Show that the Fourier transform of the function $f(t) = e^{-t^2}$ is given by*

$$\mathcal{F}[f](\omega) = \sqrt{\pi}\, e^{-\frac{\omega^2}{4}}.$$

Solution. We observe that f satisfies

$$f'(t) = -2\,t\, e^{-t^2} = -2\,t\, f(t).$$

By taking the Fourier transform, from (5.3.4) we thus get

$$i\,\omega\, \mathcal{F}[f](\omega) = \mathcal{F}[f'](\omega) = -2\, \mathcal{F}[t\, f](\omega).$$

By using formula (5.2.3), we thus get

$$\frac{d}{d\omega} \mathcal{F}[f](\omega) = -i\, \mathcal{F}[t\, f](\omega) = -\frac{\omega}{2}\, \mathcal{F}[f](\omega).$$

Also observe that

$$\mathcal{F}[f](0) = \int_{\mathbb{R}} e^{-t^2}\, dt = \sqrt{\pi}.$$

Thus the function $\omega \mapsto \mathcal{F}[f](\omega)$ solves the linear differential equation

$$\left\{ \begin{array}{rcl} y'(\omega) + \dfrac{\omega}{2}\, y(\omega) & = & 0 \\ y(0) & = & \sqrt{\pi}. \end{array} \right.$$

The solution of this problem can be easily computed to be (see Example B.1.1 of Appendix B below)

$$y(\omega) = \sqrt{\pi}\, e^{-\frac{\omega^2}{4}},$$

which thus coincides with the Fourier transform of f. $\square$

Exercise 5.8.6 (Fourier transform of a Gaussian function)**.** *Let $a > 0$ and $t_0 \in \mathbb{R}$, show that the Fourier transform of the function $f(t) = e^{-a\,(t-t_0)^2}$ is given by*

$$\mathcal{F}[f](\omega) = \sqrt{\frac{\pi}{a}}\, e^{-\frac{\omega^2}{4\,a}} \Big(\cos(t_0\,\omega) - i\,\sin(t_0\,\omega)\Big).$$

Solution. We can use the previous Exercise and Corollary 5.3.4. Indeed, observe that if we set $g(x) = e^{-x^2}$, then

$$f(t) = g(\sqrt{a}\,(t - t_0)) = g(\sqrt{a}\,t - \sqrt{a}\,t_0).$$

Thus from Corollary 5.3.4 we get

$$\mathcal{F}[f](\omega) = \frac{e^{-i\,t_0\,\omega}}{\sqrt{a}}\,\mathcal{F}[g]\left(\frac{\omega}{\sqrt{a}}\right) = \sqrt{\frac{\pi}{a}}\, e^{-i\,t_0\,\omega}\, e^{-\frac{\omega^2}{4\,a}}.$$

This gives the desired formula, by recalling that $e^{i\,\vartheta} = \cos\vartheta + i\,\sin\vartheta$. $\square$

Exercise 5.8.7. *Let $f \in L^1(\mathbb{R})$, prove that a solution u to the equation*

$$-\,u''(t) + u(t) = f(t), \qquad t \in \mathbb{R}, \tag{5.8.1}$$

can be written in the form

$$u(t) = G * f(t), \qquad \textit{with} \quad G(t) = \frac{1}{2}\, e^{-|t|}.$$

Solution. We take the Fourier transform of the equation, so by Corollary 5.3.7 we get

$$\omega^2\,\mathcal{F}[u](\omega) + \mathcal{F}[u](\omega) = \mathcal{F}[f](\omega).$$

This in turn implies that

$$\mathcal{F}[u](\omega) = \frac{1}{1+\omega^2}\,\mathcal{F}[f](\omega).$$

By using Exercise 5.1.5, we know that

$$\frac{1}{1+\omega^2} = \mathcal{F}[G](\omega),$$

thus we obtained

$$\mathcal{F}[u](\omega) = \mathcal{F}[G](\omega)\,\mathcal{F}[f](\omega) = \mathcal{F}[G * f](\omega).$$

In the last identity, we used Proposition 5.3.9. This finally gives that

$$u(t) = G * f(t),$$

as desired. $\square$

Exercise 5.8.8. *Let $f, g \in L^1(\mathbb{R})$, show that*

$$\left(\int_{\mathbb{R}} f(t)\,dt\right)\left(\int_{\mathbb{R}} g(t)\,dt\right) = \int_{\mathbb{R}} f * g(t)\,dt.$$

Solution. By definition, we have

$$\int_{\mathbb{R}} f(t)\,dt = \mathcal{F}[f](0) \qquad \text{and} \qquad \int_{\mathbb{R}} g(x)\,dt = \mathcal{F}[g](0).$$

From Proposition 5.3.9, we know

$$\left(\int_{\mathbb{R}} f(t)\,dt\right)\left(\int_{\mathbb{R}} g(x)\,dt\right) = \mathcal{F}[f](0)\,\mathcal{F}[g](0) = \mathcal{F}[f*g](0) = \int_{\mathbb{R}} f*g\,dt.$$

This concludes the proof. □

Exercise 5.8.9. *Let $f \in L^1(\mathbb{R}) \cap C^2(\mathbb{R})$ be such that $f'' \in L^1(\mathbb{R})$. Show that $\mathcal{F}[f] \in L^1(\mathbb{R})$.*

Proof. By using the *Kallman-Rota inequality* (see Exercise 3.7.7), the hypotheses imply that we have $f' \in L^1(\mathbb{R})$, as well. By Corollary 5.3.7, we have that

$$\lim_{|\omega|\to+\infty} \omega^2\,\mathcal{F}[f](\omega) = 0.$$

This implies in particular that, by using the definition of limit, there exists $M_1 > 0$ such that

$$|\omega^2\,\mathcal{F}[f](\omega)| < 1, \qquad \text{for } |\omega| > M_1.$$

We can thus write

$$\begin{aligned}\int_{\mathbb{R}} |\mathcal{F}[f](\omega)|\,d\omega &= \int_{|\omega|>M_1} |\mathcal{F}[f](\omega)|\,d\omega + \int_{-M_1}^{M_1} |\mathcal{F}[f](\omega)|\,d\omega\\ &\le \int_{|\omega|>M_1} \frac{1}{\omega^2}\,d\omega + \|\mathcal{F}[f]\|_{L^\infty(\mathbb{R})}\,2\,M_1.\end{aligned}$$

By observing that the integral of $1/\omega^2$ converges, we get the desired conclusion. □

Exercise 5.8.10. *Show that for every $\varphi \in \mathcal{S}$ we have the estimate*

$$\|\varphi\|_{L^1(\mathbb{R})} \le 4\,\sqrt{[\varphi]_{0,0}\,[\varphi]_{2,0}}. \tag{5.8.2}$$

Solution. We fix $M > 0$, then we write

$$\begin{aligned}\int_{\mathbb{R}} |\varphi(t)|\,dt &= \int_{-M}^{M} |\varphi(t)|\,dt + \int_{|t|>M} |\varphi(t)|\,t^2\,\frac{dt}{t^2}\\ &\le 2\,\sup_{t\in\mathbb{R}} |\varphi(t)|\,M + \sup_{t\in\mathbb{R}} |t^2\,\varphi(t)|\,\int_{|t|>M} \frac{dt}{t^2}\\ &= 2\,[\varphi]_{0,0}\,M + 2\,[\varphi]_{2,0}\,\int_{M}^{+\infty} \frac{dt}{t^2}\\ &= 2\,[\varphi]_{0,0}\,M + \frac{2}{M}\,[\varphi]_{2,0}.\end{aligned}$$

The previous estimate is valid for every $M > 0$ positive. In particular, we get

$$\int_{\mathbb{R}} |\varphi(t)|\,dt \le 2\,\inf_{M>0}\left([\varphi]_{0,0}\,M + \frac{1}{M}\,[\varphi]_{2,0}\right).$$

It is not difficult to see that the quantity

$$[\varphi]_{0,0}\,M + \frac{1}{M}\,[\varphi]_{2,0},$$

is minimal for

$$M=\sqrt{\frac{[\varphi]_{2,0}}{[\varphi]_{0,0}}}.$$

By replacing above, we finally get the desired result. □

Exercise 5.8.11. *Prove that for every $\varphi\in\mathcal{S}$, we have*

$$\int_{\mathbb{R}}|\mathcal{F}[\varphi](\omega)|^2\,d\omega\le 8\,\pi\,\left([\varphi]_{0,0}\right)^{\frac{3}{2}}\,\left([\varphi]_{2,0}\right)^{\frac{1}{2}}.$$

Solution. Observe that by definition, we have

$$\|\varphi\|_{L^\infty(\mathbb{R})}=\sup_{t\in\mathbb{R}}|\varphi(t)|=[\varphi]_{0,0}. \tag{5.8.3}$$

By using the interpolation inequality (3.7.1) with

$$r=2,\quad p=1,\quad q=\infty,$$

we obtain

$$\|\varphi\|_{L^2(\mathbb{R})}\le\sqrt{\|\varphi\|_{L^1(\mathbb{R})}\,\|\varphi\|_{L^\infty(\mathbb{R})}}.$$

We can use (5.8.2) and (5.8.3), so to obtain

$$\|\varphi\|_{L^2(\mathbb{R})}\le 2\sqrt{\sqrt{[\varphi]_{0,0}\,[\varphi]_{2,0}}\,[\varphi]_{0,0}}.$$

By taking the square on both sides and using Plancherel's formula (5.6.3), we get the conclusion. □

Exercise 5.8.12. *Let $0<\tau<1$, show that for every $f\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$ we have*

$$\int_{\mathbb{R}}\frac{1}{|\omega|^\tau}\,|\mathcal{F}[f](\omega)|^2\,d\omega<+\infty.$$

More precisely, prove that we have the estimate

$$\left(\int_{\mathbb{R}}\frac{1}{|\omega|^\tau}\left|\mathcal{F}[f](\omega)\right|^2d\omega\right)^{\frac{1}{2}}\le C\,\left(\int_{\mathbb{R}}|f(t)|\,dt\right)^{\tau}\,\left(\int_{\mathbb{R}}|f(t)|^2\,dt\right)^{\frac{1-\tau}{2}},$$

for a constant C depending on τ and which blows-up as $\tau\nearrow 1$.

Solution. We take $\lambda>0$ and decompose the integral in the left-hand side as follows

$$\int_{\mathbb{R}}|\omega|^{-\tau}\left|\mathcal{F}[f](\omega)\right|^2d\omega=\int_{\{|\omega|\le\lambda\}}|\omega|^{-\tau}\left|\mathcal{F}[f](\omega)\right|^2d\omega+\int_{\{|\omega|>\lambda\}}|\omega|^{-\lambda}\left|\mathcal{F}[f](\omega)\right|^2d\omega.$$

Since $f\in L^1(\mathbb{R}^N)$, by Theorem 5.2.1 we have that its Fourier transform is in $L^\infty(\mathbb{R})$, thus we obtain

$$\begin{aligned}\int_{\mathbb{R}}|\omega|^{-\tau}\left|\mathcal{F}[f](\omega)\right|^2d\omega\le{}&\left\|\mathcal{F}[f]\right\|^2_{L^\infty(\mathbb{R})}\int_{-\lambda}^{\lambda}|\omega|^{-\tau}\,d\omega\\&+\lambda^{-\tau}\int_{\mathbb{R}}\left|\mathcal{F}[f](\omega)\right|^2d\omega.\end{aligned}$$

By Theorem 5.6.8, we know that

$$\mathcal{F}_{L^2}[f]=\mathcal{F}[f],$$

since $f\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$. Thus we can use the Plancherel's identity (5.6.9) and the fact that

$$\left\|\mathcal{F}[f]\right\|_{L^\infty(\mathbb{R})}\le\|f\|_{L^1(\mathbb{R})},$$

thanks to (5.2.1). We then arrive at

$$\int_{\mathbb{R}} |\omega|^{-\tau} \left|\mathcal{F}[f](\omega)\right|^2 d\omega \le \frac{2}{1-\tau}\, \lambda^{1-\tau}\, \|f\|^2_{L^1(\mathbb{R})} + 2\,\pi\,\lambda^{-\tau}\,\|f\|_{L^2(\mathbb{R})}.$$

This is valid for every $\lambda > 0$ and the right-hand side is minimal for

$$\lambda = \pi\,\tau\,\frac{\|f\|^2_{L^2(\mathbb{R})}}{\|f\|^2_{L^1(\mathbb{R})}}.$$

This in turn gives

$$\int_{\mathbb{R}} |\omega|^{-\tau} \left|\mathcal{F}[f](\omega)\right|^2 d\omega \le \frac{2}{\tau^\tau\,(1-\tau)}\, \pi^{1-\tau}\, \left(\|f\|^2_{L^1(\mathbb{R})}\right)^{\tau} \left(\|f\|^2_{L^2(\mathbb{R})}\right)^{1-\tau},$$

as desired. □

Exercise 5.8.13. *Let us take the two signals*

$$g(t) = \left(\operatorname{sinc}\left(\frac{t}{2\,\pi}\right)\right)^2 \qquad \textit{and} \qquad h(t) = \cos(2\,t)\,g(t).$$

Show that $g * h = 0$.

Solution. We observe that $g, h \in L^1(\mathbb{R})$, then $g * h \in L^1(\mathbb{R})$. We use that

$$\cos(2\,t) = \frac{e^{2\,i\,t} + e^{-2\,i\,t}}{2},$$

thus we have

$$\begin{aligned}\mathcal{F}[g*h](\omega) &= \mathcal{F}[g](\omega)\,\mathcal{F}[\cos(2\,t)\,g](\omega)\\ &= \mathcal{F}[g](\omega)\,\mathcal{F}\left[\frac{e^{2\,i\,t}+e^{-2\,i\,t}}{2}\,g\right](\omega)\\ &= \frac{1}{2}\,\mathcal{F}[g](\omega)\,\Big(\mathcal{F}\left[e^{2\,i\,t}\,g\right](\omega) + \mathcal{F}\left[e^{-2\,i\,t}\,g\right](\omega)\Big).\end{aligned}$$

We now recall that by (5.7.1)

$$\mathcal{F}[g](\omega) = 2\,\pi\,\operatorname{tri}(\omega),$$

and by Proposition 5.3.5

$$\mathcal{F}\left[e^{2\,i\,t}\,g\right](\omega) = 2\,\pi\,\operatorname{tri}(\omega-2) \qquad \text{and} \qquad \mathcal{F}\left[e^{-2\,i\,t}\,g\right](\omega) = 2\,\operatorname{tri}(\omega+2).$$

Thus we obtained

$$\begin{aligned}\mathcal{F}[g*h](\omega) &= \pi\,\operatorname{tri}(\omega)\,\Big(2\,\pi\,\operatorname{tri}(\omega-2) + 2\,\pi\,\operatorname{tri}(\omega+2)\Big)\\ &= 2\,\pi^2\,\operatorname{tri}(\omega)\,\operatorname{tri}(\omega-2) + 2\,\pi^2\,\operatorname{tri}(\omega)\,\operatorname{tri}(\omega+2).\end{aligned}$$

The last two products identically vanish, since the functions have disjoint supports. By using the Inversion Formula, we thus obtain

$$g*h(t) = \frac{1}{2\,\pi}\int_{\mathbb{R}} \mathcal{F}[g*h](\omega)\,e^{i\,t\,\omega}\,dt = 0,$$

as desired. □

Exercise 5.8.14. *Let $f \in L^1(\mathbb{R})$ be a band-limited signal, with band limit $\omega_f > 0$. Let us take the two signals*
$$g(t) = \cos(2\,\omega_f\,t)\,f(t) \qquad \text{and} \qquad h(t) = \sin(2\,\omega_f\,t)\,f(t).$$
Prove that $f * g = f * h = 0$.

Solution. This is similar to the previous exercise. Indeed, by using that
$$\cos(2\,\omega_f\,t) = \frac{e^{2\,i\,\omega_f\,t} + e^{-2\,i\,\omega_f\,t}}{2},$$
and Propositions 5.3.9 and 5.3.5, we get
$$\begin{aligned}\mathcal{F}[f*g](\omega) = \mathcal{F}[f](\omega)\,\mathcal{F}[g](\omega) &= \frac{1}{2}\,\mathcal{F}[f](\omega)\,\Big(\mathcal{F}[e^{2\,i\,\omega_f\,t}\,f](\omega) + \mathcal{F}[e^{-2\,i\,\omega_f\,t}\,f](\omega)\Big)\\ &= \frac{1}{2}\,\mathcal{F}[f](\omega)\,\mathcal{F}[f](\omega - 2\,\omega_f)\\ &+ \frac{1}{2}\,\mathcal{F}[f](\omega)\,\mathcal{F}[f](\omega + 2\,\omega_f).\end{aligned}$$
We now observe that, since $\mathcal{F}[f]$ identically vanishes outside $[-\omega_f, \omega_f]$, we have
$$\mathcal{F}[f](\omega)\,\mathcal{F}[f](\omega - 2\,\omega_f) = \mathcal{F}[f](\omega)\,\mathcal{F}[f](\omega + 2\,\omega_f) = 0, \qquad \text{for every } \omega \in \mathbb{R}.$$
Then we can conclude as in the previous exercise. □

9. Advanced exercises

Exercise 5.9.1. *For $a > 0$ and $b, c \in \mathbb{R}$, we consider the second order polynomial*
$$P(t) = a\,t^2 + b\,t + c.$$
Let us suppose that P does not have real roots, i.e. $b^2 - 4\,a\,c < 0$. Compute the Fourier transform of the function
$$g(t) = \frac{1}{P(t)}.$$

Solution. This can be computed starting from the one of Exercise 5.8.3. Indeed, let us set
$$\Delta = 4\,a\,c - b^2,$$
then we observe that
$$\begin{aligned}P(t) = a\,t^2 + b\,t + c = a\,\left(t^2 + \frac{b}{a}\,t + \frac{c}{a}\right) &= a\,\left[\left(t + \frac{b}{2\,a}\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4\,a^2}\right)\right]\\ &= a\,\left[\left(t + \frac{b}{2\,a}\right)^2 + \frac{\Delta}{4\,a^2}\right]\\ &= \frac{\Delta}{4\,a}\,\left[\frac{4\,a^2}{\Delta}\left(t + \frac{b}{2\,a}\right)^2 + 1\right]\\ &= \frac{\Delta}{4\,a}\,\left[\left(\frac{2\,a}{\sqrt{\Delta}}\,t + \frac{b}{\sqrt{\Delta}}\right)^2 + 1\right]\end{aligned}$$
Thus we can write
$$g(t) = \frac{1}{P(t)} = \frac{4\,a}{\Delta}\,\frac{1}{\left(\dfrac{2\,a}{\sqrt{\Delta}}\,t + \dfrac{b}{\sqrt{\Delta}}\right)^2 + 1}.$$

If we set
$$f(t) = \frac{1}{1+t^2},$$
then the previous identity implies that
$$g(t) = \frac{4\,a}{\Delta}\, f\left(\frac{2\,a}{\sqrt{\Delta}}\,t + \frac{b}{\sqrt{\Delta}}\right).$$
We can now apply Corollary 5.3.4 with
$$h = \frac{b}{\sqrt{\Delta}} \qquad \text{and} \qquad \lambda = \frac{2\,a}{\sqrt{\Delta}},$$
so to get
$$\mathcal{F}[g](\omega) = \frac{4\,a}{\Delta}\,\frac{e^{i\,\frac{b}{2\,a}\,\omega}}{2\,a}\,\sqrt{\Delta}\,\mathcal{F}[f]\left(\frac{\sqrt{\Delta}}{2\,a}\,\omega\right).$$
If we now use Exercise 5.8.3 to compute the last transform and recall the definition of Δ, we finally get
$$\mathcal{F}[g](\omega) = \frac{2\,\pi}{\sqrt{4\,a\,c-b^2}}\,e^{i\,\frac{b}{2\,a}\,\omega}\,e^{-\frac{\sqrt{4\,a\,c-b^2}}{2\,a}\,|\omega|}.$$
This concludes the exercise. $\square$

Exercise 5.9.2 (Heat equation)**.** *Let* $\varphi \in L^1(\mathbb{R})$ *and let us consider the following initial value problem for the* heat equation *in* $\mathbb{R}$
$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, & \text{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R} \\[2ex] u(0,x) = \varphi(x), & x \in \mathbb{R}. \end{cases}$$
Show that the solution u can be written as
$$u(t,x) = G_t * \varphi(x) = \int_{\mathbb{R}} G_t(x-y)\,\varphi(y)\,dy,$$
where the function G_t is given by
$$G_t(x) = \frac{1}{\sqrt{4\,\pi\,t}}\,e^{-\frac{x^2}{4\,t}}, \qquad x \in \mathbb{R},\ t > 0.$$

Solution. We have to pay attention to the fact that we have 2 variables, i.e. we are dealing with a *partial differential equation.* We set
$$y(t,\omega) = \mathcal{F}[u](\omega) = \int_{\mathbb{R}} e^{-i\,x\,\omega}\,u(t,x)\,dx,$$
then we take the Fourier transform of the equation in the spatial variable x, so to obtain
$$\frac{\partial}{\partial t} y(t,\omega) = -\omega^2\,y(t,\omega),$$
with initial condition
$$y(0,\omega) = \int_{\mathbb{R}} e^{-i\,x\,\omega}\,u(0,x)\,dx = \int_{\mathbb{R}} e^{-i\,x\,\omega}\,\varphi(x)\,dx = \mathcal{F}[\varphi](\omega).$$
This means that for every fixed $\omega \in \mathbb{R}$, the function $t \mapsto y(t,\omega)$ is a solution of the first order linear differential equations
$$y'(t) = -\omega^2\,y(t)$$

with initial condition $\mathcal{F}[\varphi](\omega)$. It is not difficult to see that such a solution is given by

$$t \mapsto \mathcal{F}[\varphi](\omega)\, e^{-\omega^2 t},$$

thus

$$\mathcal{F}[u](\omega) = y(t,\omega) = \mathcal{F}[\varphi](\omega)\, e^{-\omega^2 t}.$$

We now observe that for every fixed $t > 0$, the function

$$\omega \mapsto e^{-\omega^2 t},$$

is a Gaussian function. By using Exercise 5.8.6 with

$$a = \frac{1}{4\,t} \qquad \text{and} \qquad t_0 = 0,$$

we have that

$$\omega \mapsto e^{-\omega^2 t},$$

is the Fourier transform (with respect to the variable x) of the function

$$G_t(x) = \frac{1}{\sqrt{4\,\pi\,t}}\, e^{-\frac{x^2}{4\,t}}, \qquad x \in \mathbb{R}. \tag{5.9.1}$$

We thus obtained that

$$\mathcal{F}[u](\omega) = \mathcal{F}[\varphi](\omega)\,\mathcal{F}[G_t](\omega) = \mathcal{F}[G_t * \varphi](\omega).$$

This finally gives the desired conclusion. □

Remark 5.9.3. The function u of the previous exercise represents the evolution in time of the temperature of an infinite thin bar (modeled by $\mathbb{R}$), starting from the initial temprature φ. In other words, we have

$$u(t,x) = \text{"temperature of the point } x \text{ at the time } t\text{"}.$$

Such a temperature evolves in time and space according to the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

which is called *heat equation*. The time-dependent function G_t defined in (5.9.1) is called *heat kernel*.

Exercise 5.9.4. *Compute the Fourier transform of the function*

$$f(t) = \frac{1}{1+t^4}.$$

Solution. We will rely on Exercise 4.10.1. We first observe that

$$1 + t^4 = (t^2 - i)\,(t^2 + i),$$

thus we get

$$f(t) = \frac{1}{(t^2 - i)\,(t^2 + i)} = \frac{1}{2\,i}\left(\frac{1}{t^2 - i} - \frac{1}{t^2 + i}\right).$$

We now observe that i and $-i$ can be written as

$$i = \left(e^{\frac{\pi}{4}\,i}\right)^2 \qquad \text{and} \qquad -\,i = \left(e^{-\frac{\pi}{4}\,i}\right)^2.$$

By using Exercise 4.10.1 with $a = e^{\frac{\pi}{4}\,i}$, we then obtain that

$$\frac{1}{t^2 + i} = \frac{1}{i - (i\,t)^2} = \frac{1}{a^2 - (i\,t)^2} = \frac{1}{2\,a}\,\mathcal{B}[g](i\,t) = \frac{e^{-\frac{\pi}{4}\,i}}{2}\,\mathcal{B}[g](i\,t),$$

where g is defined by

$$g(t) = \begin{cases} e^{-e^{\frac{\pi}{4}i}t}, & \text{for } t \geq 0, \\ e^{e^{\frac{\pi}{4}i}t}, & \text{for } t < 0. \end{cases}$$

In a similar way, by using Exercise 4.10.1 with $a = e^{-\frac{\pi}{4}i}$, we obtain

$$\frac{1}{t^2 - i} = \frac{1}{-i - (i\,t)^2} = \frac{1}{a^2 - (i\,t)^2} = \frac{e^{\frac{\pi}{4}i}}{2}\,\mathcal{B}[h](i\,t),$$

where h now is defined by

$$h(t) = \begin{cases} e^{-e^{-\frac{\pi}{4}i}t}, & \text{for } t \geq 0, \\ e^{e^{-\frac{\pi}{4}i}t}, & \text{for } t < 0. \end{cases}$$

By putting everything together, we obtained

$$f(t) = \frac{1}{2\,i}\left(\frac{e^{-\frac{\pi}{4}i}}{2}\,\mathcal{B}[g](i\,t) - \frac{e^{\frac{\pi}{4}i}}{2}\,\mathcal{B}[h](i\,t)\right) = \frac{1}{2\,i}\,\mathcal{B}\left[\frac{e^{-\frac{\pi}{4}i}}{2}\,g - \frac{e^{\frac{\pi}{4}i}}{2}\,h\right](i\,t).$$

We now recall the relation between the bilateral Laplace transform and the Fourier transform (see Remark 5.1.3), so to obtain

$$f(t) = \frac{1}{2\,i}\,\mathcal{F}\left[\frac{e^{-\frac{\pi}{4}i}}{2}\,g - \frac{e^{\frac{\pi}{4}i}}{2}\,h\right](t).$$

By taking the Fourier transform on both sides and using the Duality Formula, we thus get

$$\mathcal{F}[f](\omega) = \frac{\pi}{i}\left(\frac{e^{-\frac{\pi}{4}i}}{2}\,g(-\omega) - \frac{e^{\frac{\pi}{4}i}}{2}\,h(-\omega)\right).$$

By recalling the definitions of g and h, we get

$$\left(\frac{e^{-\frac{\pi}{4}i}}{2}\,g(-\omega) - \frac{e^{\frac{\pi}{4}i}}{2}\,h(-\omega)\right) = \begin{cases} \dfrac{e^{-\frac{\pi}{4}i}}{2}\,e^{e^{\frac{\pi}{4}i}\omega} - \dfrac{e^{\frac{\pi}{4}i}}{2}\,e^{e^{-\frac{\pi}{4}i}\omega}, & \text{for } \omega < 0, \\[2ex] \dfrac{e^{-\frac{\pi}{4}i}}{2}\,e^{-e^{\frac{\pi}{4}i}\omega} - \dfrac{e^{\frac{\pi}{4}i}}{2}\,e^{-e^{-\frac{\pi}{4}i}\omega}, & \text{for } \omega \geq 0. \end{cases}$$

$$= i\,e^{-\frac{\sqrt{2}}{2}|\omega|}\,\sin\left(\frac{\sqrt{2}}{2}\,|\omega| + \frac{\pi}{4}\right).$$

In conclusion, we obtain

$$\mathcal{F}[f](\omega) = \pi\,e^{-\frac{\sqrt{2}}{2}|\omega|}\,\sin\left(\frac{\sqrt{2}}{2}\,|\omega| + \frac{\pi}{4}\right).$$

This concludes the exercise. □

Exercise 5.9.5. *Show that*

$$\mathcal{F}_{L^2}[\mathrm{sinc}] = 1_{[-\pi,\pi]}.$$

Solution. We recall that by Remark 5.8.2, we have

$$\mathcal{F}[1_{[-\pi,\pi]}](\omega) = 2\,\pi\,\mathrm{sinc}\,\omega.$$

Moreover, the function $1_{[-\pi,\pi]}$ satisfies the hypotheses of the inversion formula of Theorem 5.4.2, thus we have

$$2\,\pi\,1_{[-\pi,\pi]}(t) = \lim_{L\to+\infty}\int_{-L}^{L}\mathcal{F}[1_{[-\pi,\pi]}](\omega)\,e^{i\,t\,\omega}\,d\omega = 2\,\pi\lim_{L\to+\infty}\int_{-L}^{L}\mathrm{sinc}(\omega)\,e^{i\,t\,\omega}\,d\omega.$$

In other words, we get

$$1_{[-\pi,\pi]}(t) = \lim_{L\to+\infty} \mathcal{F}[\mathrm{sinc}\cdot 1_{[-L,L]}](-t), \qquad \text{for every } t\in\mathbb{R}.$$

Observe that $\mathrm{sinc}\cdot 1_{[-L,L]} \in L^1(\mathbb{R})\cap L^2(\mathbb{R})$, thus we can use Plancherel's formula (5.6.9) and obtain

$$\begin{aligned}\int_{\mathbb{R}} \left|\mathcal{F}[\mathrm{sinc}\cdot 1_{[-L,L]}](t) - 1_{[-\pi,\pi]}(t)\right|^2 dt &= \frac{1}{2\,\pi}\int_{\mathbb{R}} \left|\mathcal{F}_{L^2}\Big[\mathcal{F}[\mathrm{sinc}\cdot 1_{[-L,L]}]\Big](\omega) - \mathcal{F}_{L^2}[1_{[-\pi,\pi]}](\omega)\right|^2 d\omega\\ &= \frac{1}{2\,\pi}\int_{\mathbb{R}} \left|\mathcal{F}\Big[\mathcal{F}[\mathrm{sinc}\cdot 1_{[-L,L]}]\Big](\omega) - \mathcal{F}[1_{[-\pi,\pi]}(\omega)\right|^2 d\omega\\ &= \frac{1}{2\,\pi}\int_{\mathbb{R}} |2\,\pi\,\mathrm{sinc}(\omega)\cdot 1_{[-L,L]}(\omega) - 2\,\pi\,\mathrm{sinc}(\omega)|^2\, d\omega\\ &= 2\,\pi\int_{\mathbb{R}} |\,\mathrm{sinc}(\omega)\cdot 1_{[-L,L]}(\omega) - \mathrm{sinc}(\omega)|^2\, d\omega.\end{aligned}$$

In the second equality we used the property (5) of Theorem 5.6.8. In the third equality we used the duality formula (5.4.6), for the even function $\mathrm{sinc}\cdot 1_{[-L,L]} \in L^1(\mathbb{R})\cap L^2(\mathbb{R})$. We now observe that

$$\lim_{L\to+\infty} \mathrm{sinc}(\omega)\cdot 1_{[-L,L]}(\omega) = \mathrm{sinc}(\omega), \qquad \text{for every } \omega\in\mathbb{R}.$$

Moreover, for every $L>0$ we have

$$|\,\mathrm{sinc}(\omega)\cdot 1_{[-L,L]}(\omega) - \mathrm{sinc}(\omega)|^2 = |\,\mathrm{sinc}(\omega)\, 1_{\mathbb{R}\setminus[-L,L]}(\omega)|^2 \le |\,\mathrm{sinc}(\omega)|^2, \qquad \text{for every } \omega\in\mathbb{R}.$$

Since the last function is in $L^1(\mathbb{R})$ and independent of L, we can use the Lebesgue Dominated Convergence Theorem (see Theorem 3.2.5) and obtain

$$\lim_{L\to+\infty}\int_{\mathbb{R}} \left|\mathcal{F}[\mathrm{sinc}\cdot 1_{[-L,L]}](t) - 1_{[-\pi,\pi]}(t)\right|^2 dt = \lim_{L\to+\infty} 2\,\pi\int_{\mathbb{R}} |\,\mathrm{sinc}(\omega)\cdot 1_{[-L,L]}(\omega) - \mathrm{sinc}(\omega)|^2\, d\omega = 0,$$

that is

$$\lim_{L\to+\infty} \left\|\mathcal{F}[\mathrm{sinc}\cdot 1_{[-L,L]}] - 1_{[-\pi,\pi]}\right\|_{L^2(\mathbb{R})} = 0. \tag{5.9.2}$$

Finally, as in the proof of point (1) of Theorem 5.6.8, we take $\varphi\in C_0^\infty(\Omega)$ such that

$$\int_{\mathbb{R}} \varphi(t)\, dt = 1,$$

and define $\varphi_n(t) = n\,\varphi(n\,t)$. We have seen in the proof of Theorem 5.6.8 that

$$\lim_{n\to\infty} \left\|(\mathrm{sinc}\cdot 1_{[-n,n]}) * \varphi_n - \mathrm{sinc}\right\|_{L^2(\mathbb{R})} = 0,$$

and

$$\lim_{n\to\infty} \left\|\mathcal{F}[(\mathrm{sinc}\cdot 1_{[-n,n]}) * \varphi_n] - \mathcal{F}_{L^2}[\mathrm{sinc}]\right\|_{L^2(\mathbb{R})} = 0. \tag{5.9.3}$$

By Minkowski's inequality (see Proposition 3.3.7), we have for every $n\in\mathbb{N}$

$$\begin{aligned}\left\|\mathcal{F}_{L^2}[\mathrm{sinc}] - 1_{[-\pi,\pi]}\right\|_{L^2(\mathbb{R})} &= \left\|\Big(\mathcal{F}_{L^2}[\mathrm{sinc}] - \mathcal{F}[\mathrm{sinc}\cdot 1_{[-n,n]}]\Big) - \Big(\mathcal{F}[\mathrm{sinc}\cdot 1_{[-n,n]}] - 1_{[-\pi,\pi]}\Big)\right\|_{L^2(\mathbb{R})}\\ &\le \left\|\mathcal{F}_{L^2}[\mathrm{sinc}] - \mathcal{F}[\mathrm{sinc}\cdot 1_{[-n,n]}]\right\|_{L^2(\mathbb{R})}\\ &+ \left\|\Big(\mathcal{F}[\mathrm{sinc}\cdot 1_{[-n,n]}] - 1_{[-\pi,\pi]}\Big)\right\|_{L^2(\mathbb{R})}.\end{aligned}$$

By taking the limit as n goes to ∞ and using (5.9.2), (5.9.3), we finally get

$$\left\| \mathcal{F}_{L^2}[\mathrm{sinc}] - 1_{[-\pi,\pi]} \right\|_{L^2(\mathbb{R})} = 0.$$

This concludes the exercise. $\square$

Chapter 6

Tempered distributions

1. A brief and rough introduction

The concept of *distribution* is a fundamental tool in Physics and Engineering. It can be seen as a generalization of the concept of function. Such a generalization is useful in order to extend some usual operations like *derivatives* or *integral transforms* beyond their natural domain of definition.

The central idea behind the definition of distributions can (very roughly) be summarized as follows:

"try to define a function NOT through its pointwise values
but through the effects it makes when tested against good functions"

Of course, this is NOT the mathematical definition of a distribution. To clarify this point, let us start with a concrete example.

Example 6.1.1 (Derivative of a step function?)**.** We considered many times the Heaviside step function H. We know that this is a piecewise constant function, which assumes only two values and has a unit jump at $t = 0$. In particular, we have

$$H'(t) = 0 \qquad \text{for } t \neq 0,$$

while for $t = 0$ the function is not derivable. Indeed, we know that

$$\lim_{h\to 0} \frac{H(h) - H(0)}{h},$$

does not exist. Let us try to apply the rough idea presented above: rather than trying to define the derivative at $t = 0$ by computing the limit of the incremental ratio (as we have seen, this is not possible), let us "test" the incremental ratio against a "good" function, for example a function $\varphi \in C_0^\infty(\mathbb{R})$. More precisely, we consider

$$\int_{\mathbb{R}} \frac{H(t+h) - H(t)}{h}\, \varphi(t)\, dt, \qquad h \neq 0 \text{ and } \varphi \in C_0^\infty(\mathbb{R}).$$

We can make a simple change of variable as follows

$$\begin{aligned}\int_{\mathbb{R}} \frac{H(t+h)-H(t)}{h}\,\varphi(t)\,dt &= \int_{\mathbb{R}} \frac{H(t+h)}{h}\,\varphi(t)\,dt - \int_{\mathbb{R}} \frac{H(t)}{h}\,\varphi(t)\,dt \\ &= \int_{\mathbb{R}} \frac{H(s)}{h}\,\varphi(s-h)\,ds - \int_{\mathbb{R}} \frac{H(s)}{h}\,\varphi(s)\,ds \\ &= \int_{\mathbb{R}} H(s)\,\frac{\varphi(s-h)-\varphi(s)}{h}\,ds \\ &= \int_0^{+\infty} \frac{\varphi(s-h)-\varphi(s)}{h}\,ds.\end{aligned}$$

Finally, we observe that since $\varphi \in C_0^\infty(\mathbb{R})$, we can pass to the limit under the integral sign in the last expression (as always, this can be justified by appealing to the Dominated Convergence Theorem). Thus we obtain

$$\lim_{h\to 0} \int_{\mathbb{R}} \frac{H(t+h)-H(t)}{h}\,dt = -\int_0^{+\infty} \varphi'(s)\,ds = -\Big[\varphi(s)\Big]_0^{+\infty} = \varphi(0).$$

In other words, while we can not always compute the pointwise limit

$$\lim_{h\to 0} \frac{H(t+h)-H(t)}{h}, \qquad t \in \mathbb{R},$$

the limit of this incremental ratio "tested" against a smooth compactly supported function can be always computed. This defines the derivative of H "in the sense of distributions". Observe that (as announced above) this does NOT define a function in the usual sense: rather, it defines a "functional" defined on the space $C_0^\infty(\mathbb{R})$ and with values in $\mathbb{C}$. More precisely, this is the functional

$$\begin{array}{ccc} C_0^\infty(\mathbb{R}) & \to & \mathbb{C} \\ \varphi & \mapsto & \varphi(0), \end{array}$$

called *Dirac delta centered at* 0. Thus one could say that

$$\text{“}H'(t) = \text{Dirac delta centered at } 0\text{''} \qquad \text{in the sense of distributions.}$$

We will come back on this in the next sections, by giving a precise mathematical framework for the ideas presented above.

2. Definitions and examples

As a space of "test functions" we want to use the Schwartz class $\mathcal{S}$ presented in Chapter 5. We first need to introduce a notion of convergence on this space.

Definition 6.2.1 (Convergence in the Schwartz class $\mathcal{S}$)**.** Let $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ and $\varphi \in \mathcal{S}$. We say that $\{\varphi_n\}_{n\in\mathbb{N}}$ *converges to* φ *in* $\mathcal{S}$ if we have

$$\lim_{n\to\infty} [\varphi_n - \varphi]_{m,k} = 0, \qquad \text{for every } m, k \in \mathbb{N}.$$

We recall that for every $m, k \in \mathbb{N}$ the quantities $[\,\cdot\,]_{m,k}$ are defined by

$$[\varphi]_{m,k} = \sup_{t\in\mathbb{R}} \left| t^m\, \varphi^{(k)}(t) \right| < +\infty.$$

We will use the notation $\varphi_n \xrightarrow{\mathcal{S}} \varphi$ for this convergence.

For a *functional* $F : \mathcal{S} \to \mathbb{C}$, we use the notation

$$\langle F, \varphi \rangle,$$

for the value of F computed at $\varphi \in \mathcal{S}$. We recall that F is said to be *linear* if

$$\langle F, \alpha\, \varphi + \beta\, \psi \rangle = \alpha\, \langle F, \varphi \rangle + \beta\, \langle F, \psi \rangle, \qquad \text{for every } \alpha, \beta \in \mathbb{C},\ \varphi, \psi \in \mathcal{S}.$$

We can now give the definition of tempered distribution.

Definition 6.2.2. Let $F : \mathcal{S} \to \mathbb{C}$ be a functional on $\mathcal{S}$. We say that F is a *tempered distribution* if:

- it is linear;
- it is continuous on $\mathcal{S}$, i.e. if for every sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ such that

$$\varphi_n \xrightarrow{\mathcal{S}} \varphi, \qquad \text{as } n \text{ goes to } \infty,$$

  we have

$$\lim_{n\to\infty} \langle F, \varphi_n \rangle = \langle F, \varphi \rangle.$$

We indicate with $\mathcal{S}'$ the collection of all tempered distributions.

Remark 6.2.3. Observe that by linearity of F, we have

$$\lim_{n\to\infty} \langle F, \varphi_n \rangle = \langle F, \varphi \rangle \quad \Longleftrightarrow \quad \lim_{n\to\infty} \langle F, \varphi_n - \varphi \rangle = 0,$$

and the sequence $\varphi_n - \varphi$ converges to 0 in $\mathcal{S}$. Thus, if we want to verify that a linear function $F : \mathcal{S} \to \mathbb{C}$ is a tempered distribution, it is sufficient to show that

$$\lim_{n\to\infty} \langle F, \varphi_n \rangle = 0,$$

for every sequence $\varphi_n \xrightarrow{\mathcal{S}} 0$.

Example 6.2.4 (Dirac delta)**.** Let $t_0 \in \mathbb{R}$, we define the linear functional $\delta_{t_0} : \mathcal{S} \to \mathbb{C}$ by

$$\langle \delta_{t_0}, \varphi \rangle = \varphi(t_0), \qquad \text{for every } \varphi \in \mathcal{S}.$$

This is called *Dirac delta centered at* t_0. Let us verify that $\delta_{t_0} \in \mathcal{S}'$.

We first verify that δ_{t_0} is a linear functional: for every $\alpha, \beta \in \mathbb{C}$ and $\varphi, \psi \in \mathcal{S}$ we have

$$\langle \delta_{t_0}, \alpha\, \varphi + \beta\, \psi \rangle = \alpha\, \varphi(t_0) + \beta\, \psi(t_0) = \alpha\, \langle \delta_{t_0}, \varphi \rangle + \beta\, \langle \delta_{t_0}, \psi \rangle.$$

We now show that δ_{t_0} is continuous on $\mathcal{S}$. We take a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ such that $\varphi_n \xrightarrow{\mathcal{S}} 0$. In particular, this implies that

$$\lim_{n\to\infty} \left(\sup_{t\in\mathbb{R}} |\varphi_n(t)| \right) = \lim_{n\to\infty} [\varphi_n]_{0,0} = 0.$$

We thus obtain

$$\lim_{n\to\infty} |\langle \delta_{t_0}, \varphi_n \rangle| = \lim_{n\to\infty} |\varphi_n(t_0)| \le \lim_{n\to\infty} \left(\sup_{t\in\mathbb{R}} |\varphi_n(t)| \right) = 0.$$

By taking into account Remark 6.2.3, this shows that δ_{t_0} is a tempered distribution.

Example 6.2.5 (Regular tempered distributions)**.** Let $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ be a locally summable function such that there exists $m \in \mathbb{N}$ for which

$$\int_{\mathbb{R}} \frac{|f(t)|}{1+|t|^m}\, dt < +\infty.$$

We then say that f is a *slowly growing function.* To such a function f we associate a linear functional $F_f : \mathcal{S} \to \mathbb{C}$, defined by

$$\langle F_f, \varphi\rangle = \int_{\mathbb{R}} f(t)\,\varphi(t)\, dt, \qquad \varphi \in \mathcal{S}.$$

Observe that the integral is well-defined for every $\varphi \in \mathcal{S}$, since

$$\begin{aligned}\left|\int_{\mathbb{R}} f(t)\,\varphi(t)\,dt\right| &= \left|\int_{\mathbb{R}} \frac{f(t)}{1+|t|^m}\,(1+|t|^m)\,\varphi(t)\,dt\right| \\ &\le \int_{\mathbb{R}} \frac{|f(t)|}{1+|t|^m}\,(1+|t|^m)\,|\varphi(t)|\,dt \\ &\le \Big([\varphi]_{0,0} + [\varphi]_{m,0}\Big) \int_{\mathbb{R}} \frac{|f(t)|}{1+|t|^m}\,dt < +\infty.\end{aligned}$$

Moreover, the linearity of F_f is a straightforward consequence of the linearity of the Lebesgue integral.

We call F_f *regular tempered distribution generated by* f. We can easily verify that F_f is indeed a tempered distribution. In order to verify the continuity on $\mathcal{S}$, we take a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ such that $\varphi_n \xrightarrow{\mathcal{S}} 0$. We get

$$\begin{aligned}\lim_{n\to\infty}\left|\int_{\mathbb{R}} f(t)\,\varphi_n(t)\,dt\right| &= \lim_{n\to\infty}\left|\int_{\mathbb{R}} \frac{f(t)}{1+|t|^m}\,(1+|t|^m)\,\varphi_n(t)\,dt\right| \\ &\le \lim_{n\to\infty}\int_{\mathbb{R}} \frac{|f(t)|}{1+|t|^m}\,(1+|t|^m)\,|\varphi_n(t)|\,dt \\ &\le \lim_{n\to\infty}\Big([\varphi_n]_{0,0} + [\varphi_n]_{m,0}\Big) \int_{\mathbb{R}} \frac{|f(t)|}{1+|t|^m}\,dt = 0,\end{aligned}$$

thanks to the fact that

$$\int_{\mathbb{R}} \frac{|f(t)|}{1+|t|^m}\,dt < +\infty \qquad \text{and} \qquad \lim_{n\to\infty}\Big([\varphi_n]_{0,0} + [\varphi_n]_{m,0}\Big) = 0.$$

Proposition 6.2.6 (L^p functions are slowly growing functions)**.** *Let* $1 \le p \le \infty$ *and let* $f \in L^p(\mathbb{R})$. *Then* f *is a slowly growing function and thus, in particular,* $F_f \in \mathcal{S}'$.

Proof. Let us start with the case $p = \infty$. Then we have

$$\int_{\mathbb{R}} \frac{|f(t)|}{1+t^2}\,dt \le \|f\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \frac{dt}{1+t^2} < +\infty,$$

which shows that f is slowly growing.

Let us now consider the case $1 < p < \infty$, then by Hölder's inequality (see Proposition 3.3.5)

$$\int_{\mathbb{R}} \frac{|f(t)|}{1+|t|}\,dt \le \left(\int_{\mathbb{R}} |f(t)|^p\,dt\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} \frac{dt}{(1+|t|)^{p'}}\right)^{\frac{1}{p'}} < +\infty,$$

and the last integral is finite, since $1 < p' < +\infty$.

Finally, for the case $p = 1$, the function $f \in L^1(\mathbb{R})$ verifies the definition of slowly growing function with $m = 0$. $\square$

Example 6.2.7 (Principal value of $1/t$)**.** An important example of tempered distribution is the one generated by the function

$$f(t) = \frac{1}{t}, \qquad t \in \mathbb{R} \setminus \{0\}.$$

Observe that this function does NOT fall in the class of slowly growing functions, since $f \notin L^1_{\mathrm{loc}}(\mathbb{R})$ (the singularity of $1/t$ is not summable near the origin). However, we can associate to this function a tempered distribution defined by

$$\left\langle \mathrm{P.V.}\frac{1}{t}, \varphi \right\rangle = \lim_{\varepsilon \to 0^+} \int_{|t|>\varepsilon} \frac{\varphi(t)}{t}\, dt, \qquad \text{for every } \varphi \in \mathcal{S}. \tag{6.2.1}$$

This is called *principal value of* $1/t$. We first observe that for every $\varepsilon > 0$, we have

$$\left| \int_{|t|>\varepsilon} \frac{\varphi(t)}{t}\, dt \right| < +\infty.$$

Indeed, it holds

$$\left| \int_{|t|>\varepsilon} \frac{\varphi(t)}{t}\, dt \right| \le \int_{|t|>\varepsilon} \frac{|\varphi(t)|}{|t|}\, dt \le \frac{1}{\varepsilon} \int_{\mathbb{R}} |\varphi(t)|\, dt,$$

and the latter is finite, since $\varphi \in \mathcal{S} \subset L^1(\mathbb{R})$ (see Proposition 5.6.4). In order to verify that (6.2.1) defines a tempered distribution, we want to rewrite it in a different form, which is easier to handle. We then fix $0 < \varepsilon < 1$ and write

$$\begin{aligned}\int_{|t|>\varepsilon} \frac{\varphi(t)}{t}\, dt &= \int_{\varepsilon}^{+\infty} \frac{\varphi}{t}\, dt + \int_{-\infty}^{-\varepsilon} \frac{\varphi(t)}{t}\, dt \\ &= \int_{\varepsilon}^{+\infty} \frac{\varphi(t)}{t}\, dt - \int_{\varepsilon}^{+\infty} \frac{\varphi(-t)}{t}\, dt = \int_{\varepsilon}^{+\infty} \frac{\varphi(t) - \varphi(-t)}{t}\, dt,\end{aligned}$$

where we used the change of variable $t \mapsto -t$ in the integral performed on $(-\infty, -\varepsilon)$. We now split the last integral as follows

$$\begin{aligned}\int_{\varepsilon}^{+\infty} \frac{\varphi(t) - \varphi(-t)}{t}\, dt &= \int_{\varepsilon}^{1} \frac{\varphi(t) - \varphi(-t)}{t}\, dt + \int_{1}^{+\infty} \frac{\varphi(t) - \varphi(-t)}{t}\, dt \\ &= \int_{0}^{1} \frac{\varphi(t) - \varphi(-t)}{t}\, 1_{[\varepsilon,1]}(t)\, dt + \int_{1}^{+\infty} \frac{\varphi(t) - \varphi(-t)}{t}\, dt.\end{aligned}$$

For the first integral, we have[1]

$$\left| \frac{\varphi(t) - \varphi(-t)}{t}\, 1_{[\varepsilon,1]}(t) \right| \le \left| \frac{\varphi(t) - \varphi(-t)}{t} \right| \le 2\, [\varphi]_{0,1}, \qquad \text{for } t \in [0, 1], \tag{6.2.2}$$

[1] By the Mean Value Theorem (i.e. *Lagrange's Theorem*, for italian readers), we have

$$\varphi(t) - \varphi(-t) = \varphi'(\xi)\, (t - (-t)) = 2\, t\, \varphi'(\xi),$$

for some $\xi \in [-t, t]$. Since we are working with $t \in [0, 1]$, this in particular gives

$$|\varphi(t) - \varphi(-t)| \le 2\, t \sup_{\xi \in [-1,1]} |\varphi'(\xi)| \le 2\, t \sup_{\xi \in \mathbb{R}} |\varphi'(\xi)| = 2\, t\, [\varphi]_{0,1}.$$

thus by the Dominated Converge Theorem, we can infer

$$\lim_{\varepsilon\to 0^+}\int_\varepsilon^1 \frac{\varphi(t)-\varphi(-t)}{t}\,dt = \lim_{\varepsilon\to 0^+}\int_0^1 \frac{\varphi(t)-\varphi(-t)}{t}\,1_{[\varepsilon,1]}(t)\,dt = \int_0^1 \frac{\varphi(t)-\varphi(-t)}{t}\,dt.$$

In conclusion, we can write

$$\left\langle \mathrm{P.V.}\frac{1}{t},\varphi\right\rangle = \int_0^{+\infty} \frac{\varphi(t)-\varphi(-t)}{t}\,dt. \tag{6.2.3}$$

We can take the latter as definition of the principal value of $1/t$. Before going further, we observe that

$$\int_0^{+\infty} \left|\frac{\varphi(t)-\varphi(-t)}{t}\right|\,dt < +\infty, \tag{6.2.4}$$

i.e. the function $t \mapsto (\varphi(t)+\varphi(-t))/t$ is in $L^1(\mathbb{R}_+)$.

With this definition, it is now easy to verify that this is a tempered distribution. Linearity is trivial and it just follows from linearity of the Lebesgue integral. Let us verify that (6.2.3) defines a continuous functional on $\mathcal{S}$. We take a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ such that $\varphi_n \xrightarrow{\mathcal{S}} 0$, then we have

$$\begin{aligned}
\left|\left\langle \mathrm{P.V.}\frac{1}{t},\varphi_n\right\rangle\right| &= \left|\int_0^1 \frac{\varphi_n(t)-\varphi_n(-t)}{t}\,dt + \int_1^{+\infty} \frac{\varphi_n(t)-\varphi_n(-t)}{t}\right| \\
&\le \left|\int_0^1 \frac{\varphi_n(t)-\varphi_n(-t)}{t}\,dt\right| + \left|\int_1^{+\infty} \frac{\varphi_n(t)-\varphi_n(-t)}{t}\,dt\right| \\
&\le \int_0^1 \left|\frac{\varphi_n(t)-\varphi_n(-t)}{t}\right|\,dt + \int_1^{+\infty} \left|\frac{\varphi_n(t)-\varphi_n(-t)}{t}\right|\,dt.
\end{aligned}$$

We now observe that by (6.2.2)

$$\lim_{n\to\infty}\int_0^1 \left|\frac{\varphi_n(t)-\varphi_n(-t)}{t}\right|\,dt \le 2\,\lim_{n\to\infty}[\varphi_n]_{0,1} = 0.$$

As for the integral on $[1,+\infty)$, we proceed as follows

$$\begin{aligned}
\lim_{n\to\infty}\int_1^{+\infty} \left|\frac{\varphi_n(t)-\varphi_n(-t)}{t}\right|\,dt &\le \lim_{n\to\infty}\int_1^{+\infty} \frac{|\varphi_n(t)|+|\varphi_n(-t)|}{t}\,dt \\
&= \lim_{n\to\infty}\int_1^{+\infty} \frac{t\left[|\varphi_n(t)|+|\varphi_n(-t)|\right]}{t^2}\,dt \\
&\le 2\,\lim_{n\to\infty}\sup_{t\in\mathbb{R}}\left|t\,\varphi_n(t)\right|\int_1^{+\infty}\frac{dt}{t^2} \\
&= 2\,\lim_{n\to\infty}[\varphi_n]_{1,0} = 0.
\end{aligned}$$

This finally gives

$$\lim_{n\to\infty}\left\langle \mathrm{P.V.}\frac{1}{t},\varphi_n\right\rangle = 0,$$

as desired.

Example 6.2.8 (Series of Dirac deltas)**.** Let $\tau > 0$ be a given time step and let $\{c_k\}_{k\in\mathbb{Z}} \subset \mathbb{C}$ be a bounded sequence, i.e.

$$|c_k| \le C, \qquad \text{for every } k \in \mathbb{Z}.$$

The functional $F : \mathcal{S} \to \mathbb{C}$ defined by

$$F = \sum_{k\in\mathbb{Z}} c_k\, \delta_{\tau\, k},$$

is a tempered distribution. We first verify that the definition is well-posed, i.e. for every $\varphi \in \mathcal{S}$ the series

$$\langle F, \varphi\rangle = \sum_{k\in\mathbb{Z}} c_k\, \varphi(\tau\, k),$$

is converging. Indeed, we have

$$\begin{aligned}\left|\sum_{k\in\mathbb{Z}} c_k\, \varphi(\tau\, k)\right| \le \sum_{k\in\mathbb{Z}} |c_k|\, |\varphi(\tau\, k)| &= \sum_{k\in\mathbb{Z}} \frac{|c_k|}{1+(\tau\, k)^2}\, (1+(\tau\, k)^2)\, |\varphi(\tau\, k)| \\ &\le C\left(\sup_{t\in\mathbb{R}} |\varphi(t)| + \sup_{t\in\mathbb{R}} |t^2\, \varphi(t)|\right) \sum_{k\in\mathbb{Z}} \frac{1}{1+\tau^2\, k^2},\end{aligned}$$

which implies

$$|\langle F, \varphi\rangle| \le C\left([\varphi]_{0,0} + [\varphi]_{2,0}\right) \sum_{k\in\mathbb{Z}} \frac{1}{1+\tau^2\, k^2} < +\infty. \tag{6.2.5}$$

The fact that F is linear is straightforward, let us verify that F is continuous on $\mathcal{S}$. We take a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ such that $\varphi_n \xrightarrow{\mathcal{S}} 0$, then by formula (6.2.5)

$$\lim_{n\to\infty} |\langle F, \varphi_n\rangle| \le C\left(\sum_{k\in\mathbb{Z}} \frac{1}{1+\tau^2\, k^2}\right) \lim_{n\to\infty}\left([\varphi_n]_{0,0} + [\varphi_n]_{2,0}\right) = 0.$$

This shows that F is continuous on $\mathcal{S}$.

3. Elementary operations on distributions

3.1. Linear combinations. Given $F, G \in \mathcal{S}'$ and $\alpha, \beta \in \mathbb{C}$, we can define their linear combination $\alpha\, F + \beta\, G$ by simply posing

$$\langle \alpha\, F + \beta\, G, \varphi\rangle = \alpha\, \langle F, \varphi\rangle + \beta\, \langle G, \varphi\rangle, \qquad \text{for every } \varphi \in \mathcal{S}.$$

It is left as an (easy!) exercise to verify that $\alpha\, F + \beta\, G$ is still a tempered distribution. This in particular entails that $\mathcal{S}'$ has a stucture of vector space over the field $\mathbb{C}$.

Remark 6.3.1. It is not difficult to see that if $\alpha, \beta \in \mathbb{C}$ and f, g are two slowly growing function, then we have

$$\alpha\, F_f + \beta\, F_g = F_{\alpha\,\beta+\beta\, g}.$$

Indeed, by using the definitions of linear combination and of regular distribution, we have for every $\varphi \in \mathcal{S}$

$$\begin{aligned}\langle \alpha\, F_f + \beta\, F_g, \varphi\rangle &= \alpha\, \langle F_f, \varphi\rangle + \beta\, \langle F_g, \varphi\rangle \\ &= \alpha \int_{\mathbb{R}} f(t)\, \varphi(t)\, dt + \beta \int_{\mathbb{R}} g(t)\, \varphi(t)\, dt \\ &= \int_{\mathbb{R}} \Big[\alpha\, f(t)\, + \beta\, g(t)\Big]\, \varphi(t)\, dt = \langle F_{\alpha\, f+\beta\, g}, \varphi\rangle.\end{aligned}$$

3.2. Change of variable. Let $\lambda \in \mathbb{R} \setminus \{0\}$ and $h \in \mathbb{R}$, we define the affine change of variable $\mathcal{A}_{\lambda,h} : \mathbb{R} \to \mathbb{R}$ such that

$$\mathcal{A}_{\lambda,h}(t) = \lambda\, t + h, \qquad \text{for every } t \in \mathbb{R}.$$

If $F \in \mathcal{S}'$, we can then define its "change of variable" as the linear functional $F \circ \mathcal{A}_{\lambda,h} : \mathcal{S} \to \mathcal{C}$ given by

$$\langle F \circ \mathcal{A}_{\lambda,h}, \varphi\rangle = \frac{1}{|\lambda|} \left\langle F, \varphi \circ \mathcal{A}_{\frac{1}{\lambda},-\frac{h}{\lambda}} \right\rangle, \qquad \text{for every } \varphi \in \mathcal{S}, \tag{6.3.1}$$

where the symbol $\circ$ on the right-hand side the usual composition of functions, i.e.

$$\varphi \circ \mathcal{A}_{\frac{1}{\lambda},-\frac{h}{\lambda}}(t) = \varphi\left(\mathcal{A}_{\frac{1}{\lambda},-\frac{h}{\lambda}}(t)\right) = \varphi\left(\frac{t-h}{\lambda}\right), \qquad \text{for every } t \in \mathbb{R}.$$

It is easy to see that with this definition $F \circ \mathcal{A}_{\lambda,h}$ is still a tempered distribution.

The previous definition is better appreciated with an example.

Example 6.3.2 (Change of variable for regular distributions)**.** If F_f is a regular tempered distribution, generated by the slowly growing function f, from the previous definition (6.3.1) we have

$$\begin{aligned}\langle F_f \circ \mathcal{A}_{\lambda,h}, \varphi\rangle = \frac{1}{|\lambda|} \left\langle F_f, \varphi \circ \mathcal{A}_{\frac{1}{\lambda},-\frac{h}{\lambda}} \right\rangle &= \frac{1}{|\lambda|} \int_{\mathbb{R}} f(t)\, \varphi\left(\frac{t-h}{\lambda}\right) dt \\ &= \int_{\mathbb{R}} f(\lambda\, s + h)\, \varphi(s)\, ds,\end{aligned}$$

thanks to the change of variable $t = \mathcal{A}_{\lambda,h}(s) = \lambda\, s + h$. This shows that in this case $F_f \circ \mathcal{A}_{\lambda,h}$ coincides with the regular tempered distribution defined by the function $f \circ \mathcal{A}_{\lambda,h}$, i.e. $t \mapsto f(\lambda\, t + h)$, i.e.

$$F_f \circ \mathcal{A}_{\lambda,h} = F_{f \circ \mathcal{A}_{\lambda,h}}. \tag{6.3.2}$$

Example 6.3.3 (Change of variable for a Dirac delta)**.** Let δ_{t_0} be the Dirac delta centered at $t_0 \in \mathbb{R}$, then for every $\lambda \in \mathbb{R} \setminus \{0\}$ and every $h \in \mathbb{R}$, we have

$$\langle \delta_{t_0} \circ \mathcal{A}_{\lambda,h}, \varphi\rangle = \frac{1}{|\lambda|} \langle \delta_{t_0}, \varphi \circ \mathcal{A}_{\frac{1}{\lambda},-\frac{h}{\lambda}} \rangle = \frac{1}{|\lambda|}\, \varphi\left(\frac{t_0-h}{\lambda}\right).$$

Thus we get that

$$\delta_{t_0} \circ \mathcal{A}_{\lambda,h} = \frac{1}{|\lambda|}\, \delta_{\frac{t_0-h}{\lambda}},$$

i.e. this is still a Dirac delta, this time centered at the point $(t_0 - h)/\lambda$ and multiplied by the factor $1/|\lambda|$.

3.3. Multiplication by a function. We first need to define a suitable class of functions.

Definition 6.3.4. We say that $\psi \in C^\infty(\mathbb{R})$ is a *multiplier of the class* $\mathcal{S}$ if for every $k \in \mathbb{N}$, there exists a constant $C_k > 0$ and an index $m_k \in \mathbb{N}$ such that

$$\left|\psi^{(k)}(t)\right| \le C_k\, (1 + |t|^{m_k}), \qquad \text{for every } t \in \mathbb{R}. \tag{6.3.3}$$

In other words, every derivative of ψ has at most polynomial growth. We indicate by $\mathcal{O}_M$ the collection of all functions with this property.

The importance of the class $\mathcal{O}_M$ lies in the following technical result, which also explains the terminology we used.

Lemma 6.3.5. *Let $\psi \in \mathcal{O}_M$ and $\varphi \in \mathcal{S}$. Then we have*

$$\psi\,\varphi \in \mathcal{S}.$$

Proof. We first observe that $\psi\,\varphi \in C^\infty(\mathbb{R})$, since both functions are infinitely times differentiable. We fix $m, k \in \mathbb{N}$ and observe that we have

$$\begin{aligned}
\left| t^m \,(\psi(t)\,\varphi(t))^{(k)} \right| &= \left| t^m \sum_{j=0}^{k} \binom{k}{j} \psi^{(j)}(t)\,\varphi^{(k-j)}(t) \right| \\
&\le |t|^m \sum_{j=0}^{k} \binom{k}{j} |\psi^{(j)}(t)|\,|\varphi^{(k-j)}(t)| \\
&\le |t|^m \sum_{j=0}^{k} \binom{k}{j} C_j\,(1 + |t|^{m_j})\,|\varphi^{(k-j)}(t)| \\
&= \sum_{j=0}^{k} \binom{k}{j} C_j\,(|t|^m + |t|^{m_j+m})\,|\varphi^{(k-j)}(t)|.
\end{aligned}$$

We now take the supremum over $t \in \mathbb{R}$, so to get

$$\begin{aligned}
[\psi\,\varphi]_{m,k} &\le \sup_{t\in\mathbb{R}} \left[\sum_{j=0}^{k} \binom{k}{j} C_j\,(|t|^m + |t|^{m_j+m})\,|\varphi^{(k-j)}(t)| \right] \\
&\le \sum_{j=0}^{k} \binom{k}{j} C_j \left([\varphi]_{m,k-j} + [\varphi]_{m_j+m,k-j} \right) < +\infty,
\end{aligned} \tag{6.3.4}$$

thanks to the fact that $\varphi \in \mathcal{S}$. □

Example 6.3.6. It is easy to see that $\mathcal{S} \subset \mathcal{O}_M$, i.e. every function of the Schwartz class $\mathcal{S}$ is a multiplier of the class $\mathcal{S}$. Indeed, if $\varphi \in \mathcal{S}$, then in particular we get

$$[\varphi]_{0,k} = \sup_{t\in\mathbb{R}} |\varphi^{(k)}(t)| < +\infty.$$

Thus we have

$$|\varphi^{(k)}(t)| \le [\varphi]_{0,k}, \qquad \text{for every } t \in \mathbb{R},$$

i.e. φ satisfies (6.3.3) with $C_k = [\varphi]_{0,k}/2$ and $m_k = 0$.

Example 6.3.7. Every polinomial is a multiplier of the class $\mathcal{S}$. Indeed, if

$$\varphi(t) = \sum_{j=0}^{m} a_j\,t^j, \qquad \text{for some } a_0, \ldots, a_m \in \mathbb{C},$$

then for $k \in \{0, \dots, m\}$ we have[2]

$$\left|\varphi^{(k)}(t)\right| = \left|\sum_{j=k}^{m} a_j \, \frac{j!}{(j-k)!} \, t^{j-k}\right| \le \sum_{j=k}^{m} \left|a_j \, \frac{j!}{(j-k)!}\right| \, |t|^{j-k}$$
$$\le \left(\sum_{j=k}^{m} \left|a_j \, \frac{j!}{(j-k)!}\right|\right) \, (1 + |t|^{m-k}),$$

thus definition (6.3.3) is satisfied with

$$C_k = \left(\sum_{j=k}^{m} \left|a_j \, \frac{j!}{(j-k)!}\right|\right) \qquad \text{and} \qquad m_k = m - k.$$

On the other hand, for $k > m$ we directly have $\varphi^{(k)}(t) \equiv 0$.

Definition 6.3.8. Let $F \in \mathcal{S}'$ and $\psi \in \mathcal{O}_M$, we define the *multiplication* $\psi \, F$ as the linear functional $\psi \, F : \mathcal{S} \to \mathbb{C}$ defined by

$$\langle \psi \, F, \varphi \rangle = \langle F, \psi \, \varphi \rangle, \qquad \text{for every } \varphi \in \mathcal{S}.$$

We observe that this is well-defined, thanks to the fact that $\psi \, \varphi \in \mathcal{S}$, see Lemma 6.3.5.

It is not difficult to see that $\psi \, F \in \mathcal{S}'$. Indeed, the linearity is straightforward. In order to verify that it is continuous on $\mathcal{S}$, it is sufficient to use the estimate (6.3.4) (*the reader should try to write the details, as an exercise*).

Example 6.3.9. Let us compute the multiplication of the tempered distribution $\mathrm{P.V.}(1/t)$ with the function $\psi(t) = t$. Observe that we have $\psi \in \mathcal{O}_M$ by Example 6.3.7, thus the multiplication is well-defined. By recalling (6.2.3), we have

$$\begin{aligned}
\left\langle t \, \mathrm{P.V.}\frac{1}{t}, \varphi \right\rangle = \left\langle \mathrm{P.V.}\frac{1}{t}, t \, \varphi \right\rangle &= \int_0^{+\infty} \frac{t \, \varphi(t) - (-t \, \varphi(-t))}{t} \, dt \\
&= \int_0^{+\infty} \Big(\varphi(t) + \varphi(-t)\Big) \, dt \\
&= \int_0^{+\infty} \varphi(t) \, dt + \int_0^{+\infty} \varphi(-t) \, dt \\
&= \int_0^{+\infty} \varphi(t) \, dt + \int_{-\infty}^{0} \varphi(s) \, ds = \int_{\mathbb{R}} \varphi(t) \, dt.
\end{aligned}$$

In other words, the product $t \, \mathrm{P.V.}(1/t)$ coincides with the regular tempered distribution F_1, generated by the constant function $f(t) = 1$. We can rewrite this result informally as

$$t \, \mathrm{P.V.}\frac{1}{t} = 1.$$

Written in this way, this result looks of course very natural...

[2] In the second inequality, we use that

$$|t|^{\alpha} \le (1 + |t|^{\beta}), \qquad \text{for every } t \in \mathbb{R},$$

whenever $0 \le \alpha \le \beta$.

Example 6.3.10. We now compute the multiplication of the Dirac delta δ_0 by the function $\psi(t) = t^k$, where $k \in \mathbb{N} \setminus \{0\}$. We first observe that still by Example (6.3.7) we have $\psi \in \mathcal{O}_M$, thus the multiplication is well-defined. For every $\varphi \in \mathcal{S}$ we have

$$\langle t^k\, \delta_0, \varphi\rangle = \langle \delta_0, t^k\, \varphi\rangle = \Big(t^k\, \varphi(t)\Big)_{|t=0} = 0.$$

In other words, $t^k\, \delta_0$ *is the zero distribution*, for every $k \in \mathbb{N} \setminus \{0\}$. Of course, the same is still true for $\psi\, \delta_0$, for every $\psi \in \mathcal{O}_M$ such that $\psi(0) = 0$.

3.4. Convolution with a function.

Definition 6.3.11. We say that a measurable function $\psi : \mathbb{R} \to \mathbb{C}$ is a *convolver of the class $\mathcal{S}$* if we have

$$t^k\, \psi \in L^1(\mathbb{R}), \qquad \text{for every } k \in \mathbb{N}. \tag{6.3.5}$$

We indicate by $\mathcal{O}_C$ the collection of all functions with this property.

Example 6.3.12 (The class $\mathcal{S}$)**.** By recalling that $\mathcal{S} \subset L^1(\mathbb{R})$ (see Proposition 5.6.4) and that for every $\psi \in \mathcal{S}$ and $k \in \mathbb{N}$ it holds

$$t^k\, \psi \in L^1(\mathbb{R}),$$

by equation (5.6.4), we have $\mathcal{S} \subset \mathcal{O}_C$. Thus every function of the Schwartz class is a convolver of the class $\mathcal{S}$.

Example 6.3.13 (Compactly supported convolvers)**.** Let $\psi \in L^1(\mathbb{R})$ be a compactly supported function, i.e. such that

$$|\psi(t)| = 0, \qquad \text{for almost every } t \in \mathbb{R} \setminus [a, b].$$

Then we have $\psi \in \mathcal{O}_C$. Indeed, for every $k \in \mathbb{N}$ we have

$$\int_{\mathbb{R}} |t|^k\, |\psi(t)|\, dt = \int_a^b |t|^k\, |\psi(t)|\, dt \le \max\{|a|^k,\, |b|^k\} \int_a^b |\psi(t)|\, dt < +\infty.$$

Remark 6.3.14. We observe that if $\psi \in \mathcal{O}_C$, then by Corollary 5.2.4 we get in particular

$$\mathcal{F}[\psi] \in C^\infty(\mathbb{R}) \qquad \text{with } \frac{d^k}{d\omega^k}\mathcal{F}[\psi] \in L^\infty(\mathbb{R}), \qquad \text{for every } k \in \mathbb{N}.$$

This implies that

$$\psi \in \mathcal{O}_C \qquad \Longrightarrow \qquad \mathcal{F}[\psi] \in \mathcal{O}_M.$$

The following expedient result justifies the name for the class $\mathcal{O}_C$.

Proposition 6.3.15. *For every $\psi \in \mathcal{O}_C$ and $\varphi \in \mathcal{S}$, we have*

$$\varphi * \psi \in \mathcal{S}.$$

Moreover, for every $k, m \in \mathbb{N}$ it holds

$$[\varphi * \psi]_{m,k} \le C \left([\varphi]_{m,k}\, \|\psi\|_{L^1(\mathbb{R})} + [\varphi]_{0,k}\, \|t^m\, \psi\|_{L^1(\mathbb{R})}\right), \tag{6.3.6}$$

for a constant $C > 0$ depending on m only.

Proof. We first observe that $\varphi * \psi \in C^\infty(\mathbb{R})$, thanks to Proposition 5.6.7. In order to conclude, we need to prove that for every $m, k \in \mathbb{N}$ we have

$$[\varphi * \psi]_{m,k} < +\infty.$$

By recalling formula (5.6.6) and the definition of convolution between functions, we get

$$\begin{aligned}
\left|t^m \, (\varphi * \psi)^{(k)}(t)\right| = \left|t^m \left(\psi * (\varphi^{(k)})(t)\right)\right| &= |t|^m \left|\int_{\mathbb{R}} \psi(y)\, \varphi^{(k)}(t-y)\, dy\right| \\
&\le |t|^m \int_{\mathbb{R}} |\psi(y)|\, |\varphi^{(k)}(t-y)|\, dy \\
&\le C \int_{\mathbb{R}} |\psi(y)| \left(|t-y|^m + |y|^m\right) \left|\varphi^{(k)}(t-y)\right| dy \\
&= C \int_{\mathbb{R}} |\psi(y)|\, |t-y|^m \left|\varphi^{(k)}(t-y)\right| dy \\
&+ C \int_{\mathbb{R}} |\psi(y)|\, |y|^m \left|\varphi^{(k)}(t-y)\right| dy,
\end{aligned}$$

where $C > 0$ depends on m only. From the previous chain of inequalities, we then get for every $t \in \mathbb{R}$

$$\left|t^m \, (\varphi * \psi)^{(k)}(t)\right| \le C \left([\varphi]_{m,k} \int_{\mathbb{R}} |\psi(y)|\, dy + [\varphi]_{0,k} \int_{\mathbb{R}} |y^m \, \psi(y)|\, dy\right) < +\infty.$$

This shows that $\varphi * \psi \in \mathcal{S}$, as well as the validity of the estimate (6.3.6). □

We can now define the convolution of a tempered distribution with a convolver of the class $\mathcal{S}$.

Definition 6.3.16. Let $F \in \mathcal{S}'$ and $\psi \in \mathcal{O}_C$. The *convolution of* ψ *and* F is the linear functional $\psi * F : \mathcal{S} \to \mathbb{C}$ defined by

$$\langle \psi * F, \varphi\rangle = \langle F, (\psi \circ \mathcal{A}_{-1,0}) * \varphi\rangle, \qquad \text{for every } \varphi \in \mathcal{S}.$$

As usual, the definition of convolution may look weird, but it is designed so to coincide with the usual operation of convolution between functions, when F is a regular tempered distribution.

Example 6.3.17. Let $f : \mathbb{R} \to \mathbb{C}$ be a slowly growing function and let $\psi \in \mathcal{O}_C$. We denote as usual by F_f the regular tempered distribution generated by f, then for every $\varphi \in \mathcal{S}$ we have

$$\begin{aligned}
\langle \psi * F_f, \varphi\rangle = \langle F_f, (\psi \circ \mathcal{A}_{-1,0}) * \varphi\rangle &= \int_{\mathbb{R}} f(t)\, (\psi \circ \mathcal{A}_{-1,0}) * \varphi(t)\, dt \\
&= \int_{\mathbb{R}} f(t) \left(\int_{\mathbb{R}} \psi \circ \mathcal{A}_{-1,0}(t-s)\, \varphi(s)\, ds\right) dt \\
&= \int_{\mathbb{R}} f(t) \left(\int_{\mathbb{R}} \psi(s-t)\, \varphi(s)\, ds\right) dt \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(t)\, \psi(s-t)\, dt\right) \varphi(s)\, ds \\
&= \int_{\mathbb{R}} f * \psi(s)\, \varphi(s)\, ds = \langle F_{f*\psi}, \varphi\rangle,
\end{aligned}$$

where in the fifth equality we exchanged the order of integration. This shows that

$$\psi * F_f = F_{f*\psi}.$$

Proposition 6.3.18. *Let* $\psi \in \mathcal{O}_C$ *and* $F \in \mathcal{S}'$. *Then* $\psi * F \in \mathcal{S}'$.

Proof. We first observe that $\psi * F$ is well-defined, since $\psi \circ \mathcal{A}_{-1,0} \in \mathcal{O}_C$ and thus for every $\varphi \in \mathcal{S}$ we have

$$\psi \circ \mathcal{A}_{-1,0} * \varphi \in \mathcal{S},$$

by Proposition 6.3.15. Then the expression

$$\langle F, \psi \circ \mathcal{A}_{-1,0} * \varphi \rangle,$$

makes sense. Linearity of $\psi * F$ is easy to verify, in order to prove $\psi * F \in \mathcal{S}'$ we only need to check that it is continuous on $\mathcal{S}$. We take a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ such that $\varphi_n \xrightarrow{\mathcal{S}} 0$. Then by definition

$$\lim_{n\to\infty} \langle \psi * F, \varphi_n \rangle = \lim_{n\to\infty} \langle F, (\psi \circ \mathcal{A}_{-1,0}) * \varphi_n \rangle.$$

By using the estimate (6.3.6), we have

$$\begin{aligned}\lim_{n\to\infty} \Big[(\psi \circ \mathcal{A}_{-1,0}) * \varphi_n\Big]_{m,k} &\leq C \lim_{n\to\infty} [\varphi_n]_{m,k} \, \|\psi \circ \mathcal{A}_{-1,0}\|_{L^1(\mathbb{R})} \\ &\quad + C \lim_{n\to\infty} [\varphi_n]_{0,k} \, \|t^m \, \psi \circ \mathcal{A}_{-1,0}\|_{L^1(\mathbb{R})} = 0.\end{aligned}$$

This shows that the sequence $\{(\psi \circ \mathcal{A}_{-1,0}) * \varphi_n\}_{n\in\mathbb{N}}$ converges to 0 in $\mathcal{S}$. Since F is continuous on $\mathcal{S}$, we thus get

$$\lim_{n\to\infty} \langle \psi * F, \varphi_n \rangle = \lim_{n\to\infty} \langle F, (\psi \circ \mathcal{A}_{-1,0}) * \varphi_n \rangle = 0,$$

as desired. □

Example 6.3.19. Let us compute the convolution of a Dirac delta δ_{t_0} with a convolver $\psi \in \mathcal{O}_C$. By definition, for every $\varphi \in \mathcal{S}$ we have

$$\langle \psi * \delta_{t_0}, \varphi \rangle = \langle \delta_{t_0}, (\psi \circ \mathcal{A}_{-1,0}) * \varphi \rangle = (\psi \circ \mathcal{A}_{-1,0}) * \varphi(t_0) = \int_{\mathbb{R}} \psi(t - t_0)\, \varphi(t)\, dt = \langle F_{\psi \circ \mathcal{A}_{1,-t_0}}, \varphi \rangle.$$

In other words, the distribution $\psi * \delta_{t_0}$ coincides with the regular tempered distribution generated by $t \mapsto \psi(t - t_0)$. In particular, for $t_0 = 0$ we have that $\psi * \delta_0$ coincides with the regular tempered distrbution generate by ψ. Informally, we could write this as

$$\psi * \delta_0 = \psi.$$

3.5. Convergence of distributions. On the vector space of tempered distributions, we can define in a natural way a notion of convergence.

Definition 6.3.20. Let $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{S}'$ be a sequence of tempered distributions. We say that F_n *converges to* $F \in \mathcal{S}'$ if

$$\lim_{n\to\infty} \langle F_n, \varphi \rangle = \langle F, \varphi \rangle, \qquad \text{for every } \varphi \in \mathcal{S}.$$

In this case, we use the notation $F_n \xrightarrow{\mathcal{S}'} F$.

The following result shows that a Dirac delta can be obtained as limit in $\mathcal{S}'$ of regular tempered distributions.

Proposition 6.3.21 (Regular approximations of a Dirac delta)**.** *Let $t_0 \in \mathbb{R}$ and let $f \in L^1(\mathbb{R})$ be such that*

$$\int_{\mathbb{R}} f(t)\, dt = c.$$

For every $\varepsilon > 0$, we define the L^1 function

$$f_\varepsilon(t) = \frac{1}{\varepsilon} f\left(\frac{t - t_0}{\varepsilon}\right), \qquad \text{for } t \in \mathbb{R}.$$

Then we have

$$F_{f_\varepsilon} \xrightarrow{\mathcal{S}'} c\,\delta_{t_0},$$

that is

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}} f_\varepsilon(t)\,\varphi(t)\,dt = c\,\varphi(t_0), \qquad \text{for every } \varphi \in \mathcal{S}. \tag{6.3.7}$$

Proof. Let $\varphi \in \mathcal{S}$, by definition of regular tempered distribution and of f_ε, we have

$$\begin{aligned}\langle F_{f_\varepsilon}, \varphi\rangle = \int_{\mathbb{R}} f_\varepsilon(t)\,\varphi(t)\,dt &= \frac{1}{\varepsilon}\int_{\mathbb{R}} f\left(\frac{t-t_0}{\varepsilon}\right)\varphi(t)\,dt \\ &= \int_{\mathbb{R}} f(s)\,\varphi(\varepsilon\, s + t_0)\,ds.\end{aligned}$$

We now observe that

$$\lim_{\varepsilon\to 0}\Big(f(s)\,\varphi(\varepsilon\, s+t_0)\Big) = f(s)\,\varphi(t_0), \qquad \text{for a. e. } s \in \mathbb{R},$$

and that

$$\Big|f(s)\,\varphi(\varepsilon\, s+t_0)\Big| \le |f(s)|\,[\varphi]_{0,0}, \qquad \text{for a. e. } s \in \mathbb{R}.$$

The last function belongs to $L^1(\mathbb{R})$ and does not depend on $\varepsilon > 0$, thus we can apply the Dominated Convergence Theorem and obtain

$$\lim_{\varepsilon\to 0}\langle F_{f_\varepsilon}, \varphi\rangle = \lim_{\varepsilon\to 0}\int_{\mathbb{R}} f(s)\,\varphi(\varepsilon\, s+t_0)\,ds = \varphi(t_0)\int_{\mathbb{R}} f(s)\,ds = c\,\varphi(t_0),$$

as desired. □

The following result is quite sophisticated. It will be useful in order to compute the Fourier transform of some tempered distributions.

Theorem 6.3.22 (The Sochocki-Plemelj formula)**.** *For every $\alpha > 0$, let us define*

$$g_\alpha(t) = \frac{1}{t - i\,\alpha}, \qquad t \in \mathbb{R}.$$

Then the sequence of regular tempered distributions $\{F_{g_\alpha}\}_{\alpha>0} \subset \mathcal{S}'$ generated by the family $\{g_\alpha\}_{\alpha>0}$ converges in $\mathcal{S}'$ to the tempered distribution

$$\text{P.V.}\,\frac{1}{t} + i\,\pi\,\delta_0,$$

as α goes to 0. In other words, we have

$$F_{g_\alpha} \xrightarrow{\mathcal{S}'} \text{P.V.}\,\frac{1}{t} + i\,\pi\,\delta_0,$$

that is

$$\lim_{\alpha\to 0^+}\int_{\mathbb{R}} \frac{\varphi(t)}{t - i\,\alpha}\,dt = \left\langle \text{P.V.}\,\frac{1}{t}, \varphi\right\rangle + i\,\pi\,\varphi(0), \qquad \text{for every } \varphi \in \mathcal{S}.$$

Proof. Let $\varphi \in \mathcal{S}$, we use the same trick that we used to define the principal value of $1/t$. We split the integral and use a change of variable, so to get

$$\begin{aligned}\int_{\mathbb{R}} \frac{\varphi(t)}{t-i\,\alpha}\,dt &= \int_0^{+\infty} \frac{\varphi(t)}{t-i\,\alpha}\,dt + \int_{-\infty}^0 \frac{\varphi(t)}{t-i\,\alpha}\,dt \\ &= \int_0^{+\infty} \frac{\varphi(t)}{t-i\,\alpha}\,dt - \int_0^{+\infty} \frac{\varphi(-t)}{t+i\,\alpha}\,dt.\end{aligned}$$

With simple algebraic manipulations, we obtain

$$\begin{aligned}\int_{\mathbb{R}} \frac{\varphi(t)}{t-i\,\alpha}\,dt &= \int_0^{+\infty} \left[\frac{\varphi(t)}{t-i\,\alpha} - \frac{\varphi(-t)}{t+i\,\alpha}\right] dt \\ &= \int_0^{+\infty} \frac{\varphi(t)\,(t+i\,\alpha) - \varphi(-t)\,(t-i\,\alpha)}{t^2+\alpha^2}\,dt \\ &= \int_0^{+\infty} \frac{\varphi(t)-\varphi(-t)}{t^2+\alpha^2}\,t\,dt + i\int_0^{+\infty} \frac{\varphi(t)+\varphi(-t)}{t^2+\alpha^2}\,\alpha\,dt.\end{aligned}$$

We now need to take the limit as α goes to 0 in the last two integrals, i.e.

$$\mathcal{I}_1(\alpha) = \int_0^{+\infty} \frac{\varphi(t)-\varphi(-t)}{t^2+\alpha^2}\,t\,dt,$$

and

$$\mathcal{I}_2(\alpha) = \int_0^{+\infty} \frac{\varphi(t)+\varphi(-t)}{t^2+\alpha^2}\,\alpha\,dt.$$

For $\mathcal{I}_1(\alpha)$, it is sufficient to observe that

$$\left|\frac{\varphi(t)-\varphi(-t)}{t^2+\alpha^2}\,t\right| \le \left|\frac{\varphi(t)-\varphi(-t)}{t}\right| \qquad \text{for } t>0,$$

and the last function is in $L^1(\mathbb{R}_+)$ (recall (6.2.4)). Thus we can apply the Dominated Convergence Theorem and get

$$\lim_{\alpha\to 0^+} \mathcal{I}_1(\alpha) = \int_0^{+\infty} \frac{\varphi(t)-\varphi(-t)}{t}\,dt = \left\langle \mathrm{P.V.}\frac{1}{t},\varphi\right\rangle.$$

As for the second integral above, i.e. $\mathcal{I}_2(\alpha)$, it is sufficient to apply Exercise 6.8.10 below, which guarantees

$$\lim_{\alpha\to 0^+} \mathcal{I}_2(\alpha) = \langle \pi\,\delta_0,\varphi\rangle = \pi\,\varphi(0).$$

By keeping everything together, we obtained

$$\lim_{\alpha\to 0^+} \int_{\mathbb{R}} \frac{\varphi(t)}{t-i\,\alpha}\,dt = \lim_{\alpha\to 0^+} \mathcal{I}_1(\alpha) + i\,\lim_{\alpha\to 0^+} \mathcal{I}_2(\alpha) = \left\langle \mathrm{P.V.}\frac{1}{t},\varphi\right\rangle + i\,\langle \pi\,\delta_0,\varphi\rangle,$$

as desired. □

4. Distributional derivatives

Definition 6.4.1. Let $F \in \mathcal{S}'$, its *distributional derivative* is the linear functional $F' : \mathcal{S} \to \mathbb{C}$ defined by

$$\langle F',\varphi\rangle = -\langle F,\varphi'\rangle, \qquad \text{for every } \varphi \in \mathcal{S}.$$

More generally, for every $k \in \mathbb{N}\setminus\{0\}$ the $k-$th distributional derivative of F is the linear functional $F^{(k)}$ defined by

$$\langle F^{(k)},\varphi\rangle = (-1)^k\,\langle F,\varphi^{(k)}\rangle, \qquad \text{for every } \varphi \in \mathcal{S}.$$

Observe that the definitions are well-posed, since if $\varphi \in \mathcal{S}$, then $\varphi^{(k)} \in \mathcal{S}$ for every $k \in \mathbb{N}\setminus\{0\}$.

The distributional derivative of a tempered distribution still defines a tempered distribution. This is the content of the next result.

Proposition 6.4.2. *Let $F \in \mathcal{S}'$, then for every $k \in \mathbb{N}\setminus\{0\}$ we have $F^{(k)} \in \mathcal{S}'$ as well.*

Proof. Let us prove the result for $k = 1$. We take a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ such that $\varphi_n \xrightarrow{\mathcal{S}} 0$ in $\mathcal{S}$. This means that

$$\lim_{n\to\infty} [\varphi_n]_{m,\ell} = 0, \qquad \text{for every } m, \ell \in \mathbb{N}.$$

Observe that $\varphi_n' \in \mathcal{S}$ thanks to Proposition 5.6.5. Moreover, for every $m, \ell \in \mathbb{N}$ we have

$$[\varphi_n']_{m,\ell} = \sup_{t\in\mathbb{R}} \left| t^m \frac{d^\ell}{dt^\ell} \varphi_n'(t) \right| = \sup_{t\in\mathbb{R}} \left| t^m \varphi_n^{(\ell+1)}(t) \right| = [\varphi_n]_{m,\ell+1}.$$

This implies that the sequence $\{\varphi_n'\}_{n\in\mathbb{N}} \subset \mathcal{S}$ is such that $\varphi_n' \xrightarrow{\mathcal{S}} 0$ in $\mathcal{S}$. Finally, by using the definition of distributional derivative, we get

$$\lim_{n\to\infty} \langle F, \varphi_n \rangle = - \lim_{n\to\infty} \langle F, \varphi_n' \rangle = 0,$$

where in the last passage we used that $F \in \mathcal{S}'$, so in particular it is continuous on $\mathcal{S}$. $\square$

The following result is important, it enables one to compute the distributional derivative of a piecewise C^1 function.

Theorem 6.4.3. *Let $f : \mathbb{R} \to \mathbb{C}$ be a piecewise C^1 function, such that f and f' have only jump discontinuities at the points $\{t_n\}_{n\in\mathbb{N}}$, with*

$$|t_k - t_j| \geq \delta > 0, \qquad \textit{for every } k \neq j \in \mathbb{N}.$$

Let us suppose in addition that there exists $C > 0$ and $m \in \mathbb{N}$ such that

$$|f(t)| + |f'(t)| \leq C\,(1 + |t|^m), \qquad \textit{for every } t \in \mathbb{R}. \tag{6.4.1}$$

Then the distributional derivative of F_f is given by

$$F_f' = F_{f'} + \sum_{n=0}^{\infty} \big(f(t_n^+) - f(t_n^-) \big)\, \delta_{t_n}. \tag{6.4.2}$$

Proof. In order to give a better understanding of the proof, we confine ourselves to prove the result in the case where f and f' have only one jump discontinuity, in correspondence of the point t_0.

We first observe that (6.4.1) guarantees that both f and f' are slowly growing functions, thus it is possible to consider F_f and $F_{f'}$. For every $\varphi \in \mathcal{S}$, by using the definition of distributional derivative we then have

$$\begin{aligned}
\langle F_f', \varphi \rangle = -\langle F_f, \varphi' \rangle &= - \int_{\mathbb{R}} f(t)\, \varphi'(t)\, dt \\
&= - \int_{-\infty}^{t_0} f(t)\, \varphi'(t)\, dt - \int_{t_0}^{\infty} f(t)\, \varphi'(t)\, dt \\
&= - \Big[f(t)\, \varphi(t) \Big]_{-\infty}^{t_0} + \int_{-\infty}^{t_0} f'(t)\, \varphi(t)\, dt \\
&\quad - \Big[f(t)\, \varphi(t) \Big]_{t_0}^{+\infty} + \int_{t_0}^{+\infty} f'(t)\, \varphi(t)\, dt.
\end{aligned}$$

In order to conclude, we just need to observe that since $\varphi \in \mathcal{S}$ and f verifies (6.4.1), we have

$$\lim_{t\to+\infty} f(t)\, \varphi(t) = \lim_{t\to-\infty} f(t)\, \varphi(t) = 0.$$

Thus we obtain

$$\begin{aligned}\langle F_f', \varphi\rangle &= f(t_0^+)\,\varphi(t_0) - f(t_0^-)\,\varphi(t_0) + \int_{\mathbb{R}} f'(t)\,\varphi(t)\,dt \\ &= \Big\langle \big(f(t_0^+) - f(t_0^-)\big)\,\delta_{t_0}, \varphi\Big\rangle + \langle F_{f'}, \varphi\rangle.\end{aligned}$$

This concludes the proof. □

Example 6.4.4 (Distributional derivative of the Heaviside function)**.** Let us consider the regular tempered distribution F_H, generated by the Heaviside step function H. From formula (6.4.2), we find again

$$F_H' = \delta_0,$$

as we computed "by hand" in Section 1.

Example 6.4.5 (Distributional derivative of rect)**.** Let us compute the distributional derivative of the rectangular function or, more precisely, of the regular tempered distribution F_{rect} *generated* by the rectangular function. Observe that $t \mapsto \text{rect}(t)$ verifies the hypotheses of Theorem 6.4.3, since it is a piecewise constant function, with compact support. Observe that

$$\text{rect}'(t) = 0 \qquad \text{for } |t| \neq \frac{1}{2},$$

and rect has only two discontinuity points

$$t_0 = -\frac{1}{2} \qquad \text{and} \qquad t_1 = \frac{1}{2},$$

with jumps

$$\text{rect}(t_0^+) - \text{rect}(t_0^-) = 1 \qquad \text{and} \qquad \text{rect}(t_1^+) - \text{rect}(t_1^-) = -1.$$

Thus from formula (6.4.2) we get

$$F_{\text{rect}}' = \delta_{-\frac{1}{2}} - \delta_{\frac{1}{2}}.$$

In other words, for every $\varphi \in \mathcal{S}$ we have

$$\langle F_{\text{rect}}', \varphi\rangle = \varphi\left(-\frac{1}{2}\right) - \varphi\left(\frac{1}{2}\right).$$

Example 6.4.6 (Distributional derivative of the sawtooth wave)**.** Let us compute the distributional derivative of the *sawtooth wave*

$$SW(t) = \sum_{k=0}^{\infty} (t-k)\,\Big[H(t-k) - H(t-k-1)\Big], \qquad t \in \mathbb{R}.$$

Observe that this verifies the hypotheses of Theorem 6.4.3. Indeed, SW is piecewise C^1, with SW and SW' discontinuous at the points $t_n = n$ for $n \in \mathbb{N}$. More precisely, we observe that for $t_0 = 0$ the function SW is continuous, thus

$$SW(0^-) = SW(0^+),$$

while for $n \geq 1$ the jump is -1, i.e.

$$SW(n^+) - SW(n^-) = -1.$$

Also observe that

$$SW'(t) = H(t), \qquad \text{for } t \notin \mathbb{N}.$$

Thus from (6.4.2) we obtain

$$F_{SW}' = F_H - \sum_{n=1}^{\infty} \delta_n,$$

where F_H is the regular tempered distribution generated by the Heaviside step function $SW'(t) = H(t)$. In other words, for every $\varphi \in \mathcal{S}$ we have

$$\langle F'_{SW}, \varphi\rangle = \int_0^{+\infty} \varphi(t)\,dt - \sum_{n=1}^{\infty} \varphi(n).$$

Corollary 6.4.7. *Let $f : \mathbb{R} \to \mathbb{C}$ satisfy the hypothesis of Theorem 6.4.3. Let us suppose in addition that f is continuous. Then the distributional derivative of F_f is the regular tempered distribution generated by f', i.e.*

$$F'_f = F_{f'}.$$

In other words, we have

$$\langle F'_f, \varphi\rangle = \int_{\mathbb{R}} f'(t)\,\varphi(t)\,dt, \qquad \textit{for every } \varphi \in \mathcal{S}.$$

Remark 6.4.8. The previous result can be rephrased informally by saying that "*the distributional derivative of f coincides with the classical one f'*", under the previous assumptions.

Example 6.4.9 (Derivative of the ramp function)**.** We consider the unitary ramp function $R(t) = t\,H(t)$. This verifies the assumptions of Corollary 6.4.7, thus by observing that

$$R'(t) = H(t), \qquad \text{for } t \neq 0,$$

we obtain

$$F'_R = F_H.$$

Informally, this means that the Heaviside step function is the distributional derivative of the unitary ramp function.

Example 6.4.10 (Derivative of the absolute value)**.** We consider the function $f(t) = |t|$. This function verifies the assumptions of Corollary 6.4.7 and

$$\frac{d}{dt}|t| = \begin{cases} 1, & \text{if } t > 0,\\ -1, & \text{if } t < 0.\end{cases}$$

Then the distributional derivative of the regular tempered distribution $F_{|t|}$ is the regular tempered distribution generated by $\frac{d}{dt}|t|$.

Proposition 6.4.11. *Let $F \in \mathcal{S}'$ be such that $F' = 0$, i.e.*

$$\langle F', \varphi\rangle = 0, \qquad \textit{for every } \varphi \in \mathcal{S}.$$

Then F is the regular tempered distribution generated by a constant function, i.e. there exists $c \in \mathbb{C}$ such that

$$\langle F, \varphi\rangle = c\int_{\mathbb{R}} \varphi(t)\,dt, \qquad \textit{for every } \varphi \in \mathcal{S}.$$

5. The distributional Fourier transform

The following simple result suggests a way to define the Fourier transform for a tempered distribution.

Lemma 6.5.1. *Let $f, g \in L^1(\mathbb{R})$, then we have*

$$\int_{\mathbb{R}} \mathcal{F}[f](\omega)\,g(\omega)\,d\omega = \int_{\mathbb{R}} f(\omega)\,\mathcal{F}[g](\omega)\,d\omega.$$

Proof. We first observe that both sides are well-defined, since by Theorem 5.2.1 we have

$$\mathcal{F}[f] \in L^\infty(\mathbb{R}) \qquad \text{and} \qquad \mathcal{F}[g] \in L^\infty(\mathbb{R}),$$

thus by Holder's inequality

$$\mathcal{F}[f]\, g \in L^1(\mathbb{R}) \qquad \text{and} \qquad f\, \mathcal{F}[g] \in L^1(\mathbb{R}).$$

By applying Fubini's Theorem and exchanging the order of integration, we get

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{F}[f](\omega)\, g(\omega)\, d\omega &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{-i\,t\omega}\, f(t)\, dt\right) g(\omega)\, d\omega \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{-i\,t\omega}\, g(\omega)\, d\omega\right) f(t)\, dt \\ &= \int_{\mathbb{R}} f(t)\, \mathcal{F}[g](t)\, d\omega\end{aligned}$$

which is the desired formula. □

Definition 6.5.2. Let $F \in \mathcal{S}'$, the *Fourier transform of* F is the linear functional $\mathcal{F}[F] : \mathcal{S} \to \mathbb{C}$ defined by

$$\langle \mathcal{F}[F], \varphi\rangle = \langle F, \mathcal{F}[\varphi]\rangle, \qquad \text{for every } \varphi \in \mathcal{S}.$$

Observe that for every $\varphi \in \mathcal{S}$, we know by Theorem 5.6.6 that $\mathcal{F}[\varphi] \in \mathcal{S}$ as well, thus

$$\langle F, \mathcal{F}[\varphi]\rangle,$$

is well-defined.

The next result shows that this definition of Fourier transform extends to $\mathcal{S}'$ the definition we gave for $L^1(\mathbb{R})$. In other words, for regular distributions generated by L^1 functions, we are back to the usual definition.

Proposition 6.5.3. *Let $f \in L^1(\mathbb{R})$ and let F_f be the regular tempered distribution generated by f. Then we have*

$$\mathcal{F}[F_f] = F_{\mathcal{F}[f]},$$

i.e. the distributional Fourier transform of F_f coincides with the tempered distribution generated by $\mathcal{F}[f]$. This implies that

$$\langle \mathcal{F}[F_f], \varphi\rangle = \int_{\mathbb{R}} \mathcal{F}[f](\omega)\, \varphi(\omega)\, d\omega, \qquad \textit{for every } \varphi \in \mathcal{S}.$$

Proof. By using the definition of Fourier transform for a tempered distribution and Lemma 6.5.1, for every $\varphi \in \mathcal{S}$ we have

$$\begin{aligned}\langle \mathcal{F}[F_f], \varphi\rangle = \langle F_f, \mathcal{F}[\varphi]\rangle &= \int_{\mathbb{R}} f(\omega)\, \mathcal{F}[\varphi](\omega)\, d\omega \\ &= \int_{\mathbb{R}} \mathcal{F}[f](\omega)\, \varphi(\omega)\, d\omega = \langle F_{\mathcal{F}[f]}, \varphi\rangle.\end{aligned}$$

This shows the desired identity. □

The same can be said for the distributional Fourier transform of regular tempered distribution generated by a function in L^2.

Proposition 6.5.4. *Let $f \in L^2(\mathbb{R})$, then we have*

$$\mathcal{F}[F_f] = F_{\mathcal{F}_{L^2}[f]},$$

i.e. the distributional Fourier transform of F_f coincides with the tempered distribution generated by $\mathcal{F}_{L^2}[f]$ defined in Section 6 of Chapter 5.

Proof. For every $\varphi \in \mathcal{S}$, by definition of distributional Fourier transform we have

$$\langle \mathcal{F}[F_f], \varphi \rangle = \langle F_f, \mathcal{F}[\varphi] \rangle = \int_{\mathbb{R}} f(\omega)\, \mathcal{F}[\varphi](\omega)\, d\omega. \tag{6.5.1}$$

We now use Parseval's formula in L^2 (see Theorem 5.6.8) for the two functions[3]

$$f \qquad \text{and} \qquad \mathcal{F}[\varphi]^*,$$

which gives

$$\int_{\mathbb{R}} f(\omega)\, \mathcal{F}[\varphi](\omega)\, d\omega = \frac{1}{2\,\pi} \int_{\mathbb{R}} \mathcal{F}_{L^2}[f](\omega)\, \mathcal{F}\Big[\mathcal{F}[\varphi]^*\Big](\omega)^*\, d\omega.$$

We now observe that

$$\mathcal{F}\Big[\mathcal{F}[\varphi]^*\Big](\omega)^* = \left(\int_{\mathbb{R}} e^{-i\,t\,\omega}\, \mathcal{F}[\varphi](t)^*\, dt\right)^* = \int_{\mathbb{R}} e^{i\,t\,\omega}\, \mathcal{F}[\varphi](t)\, dt = 2\,\pi\, \varphi(\omega),$$

where in the last identity we used the inversion formula, i.e. Theorem 5.4.2. This shows that

$$\int_{\mathbb{R}} f(\omega)\, \mathcal{F}[\varphi](\omega)\, d\omega = \int_{\mathbb{R}} \mathcal{F}_{L^2}[f](\omega)\, \varphi(\omega)\, d\omega.$$

By using this information in (6.5.1), we get the conclusion. □

Example 6.5.5 (Fourier transform of a Dirac delta)**.** Let $t_0 \in \mathbb{R}$, we have shown that $\delta_{t_0} \in \mathcal{S}'$. Let us compute its Fourier transform. By using the definitions, we have

$$\langle \mathcal{F}[\delta_{t_0}], \varphi \rangle = \langle \delta_{t_0}, \mathcal{F}[\varphi] \rangle = \mathcal{F}[\varphi](t_0) = \int_{\mathbb{R}} e^{-i\,\omega\,t_0}\, \varphi(\omega)\, d\omega.$$

This shows that $\mathcal{F}[\delta_{t_0}]$ coincides with the tempered distribution generated by the bounded function $\omega \mapsto e^{-i\,\omega\,t_0}$. Thus we could *informally* write

$$\mathcal{F}[\delta_{t_0}](\omega) = e^{-i\,\omega\,t_0}.$$

Observe in particular that for $t_0 = 0$ we have (by still using the informal writing as above)

$$\mathcal{F}[\delta_0](\omega) = 1,$$

i.e. the Fourier transform of δ_0 is the constant function valued 1.

Example 6.5.6 (Fourier transform of a constant function)**.** We now consider the regular tempered distribution F_1 generated by the constant function, valued 1. For every $\varphi \in \mathcal{S}$ we have

$$\langle \mathcal{F}[F_1], \varphi \rangle = \langle F_1, \mathcal{F}[\varphi] \rangle = \int_{\mathbb{R}} \mathcal{F}[\varphi](\omega)\, d\omega.$$

Since $\varphi \in \mathcal{S}$, it verifies the hypothesis of the Inversion Formula of Theorem 5.4.2. Then we have

$$\varphi(t) = \frac{1}{2\,\pi} \int_{\mathbb{R}} \mathcal{F}[\varphi](\omega)\, e^{i\,t\,\omega}\, d\omega,$$

[3] Recall that for a complex number z, we have $(z^*)^* = z$.

and by taking $t=0$ we get

$$2\,\pi\,\varphi(0) = \int_{\mathbb{R}} \mathcal{F}[\varphi](\omega)\,d\omega.$$

This shows that

$$\langle \mathcal{F}[F_1],\varphi\rangle = \langle F_1,\mathcal{F}[\varphi]\rangle = \int_{\mathbb{R}} \mathcal{F}[\varphi](\omega)\,d\omega = 2\,\pi\,\varphi(0),$$

that is

$$\mathcal{F}[F_1] = 2\,\pi\,\delta_0.$$

The following result collects the properties of the Fourier transform of a tempered distribution. These are analogue to those for L^1 functions seen in Chapter 5.

Theorem 6.5.7. *Let $F \in \mathcal{S}'$, then $\mathcal{F}[F]$ is a tempered distribution as well. Moreover, the following formulas hold in the sense of distributions*

$$\frac{d^k}{d\omega^k}\mathcal{F}[F] = (-i)^k\,\mathcal{F}[t^k\,F], \qquad \text{for every } k \in \mathbb{N}, \tag{6.5.2}$$

$$\mathcal{F}[F^{(k)}] = (i\,\omega)^k\,\mathcal{F}[F], \qquad \text{for every } k \in \mathbb{N}, \tag{6.5.3}$$

$$\mathcal{F}[F \circ \mathcal{A}_{\lambda,h}] = \frac{1}{\lambda}\,e^{i\,\frac{h}{\lambda}\,\omega}\,\mathcal{F}[F] \circ \mathcal{A}_{\frac{1}{\lambda},0}, \qquad \text{for every } \lambda > 0, h \in \mathbb{R}, \tag{6.5.4}$$

$$\mathcal{F}[e^{i\,\omega_0\,t}\,F] = \mathcal{F}[F] \circ \mathcal{A}_{1,-\omega_0}, \qquad \text{for every } \omega_0 \in \mathbb{R}, \tag{6.5.5}$$

$$\mathcal{F}\Big[\mathcal{F}[F]\Big] = 2\,\pi\,F \circ \mathcal{A}_{-1,0}, \qquad (\text{duality formula}) \tag{6.5.6}$$

$$\mathcal{F}[\psi * F] = \mathcal{F}[\psi]\,\mathcal{F}[F], \qquad \text{for every } \psi \in \mathcal{O}_C. \tag{6.5.7}$$

Proof. At first, we need to show that $\mathcal{F}[F]$ is linear and continuous on $\mathcal{S}$. Linearity easily follows from its definition and the linearity of the Fourier transform for functions. In order to verify the continuity, we have to show that

$$\lim_{n\to\infty} \langle \mathcal{F}[F],\varphi_n\rangle = 0,$$

for every $\{\varphi\}_{n\in\mathbb{N}} \subset \mathcal{S}$ such that $\varphi_n \xrightarrow{\mathcal{S}} 0$. By definition of distributional Fourier transform, we have

$$\langle \mathcal{F}[F],\varphi_n\rangle = \langle F,\mathcal{F}[\varphi_n]\rangle,$$

then we get the conclusion by using that F is continuous on $\mathcal{S}$ and

$$\varphi_n \xrightarrow{\mathcal{S}} 0 \qquad \Longrightarrow \qquad \mathcal{F}[\varphi_n] \xrightarrow{\mathcal{S}} 0. \tag{6.5.8}$$

In order to prove the last result, we recall that by proceeding as in the proof of Theorem 5.6.6, we have

$$\left|\omega^m\,\frac{d^k}{d\omega^k}\mathcal{F}[\varphi_n](\omega)\right| = \left|\mathcal{F}\left[\frac{d^m}{dt^m}(t^k\,\varphi_n)\right](\omega)\right|,$$

thus by taking the supremum we get

$$\Big[\mathcal{F}[\varphi_n]\Big]_{m,k} = \left\|\mathcal{F}\left[\frac{d^m}{dt^m}(t^k\,\varphi_n)\right]\right\|_{L^\infty(\mathbb{R})}.$$

If we now use Theorem 5.2.1 and in particular the estimate (5.2.1), we obtain

$$\Big[\mathcal{F}[\varphi_n]\Big]_{m,k} \leq \left\| \frac{d^m}{dt^m}(t^k\,\varphi_n) \right\|_{L^1(\mathbb{R})}.$$

We can further use the estimate of Exercise 5.8.10 for the function

$$\frac{d^m}{dt^m}(t^k\,\varphi_n) \in \mathcal{S},$$

so to get

$$\Big[\mathcal{F}[\varphi_n]\Big]_{m,k} \leq 4\sqrt{\left[\frac{d^m}{dt^m}(t^k\,\varphi_n)\right]_{0,0}\left[\frac{d^m}{dt^m}(t^k\,\varphi_n)\right]_{2,0}}.$$

It is now quite easy to prove (6.5.8) by using this estimate.

Let us prove formula (6.5.2). By using first the definition of distributional derivative and then the definition of distributional Fourier transform, for every $\varphi \in \mathcal{S}$ we get

$$\left\langle \frac{d^k}{d\omega^k}\mathcal{F}[F], \varphi \right\rangle = (-1)^k \langle \mathcal{F}[F], \varphi^{(k)}\rangle = (-1)^k\,\langle F, \mathcal{F}[\varphi^{(k)}]\rangle.$$

We now recall that by Corollary 5.3.7, we have

$$\mathcal{F}[\varphi^{(k)}](\omega) = (i\,\omega)^k\,\mathcal{F}[\varphi](\omega),$$

and observe that the function $\omega \mapsto \omega^k$ belongs to the class $\mathcal{O}_M$ (recall Example 6.3.7). Thus we get

$$\begin{aligned}\left\langle \frac{d^k}{d\omega^k}\mathcal{F}[F], \varphi \right\rangle = (-1)^k\,\langle F, (i\,\omega)^k\,\mathcal{F}[\varphi](\omega)\rangle &= (-i)^k\,\langle \omega^k F, \mathcal{F}[\varphi]\rangle \\ &= (-i)^k\,\langle \mathcal{F}[\omega^k\,F], \varphi\rangle,\end{aligned}$$

which gives the desired result.

We prove formula (6.5.3). We take $\varphi \in \mathcal{S}$, then we get

$$\langle \mathcal{F}[F^{(k)}], \varphi\rangle = \langle F^{(k)}, \mathcal{F}[\varphi]\rangle = (-1)^k\,\left\langle F, \frac{d^k}{d\omega^k}\mathcal{F}[\varphi]\right\rangle.$$

On the other hand, by Corollary 5.2.4 we have

$$\frac{d^k}{d\omega^k}\mathcal{F}[\varphi](\omega) = (-i)^k\,\mathcal{F}[t^k\,\varphi](\omega).$$

Thus we can proceed similarly as before, i.e.

$$\begin{aligned}\langle \mathcal{F}[F^{(k)}], \varphi\rangle = \langle F^{(k)}, \mathcal{F}[\varphi]\rangle &= (-1)^k\,\left\langle F, \frac{d^k}{d\omega^k}\mathcal{F}[\varphi]\right\rangle \\ &= i^k\,\langle F, \mathcal{F}[t^k\,\varphi]\rangle \\ &= i^k\,\langle \mathcal{F}[F], t^k\,\varphi\rangle \\ &= \langle (i\,t)^k\,\mathcal{F}[F], \varphi\rangle,\end{aligned}$$

which proves the formula. Observe that the function $t \mapsto (i\,t)^k$ belongs to $\mathcal{O}_M$ (recall Remark 6.3.7), thus the multiplication is well-defined.

The proofs of (6.5.4), (6.5.5) and (6.5.6) are achieved in a similar way, by appealing to the relevant formulas for the Fourier transform of functions.

As for formula (6.5.7), we observe that by Remark 6.3.14 we have $\mathcal{F}[\psi] \in \mathcal{O}_M$ for every $\psi \in \mathcal{O}_C$. Thus the product $\mathcal{F}[\psi]\,\mathcal{F}[F]$ is well-defined in $\mathcal{S}'$ and the formula does make sense. In order to prove it, by first using the definition of distributional Fourier transform and then the definition of convolution, for every $\varphi \in \mathcal{S}$ we get

$$\langle \mathcal{F}[\psi * F], \varphi\rangle = \langle \psi * F, \mathcal{F}[\varphi]\rangle = \langle F, (\psi \circ \mathcal{A}_{-1,0}) * \mathcal{F}[\varphi]\rangle. \tag{6.5.9}$$

We now observe that

$$\begin{aligned}(\psi \circ \mathcal{A}_{-1,0}) * \mathcal{F}[\varphi](t) &= \int_{\mathbb{R}} \psi(y-t)\,\mathcal{F}[\varphi](y)\,dy\\ &= \int_{\mathbb{R}} \varphi(y)\,\mathcal{F}[\psi \circ \mathcal{A}_{1,-t}][y]\,dy,\end{aligned}$$

thanks to Lemma 6.5.1. We can use that (see Proposition 5.3.5)

$$\mathcal{F}[\psi \circ \mathcal{A}_{1,-t}](y) = e^{-i\,y\,t}\mathcal{F}[\psi](y),$$

thus in conclusion

$$\begin{aligned}(\psi \circ \mathcal{A}_{-1,0}) * \mathcal{F}[\varphi](t) &= \int_{\mathbb{R}} \psi(y-t)\,\mathcal{F}[\varphi](y)\,dy\\ &= \int_{\mathbb{R}} \varphi(y)\,\mathcal{F}[\psi \circ \mathcal{A}_{1,-t}][y]\,dy\\ &= \int_{\mathbb{R}} e^{-i\,t\,y}\,\varphi(y)\,\mathcal{F}[\psi](y)\,dy = \mathcal{F}\Big[\varphi\,\mathcal{F}[\psi]\Big](t).\end{aligned}$$

By using this in (6.5.9), we obtain

$$\langle \mathcal{F}[\psi * F], \varphi\rangle = \Big\langle F, \mathcal{F}\Big[\varphi\,\mathcal{F}[\psi]\Big]\Big\rangle.$$

If we now use the definition of distributional Fourier transform and the definition of multiplication in $\mathcal{S}'$ by the function $\mathcal{F}[\psi]$, we get the conclusion. □

Definition 6.5.8. Let $P : \mathcal{S} \to \mathcal{S}$ be a linear differential operator with constant coefficients, i.e.

$$P(u) = \sum_{k=0}^{m} a_k\,\frac{d^k u}{dt^k}, \qquad \text{for } u \in \mathcal{S},$$

where $a_0, \dots, a_m \in \mathbb{C}$. We say that a tempered distribution $F \in \mathcal{S}'$ is a *fundamental solution* of the operator P if

$$P(F) = \delta_0, \qquad \text{in } \mathcal{S}',$$

i.e. if it holds

$$\sum_{k=0}^{m} (-1)^k\,a_k\,\langle F, \varphi^{(k)}\rangle = \varphi(0), \qquad \text{for every } \varphi \in \mathcal{S}.$$

Example 6.5.9. Let us consider the operator

$$P(u) = -u'' + u.$$

We look for a fundamental solution of this operator, i.e. we look for a solution in $\mathcal{S}'$ of the equation

$$-F'' + F = \delta_0.$$

By using the Fourier transform in $\mathcal{S}'$, the previous equation gives

$$-\mathcal{F}[F''] + \mathcal{F}[F] = \mathcal{F}[\delta_0].$$

By using (6.5.3), we get

$$\left(\omega^2+1\right)\mathcal{F}[F]=F_1,$$

that is

$$\mathcal{F}[F]=\frac{1}{1+\omega^2}\,F_1.$$

This shows that the Fourier transform of F is the regular tempered distribution generated by the function

$$\omega\mapsto\frac{1}{1+\omega^2}.$$

We then observe that this function is the Fourier transform of the L^1 function

$$t\mapsto\frac{1}{2}\,e^{-|t|},$$

thanks to Exercise 5.1.5. We can thus conclude that the regular tempered distribution generated by the last function is a fundamental solution of the operator P. In other words, we can informally write

$$-\frac{d^2}{dt^2}\left(\frac{1}{2}\,e^{-|t|}\right)+\frac{1}{2}\,e^{-|t|}=\delta_0,\qquad \text{in } \mathcal{S}'.$$

We refer to Exercise 6.9.2 below for a generalization of this example.

6. Periodic distributions

In this section, we still make use of the notation $\mathcal{A}_{\lambda,h}(t)=\lambda\,t+h$.

Definition 6.6.1. Let $F\in\mathcal{S}'$ and $\tau>0$, we say that F is $\tau-$periodic if

$$\langle F\circ\mathcal{A}_{1,\tau},\varphi\rangle=\langle F,\varphi\rangle,\qquad \text{for every } \varphi\in\mathcal{S}.$$

By recalling the definition of $\mathcal{A}_{1,\tau}(t)=t+\tau$ and formula (6.3.1) for the change of variabile, this is the same as

$$\langle F,\varphi\circ\mathcal{A}_{1,-\tau}\rangle=\langle F,\varphi\rangle,\qquad \text{for every } \varphi\in\mathcal{S}.$$

Informally, this property could be written as

$$F(t+\tau)=F(t).$$

Example 6.6.2 (The Dirac comb or sampling function)**.** An important example of periodic tempered distribution is the so-called *Dirac comb* (also called *sampling function*) with time step $\tau>0$

$$P_\tau=\sum_{k\in\mathbb{Z}}\delta_{k\,\tau}.$$

This is a particular case of the family of tempered distributions encountered in Example 6.2.8. By definition, it acts as

$$\langle P_\tau,\varphi\rangle=\sum_{k\in\mathbb{Z}}\varphi(\tau\,k),\qquad \text{for every } \varphi\in\mathcal{S}.$$

Theorem 6.6.3 (Poisson's summation formula)**.** *Let $f\in\mathcal{S}$, then we have*

$$\sum_{k\in\mathbb{Z}}f(k)=\sum_{n\in\mathbb{Z}}\mathcal{F}[f](2\,\pi\,n),\tag{6.6.1}$$

where both series are absolutely convergent.

Proof. We first observe that the absolute convergence of the two series follows from the fact that both f and $\mathcal{F}[f]$ belong to $\mathcal{S}$ (recall Theorem 5.6.6). Then, we have

$$|f(k)| = (1+k^2)\,\frac{|f(k)|}{1+k^2} \le \frac{[f]_{0,0}+[f]_{2,0}}{1+k^2}, \qquad \text{for every } k\in\mathbb{Z},$$

and the latter is the $k-$th term of a converging series. The same computations apply to $\mathcal{F}[f]$.

Let us now prove (6.6.1). We define the $1-$periodic repetition of f, i.e. we consider the function

$$f_1(t) = \sum_{k\in\mathbb{Z}} f(t+k), \qquad t\in\mathbb{R}.$$

We observe that this series of functions converges totally on closed and bounded intervals and f_1 is a C^∞ function (thanks to the fact that $f\in\mathcal{S}$). Thus, by Theorem C.2.3 we can write the Fourier expansion of f_1, which is given by

$$f_1(t) = \sum_{n\in\mathbb{Z}} c_n\, e^{2\pi i n t}. \tag{6.6.2}$$

The coefficients c_n are given by

$$\begin{aligned} c_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} f_1(t)\, e^{-2\pi i n t}\,dt &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{k\in\mathbb{Z}} f(t+k)\, e^{-2\pi i n t}\,dt \\ &= \sum_{k\in\mathbb{Z}} \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t+k)\, e^{-2\pi i n t}\,dt \\ &= \sum_{k\in\mathbb{Z}} \int_{-\frac{1}{2}+k}^{\frac{1}{2}+k} f(s)\, e^{-2\pi i n s}\,ds, \end{aligned}$$

where in the last equality we used the change of variable $t+k=s$ and the fact that

$$e^{2\pi i n (s-k)} = e^{2\pi i n s}.$$

We now observe that

$$\sum_{k\in\mathbb{Z}} \int_{-\frac{1}{2}+k}^{\frac{1}{2}+k} f(s)\, e^{-2\pi i n s}\,ds = \int_{\mathbb{R}} f(s)\, e^{2\pi i n s}\,ds = \mathcal{F}[f](2\pi n),$$

that is

$$c_n = \mathcal{F}[f](2\pi n), \qquad \text{for every } n\in\mathbb{Z}.$$

By using this in (6.6.2), we get

$$f_1(t) = \sum_{n\in\mathbb{Z}} \mathcal{F}[f](2\pi n)\, e^{2\pi i n t},$$

and by recalling the definition of f_1, this is the same as

$$\sum_{k\in\mathbb{Z}} f(t+k) = \sum_{n\in\mathbb{Z}} \mathcal{F}[f](2\pi n)\, e^{2\pi i n t}.$$

If we now use this identity with $t=0$, we get (6.6.1) as desired. □

Remark 6.6.4. The hypothesis $f\in\mathcal{S}$ of the previous Theorem can be considerably relaxed. For simplicity, we avoided to state the result in its most general form.

Example 6.6.5 (Fourier transform of a Dirac comb)**.** Let us consider again the Dirac comb P_τ of Example 6.6.2. We use Poisson's summation formula (6.6.1) to compute its Fourier transform. By using the definition of Fourier transform for a tempered distrbution, i.e. Definition 6.5.2, for every $\varphi \in \mathcal{S}$ we have

$$\begin{aligned}\langle \mathcal{F}[P_\tau], \varphi\rangle = \langle P_\tau, \mathcal{F}[\varphi]\rangle &= \sum_{k\in\mathbb{Z}} \mathcal{F}[\varphi](k\,\tau)\\ &= \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} e^{i\,t\,k\,\tau}\,\varphi(t)\,dt\\ &= \frac{2\,\pi}{\tau}\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} e^{2\,\pi\,i\,s\,k}\,\varphi\left(\frac{2\,\pi}{\tau}\,s\right)\,ds\\ &= \frac{2\,\pi}{\tau}\sum_{k\in\mathbb{Z}} \mathcal{F}\left[\varphi\circ\mathcal{A}_{\frac{2\,\pi}{\tau},0}\right](2\,\pi\,k),\end{aligned}$$

where we used the change of variable $t = (2\,\pi\,s)/\tau$. We can now use Poisson's summation formula for the function $\varphi \circ \mathcal{A}_{2\pi/\tau,0}$ to infer

$$\frac{2\,\pi}{\tau}\sum_{k\in\mathbb{Z}} \mathcal{F}\left[\varphi\circ\mathcal{A}_{\frac{2\,\pi}{\tau},0}\right](2\,\pi\,k) = \frac{2\,\pi}{\tau}\sum_{n\in\mathbb{Z}} \varphi\circ\mathcal{A}_{\frac{2\,\pi}{\tau},0}(n) = \frac{2\,\pi}{\tau}\sum_{n\in\mathbb{Z}} \varphi\left(\frac{2\,\pi\,n}{\tau}\right).$$

We now observe that the last series coincides with a suitable Dirac comb applied to φ, i.e.

$$\frac{2\,\pi}{\tau}\sum_{n\in\mathbb{Z}} \varphi\left(\frac{2\,\pi\,n}{\tau}\right) = \frac{2\,\pi}{\tau}\left\langle P_{\frac{2\,\pi}{\tau}}, \varphi\right\rangle.$$

In other words, for every time step $\tau > 0$, we showed that

$$\mathcal{F}[P_\tau] = \frac{2\,\pi}{\tau}\,P_{\frac{2\,\pi}{\tau}},$$

which shows that *the Fourier transform of a Dirac comb is still a Dirac comb.*

7. Hilbert transform

In this section we give a brief treatment of the so-called *Hilbert transform*. At a formal level, this is the operator defined by

$$\mathcal{H}[\varphi](s) = \int_{\mathbb{R}} \frac{\varphi(t)}{s-t}\,dt.$$

In other words, $\mathcal{H}[\varphi]$ is the convolution between φ and the function $t \mapsto 1/t$. However, since the latter is not even in $L^1_{\text{loc}}(\mathbb{R})$ as already observed, the correct definition of $\mathcal{H}[\varphi]$ needs some care.

We have seen in Example 6.2.7 that we can treat $1/t$ as a tempered distribution, i.e. we may consider the distribution P.V.$1/t$ in place of the function $1/t$. Then, for every $\varphi \in \mathcal{O}_C$, we can define the convolution in distributional sense, as in Definition 6.3.16. This leads to the

Definition 6.7.1. The *Hilbert transform* of a function $\varphi \in \mathcal{O}_C$ is defined by

$$\mathcal{H}[\varphi] = \varphi * \left(\text{P.V.}\frac{1}{t}\right),$$

i.e. for every $\psi \in \mathcal{S}$ we have

$$\langle \mathcal{H}[\varphi], \psi\rangle = \left\langle \text{P.V.}\frac{1}{t}, (\varphi\circ\mathcal{A}_{-1,0} * \psi)\right\rangle.$$

Observe that by definition $\mathcal{H}[\varphi] \in \mathcal{S}'$. In other words, the Hilbert transform of a function $\varphi \in \mathcal{O}_C$ is a tempered distribution.

Proposition 6.7.2 (Hilbert VS. Fourier)**.** *Let us define the* sign function

$$\operatorname{sign}(t) = \begin{cases} -1, & \text{if } t < 0, \\ 1, & \text{if } t > 0. \end{cases}$$

For every $\varphi \in \mathcal{O}_C$ we have

$$\mathcal{F}\Big[\mathcal{H}[\varphi]\Big] = -\pi\, i\, F_{\text{sign}}\, \mathcal{F}[\varphi],$$

where as usual F_{sign} denotes the regular tempered distribution generated by sign*.*

Proof. By using the definition of Hilbert transform and formula (6.5.7) for the Fourier transform of a convolution, we have

$$\mathcal{F}\Big[\mathcal{H}[\varphi]\Big] = \mathcal{F}[\varphi]\, \mathcal{F}\left[\text{P.V.}\frac{1}{t}\right].$$

Then the conclusion follows by using Exercise 6.8.5, which computes the last Fourier transform. □

Remark 6.7.3. The previous result can be informally rephrased as

$$\mathcal{F}\Big[\mathcal{H}[\varphi]\Big](\omega) = -\pi\, i \operatorname{sign}(\omega)\, \mathcal{F}[\varphi](\omega).$$

Example 6.7.4. Let us take the rectangular function and recall that

$$\mathcal{F}[\text{rect}](\omega) = \operatorname{sinc}\left(\frac{\omega}{2\,\pi}\right).$$

By observing that $\text{rect} \in \mathcal{O}_C$ (thanks to Example 6.3.13), from the previous result we get that

$$\mathcal{F}\Big[\mathcal{H}[\text{rect}]\Big] = -i\,\pi\, \operatorname{sinc}\left(\frac{\omega}{2\,\pi}\right) F_{\text{sign}}.$$

In other words, the distributional Fourier transform of $\mathcal{H}[\text{rect}]$ is the regular tempered distribution generated by the slowly growing function

$$\omega \mapsto \begin{cases} -i\,\pi\, \operatorname{sinc}\left(\dfrac{\omega}{2\,\pi}\right), & \text{if } \omega > 0, \\ \\ i\,\pi\, \operatorname{sinc}\left(\dfrac{\omega}{2\,\pi}\right), & \text{if } \omega < 0. \end{cases}$$

The Hilbert transform of rect can be computed explicitly, see Exercise 6.9.8 below.

8. Exercises

Exercise 6.8.1. *Prove the following formula*

$$t\,\delta_0' = -\delta_0, \qquad \text{in } \mathcal{S}'.$$

Solution. By using the definition of multiplication, we have

$$\langle t\,\delta_0', \varphi\rangle = \langle \delta_0', t\,\varphi\rangle, \qquad \text{for every } \varphi \in \mathcal{S}.$$

We now use the definition of distributional derivative, so to get

$$\langle t\,\delta_0', \varphi\rangle = -\langle \delta_0, (t\,\varphi)'\rangle, \qquad \text{for every } \varphi \in \mathcal{S}.$$

By observing that

$$(t\,\varphi(t))' = \varphi(t) + t\,\varphi'(t),$$

and using the definition of δ_0, we now get the conclusion. $\square$

Exercise 6.8.2 (Distributional Leibniz rule). *Let $\psi \in \mathcal{O}_M$ and $F \in \mathcal{S}'$. Prove the validity of the* Leibniz rule *for the distributional derivative of the product $\psi\, F$, i.e. show that*

$$(\psi\, F)' = \psi'\, F + \psi\, F', \qquad \text{in } \mathcal{S}'.$$

Solution. We take $\varphi \in \mathcal{S}$, we have to show that

$$\langle (\psi\, F)', \varphi\rangle = \langle \psi'\, F + \psi\, F', \varphi\rangle.$$

We first observe that if $\psi \in \mathcal{O}$, then by definition we have $\psi' \in \mathcal{O}_M$. Thus the previous formula makes sense. We start computing: by using the definition of distributional derivative and that of multiplication, we get

$$\begin{aligned} \langle (\psi\, F)', \varphi\rangle &= -\langle \psi\, F, \varphi\rangle = -\langle F, \psi\, \varphi'\rangle \\ &= -\langle F, (\psi\, \varphi)'\rangle + \langle F, \psi'\, \varphi\rangle. \end{aligned} \tag{6.8.1}$$

In the last identity we used the Leibniz rule for functions, so that

$$(\psi\, \varphi)' = \psi'\, \varphi + \psi\, \varphi'.$$

We now use again the definition distributional derivative and that of multiplication, so that

$$-\langle F, (\psi\, \varphi)'\rangle = \langle F', \psi\, \varphi\rangle = \langle \psi\, F, \varphi\rangle,$$

and

$$\langle F, \psi'\, \varphi\rangle = \langle \psi'\, F, \varphi\rangle.$$

By using the last two identities in (6.8.1), we end up with

$$\langle (\psi\, F)', \varphi\rangle = \langle \psi\, F, \varphi\rangle + \langle \psi'\, F, \varphi\rangle = \langle \psi\, F' + \psi'\, F\, \varphi\rangle,$$

as desired. $\square$

Exercise 6.8.3. *Prove that if $f(t) = \operatorname{sinc}(t)$, then F_f is a tempered distribution. Show that its Fourier transform is the regular tempered distribution generated by*

$$h(\omega) = 1_{[-\pi,\pi]}(\omega).$$

Solution. We know from Example 3.3.15 that $\operatorname{sinc} \in L^p(\mathbb{R})$ for every $1 < p \le \infty$, thus it is a slowly growing function by Proposition 6.2.6. This gives that F_f is a tempered distribution. We can use two different methods to compute its Fourier transform.

First method. We recall that

$$\mathcal{F}[\mathrm{rect}](\omega) = \operatorname{sinc}\left(\frac{\omega}{2\,\pi}\right),$$

thus we get

$$\operatorname{sinc}\omega = \mathcal{F}[\mathrm{rect}](2\,\pi\,\omega) = \frac{1}{2\,\pi}\,\mathcal{F}\Big[1_{[-\pi,\pi]}\Big](\omega).$$

By using this identity and Lemma 6.5.1, for every $\varphi \in \mathcal{S}$ we get[4]

$$\begin{aligned}\langle \mathcal{F}[F_f], \varphi\rangle &= \int_{\mathbb{R}} \operatorname{sinc}(\omega)\, \mathcal{F}[\varphi](\omega)\, d\omega = \frac{1}{2\,\pi} \int_{\mathbb{R}} \mathcal{F}\Big[1_{[-\pi,\pi]}\Big](\omega)\, \mathcal{F}[\varphi](\omega)\, d\omega \\ &= \frac{1}{2\,\pi} \int_{\mathbb{R}} 1_{[-\pi,\pi]}(\omega)\, \mathcal{F}\Big[\mathcal{F}[\varphi]\Big](\omega)\, d\omega \\ &= \int_{\mathbb{R}} 1_{[-\pi,\pi]}(\omega)\, \varphi(-\omega)\, d\omega,\end{aligned}$$

thanks to the duality formula (5.4.6) applied to $\varphi \in \mathcal{S}$. By using the simple change of variable $\omega \mapsto -\omega$ and the fact that $1_{[-\pi,\pi]}$ is an even function, we get the conclusion.

Second method. We have seen in Exercise 5.9.5, that

$$\mathcal{F}_{L^2}[\operatorname{sinc}] = 1_{[-\pi,\pi]}.$$

By appealing to Proposition 6.5.4, we directly get the conclusion. □

Exercise 6.8.4. *Prove that the Fourier transform of the regular tempered distribution generated by the Heaviside step function is given by*

$$\mathcal{F}[F_H] = \frac{1}{i}\, \text{P.V.}\, \frac{1}{\omega} + \pi\, \delta_0.$$

Solution. We want to compute this Fourier transform by restricting the Laplace transform of H to the imaginary axis. However, since the imaginary axis is the critical axis for such a Laplace transform, much care is needed in this operation. We first recall that (see Example 4.1.5)

$$\mathcal{L}[H](z) = \frac{1}{z}, \qquad \text{for } \operatorname{Re}(z) > 0.$$

We now use the definition of Fourier transform for a tempered distribution and get

$$\langle \mathcal{F}[F_H], \varphi\rangle = \langle F_H, \mathcal{F}[\varphi]\rangle = \int_0^{+\infty} \mathcal{F}[\varphi](\omega)\, d\omega.$$

Then we observe that

$$\int_0^{+\infty} \mathcal{F}[\varphi](\omega)\, d\omega = \lim_{\alpha\to 0^+} \int_0^{+\infty} e^{-\alpha\,\omega}\, \mathcal{F}[\varphi](\omega)\, d\omega,$$

thanks to the Dominated Convergence Theorem. Let us take $\alpha > 0$ and consider the last integral: we have

$$\begin{aligned}\int_0^{+\infty} e^{-\alpha\,\omega}\, \mathcal{F}[\varphi](\omega)\, d\omega &= \int_0^{+\infty} e^{-\alpha\,\omega} \left(\int_{\mathbb{R}} e^{-i\,t\,\omega}\, \varphi(t)\, dt\right) d\omega \\ &= \int_{\mathbb{R}} \left(\int_0^{+\infty} e^{-(\alpha+i\,t)\,\omega}\, d\omega\right) \varphi(t)\, dt,\end{aligned}$$

where we used Fubini's and Tonelli's Theorems in order to exchange the order of integration. We can now recognize that the integral in ω is a Laplace transform: we have

$$\int_0^{+\infty} e^{-(\alpha+i\,t)\,\omega}\, d\omega = \mathcal{L}[H](\alpha + i\,t) = \frac{1}{\alpha + i\,t}, \qquad \text{for } \alpha > 0,\ t \in \mathbb{R}.$$

[4] Observe that we could also use Parseval's formula in the third passage and get directly the conclusion.

By keeping everything together, up to now we obtained

$$\langle \mathcal{F}[F_H], \varphi \rangle = \lim_{\alpha \to 0^+} \int_{\mathbb{R}} \frac{\varphi(t)}{\alpha + i\,t}\, dt.$$

By recalling that $1/i = -i$, we get

$$\int_{\mathbb{R}} \frac{\varphi(t)}{\alpha + i\,t}\, dt = -i \int_{\mathbb{R}} \frac{\varphi(t)}{t - i\,\alpha}\, dt,$$

and thus

$$\langle \mathcal{F}[F_H], \varphi \rangle = -i \lim_{\alpha \to 0^+} \int_{\mathbb{R}} \frac{\varphi(t)}{t - i\,\alpha}\, dt.$$

We can now conclude by appealing to the *Sochocki-Plemelj Formula*, i.e. Theorem 6.3.22. □

Exercise 6.8.5. *Let us consider the piecewise constant function*

$$\mathrm{sign}(t) = \begin{cases} -1, & \text{if } t < 0, \\ \;\;1 & \text{if } t > 0. \end{cases}$$

Show that the Fourier transform of the principal value of $1/t$ *is given by*

$$\mathcal{F}\left[\mathrm{P.V.}\frac{1}{t}\right] = -\pi\, i\, F_{\mathrm{sign}}.$$

Solution. We already know from Exercise 6.8.4 that

$$\mathcal{F}[F_H] = \frac{1}{i}\,\mathrm{P.V.}\,\frac{1}{\omega} + \pi\,\delta_0,$$

that is

$$\mathrm{P.V.}\,\frac{1}{\omega} = i\,\mathcal{F}[F_H] - i\,\pi\,\delta_0.$$

We take the distributional Fourier transform on both sides, so to get

$$\mathcal{F}\left[\mathrm{P.V.}\,\frac{1}{\omega}\right] = i\,\mathcal{F}\Big[\mathcal{F}[F_H]\Big] - i\,\pi\,F_1,$$

where we used that the Fourier transform of δ_0 is the regular tempered distribution generated by the constant function 1 (recall Example 6.5.5). By using the duality formula (6.5.6) in $\mathcal{S}'$, we get

$$\mathcal{F}\left[\mathrm{P.V.}\,\frac{1}{\omega}\right] = 2\,\pi\,i\,F_H \circ \mathcal{A}_{-1,0} - i\,\pi\,F_1 = i\,\pi\,\Big(2\,F_H \circ \mathcal{A}_{-1,0} - F_1\Big).$$

By recalling (6.3.2) and Remark 6.3.1, we get

$$2\,F_H \circ \mathcal{A}_{-1,0} - F_1 = 2\,F_{H \circ \mathcal{A}_{-1,0}} - F_1 = F_{2\,H \circ \mathcal{A}_{-1,0} - 1}.$$

Finally, by observing that

$$2\,H \circ \mathcal{A}_{-1,0}(t) - 1 = -\mathrm{sign}(t),$$

we get the conclusion. □

Exercise 6.8.6. *Show that the Fourier transform of* F_{sign} *is given by*

$$\mathcal{F}[F_{\mathrm{sign}}] = -2\,i\,\mathrm{P.V.}\frac{1}{\omega}.$$

Solution. Here we just need to use the previous exercise and the duality formula in $\mathcal{S}'$, i.e. formula (6.5.6). We leave the details to the reader. □

Exercise 6.8.7. *Let $\omega_0 \in \mathbb{R}$ and let us consider the regular tempered distribution $F_{e^{i\,t\,\omega_0}}$ generated by the bounded function $t \mapsto e^{i\,t\,\omega_0}$. Show that*

$$\mathcal{F}[F_{e^{i\,t\,\omega_0}}] = 2\,\pi\,\delta_{\omega_0}.$$

Solution. By using the definition of distributional Fourier transform, for every $\varphi \in \mathcal{S}$ we have

$$\langle \mathcal{F}[F_{e^{i\,t\,\omega_0}}], \varphi\rangle = \langle F_{e^{i\,t\,\omega_0}}, \mathcal{F}[\varphi]\rangle = \int_{\mathbb{R}} e^{i\,t\,\omega_0}\,\mathcal{F}[\varphi](t)\,dt.$$

By using the inversion formula, we know that

$$\int_{\mathbb{R}} e^{i\,t\,\omega_0}\,\mathcal{F}[\varphi](\omega)\,d\omega = 2\,\pi\,\varphi(\omega_0).$$

This implies that

$$\mathcal{F}[F_{e^{i\,t\,\omega_0}}] = 2\,\pi\,\delta_{\omega_0}.$$

as desired. $\square$

Exercise 6.8.8. *Let us consider the regular tempered distributions $F_{\cos}$ and $F_{\sin}$ generated by the bounded functions $t \mapsto \cos t$ and $t \mapsto \sin t$, respectively. Show that*

$$\mathcal{F}[F_{\cos}] = \pi\left(\delta_1 + \delta_{-1}\right),$$

and

$$\mathcal{F}[F_{\sin}] = \pi\,i\left(\delta_{-1} - \delta_1\right).$$

Solution. We recall that

$$\cos t = \frac{e^{i\,t} + e^{-i\,t}}{2} \qquad \text{and} \qquad \sin t = \frac{e^{i\,t} - e^{-i\,t}}{2\,i},$$

that is

$$F_{\cos} = \frac{1}{2}\,F_{e^{i\,t}} + \frac{1}{2}\,F_{e^{-i\,t}} \qquad \text{and} \qquad F_{\sin} = \frac{1}{2\,i}\,F_{e^{i\,t}} - \frac{1}{2\,i}\,F_{e^{-i\,t}}.$$

By using the linearity of the Fourier transform and the previous exercise, we thus get

$$\mathcal{F}[F_{\cos}] = \pi\,\delta_1 + \pi\,\delta_{-1},$$

and

$$\mathcal{F}[F_{\cos}] = \frac{\pi}{i}\,\delta_1 - \frac{\pi}{i}\,\delta_{-1}.$$

as desired (recall that $1/i = -i$). $\square$

Exercise 6.8.9. *For every $\alpha > 0$, we define the function*

$$f_\alpha(t) = \frac{1}{\sqrt{\alpha}}\,e^{-\frac{t^2}{\alpha}}, \qquad t \in \mathbb{R}.$$

Prove that

$$F_{f_\alpha} \xrightarrow{\mathcal{S}'} \sqrt{\pi}\,\delta_0.$$

Solution. We define the function

$$f(t) = e^{-t^2}, \qquad \text{for } t \in \mathbb{R},$$

and observe that this is in $L^1(\mathbb{R})$ (indeed, it is a function belonging to the Schwartz class $\mathcal{S}$). We also recall that

$$\int_{\mathbb{R}} f(t)\,dt = \sqrt{\pi}.$$

If we now observe that

$$f_\alpha(t) = \frac{1}{\alpha}\, f\left(\frac{t}{\sqrt{\alpha}}\right),$$

the conclusion is readily obtained by applying Proposition 6.3.21, with $\varepsilon = \sqrt{\alpha}$. $\square$

Exercise 6.8.10. *For every $\alpha > 0$, we define the function*

$$f_\alpha(t) = \frac{\alpha}{t^2 + \alpha^2}, \qquad t \in \mathbb{R}.$$

Prove that

$$F_{f_\alpha} \xrightarrow{\mathcal{S}'} \pi\, \delta_0.$$

Solution. This is very similar to the previous exercise. If we introduce the $L^1(\mathbb{R})$ function

$$f(t) = \frac{1}{1+t^2}, \qquad t \in \mathbb{R},$$

it is not difficult to see that

$$f_\alpha(t) = \frac{1}{\alpha}\, f\left(\frac{t}{\alpha}\right).$$

By observing that

$$\int_{\mathbb{R}} f(t)\, dt = \int_{\mathbb{R}} \frac{1}{1+t^2}\, dt = \Big[\arctan t\Big]_{-\infty}^{+\infty} = \pi,$$

we get the desired conclusion again by Proposition 6.3.21. $\square$

Remark 6.8.11. We refer to Exercise D.5.3 for an interesting application of the previous exercise.

9. Advanced exercises

Exercise 6.9.1. *Prove that the distributional derivative of the regular tempered distribution generated by the function $t \mapsto \log|t|$ is given by the principal value of $1/t$, i.e.*

$$F'_{\log|t|} = \mathrm{P.V.}\frac{1}{t}.$$

Solution. We first observe that $t \mapsto \log|t|$ is a slowly growing function. Indeed, we have

$$\begin{aligned}
\int_{\mathbb{R}} \frac{|\log|t||}{1+|t|^3}\, dt = 2\int_0^{+\infty} \frac{|\log t|}{1+t^3}\, dt &= 2\int_0^1 \frac{-\log t}{1+t^3}\, dt + 2\int_1^{+\infty} \frac{\log t}{1+t^3}\, dt \\
&\le 2\int_0^1 (-\log t)\, dt + 2\int_1^{+\infty} \frac{\log t}{t^3}\, dt \\
&\le 2\int_0^1 (-\log t)\, dt + 2\int_1^{+\infty} \frac{1}{t^2}\, dt,
\end{aligned}$$

where in the last integral we used that

$$\log t \le t, \qquad \text{for } t > 0.$$

By computing the last integrals, we get

$$\int_{\mathbb{R}} \frac{|\log|t||}{1+|t|^3}\, dt < +\infty.$$

Thus we know that $F_{\log|t|} \in \mathcal{S}'$, thanks to the discussion of Example 6.2.5.

In order to compute its distributional derivative, we take $\varphi \in \mathcal{S}$. Then, by using the definitions of distributional derivative and of regular tempered distribution, we get

$$\begin{aligned}
\langle F'_{\log|t|}, \varphi\rangle = -\langle F_{\log|t|}, \varphi'\rangle &= -\int_{\mathbb{R}} \log|t|\,\varphi'(t)\,dt \\
&= -\lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{+\infty} \log t\,\varphi'(t)\,dt - \lim_{\varepsilon\to 0}\int_{-\infty}^{-\varepsilon}\log(-t)\,\varphi'(t)\,dt \\
&= -\lim_{\varepsilon\to 0^+}\left(-\log\varepsilon\,\varphi(\varepsilon) - \int_{\varepsilon}^{+\infty}\frac{\varphi(t)}{t}\,dt\right) \\
&- \lim_{\varepsilon\to 0^+}\left[\log(\varepsilon)\,\varphi(-\varepsilon) - \int_{-\infty}^{-\varepsilon}\frac{\varphi(t)}{t}\,dt\right] \\
&= \lim_{\varepsilon\to 0^+}\log\varepsilon\,\Big(\varphi(\varepsilon)-\varphi(-\varepsilon)\Big) \\
&+ \lim_{\varepsilon\to 0^+}\int_{|t|>\varepsilon}\frac{\varphi(t)}{t}\,dt.
\end{aligned}$$

By using a first order Taylor expansion, we have

$$\begin{aligned}
\varphi(\varepsilon)-\varphi(-\varepsilon) &= \Big(\varphi(0)+\varphi'(0)\,\varepsilon+o(\varepsilon)\Big) - \Big(\varphi(0)-\varphi'(0)\,\varepsilon+o(\varepsilon)\Big) \\
&= 2\,\varphi'(0)\,\varepsilon + o(\varepsilon), \qquad \text{for } \varepsilon\to 0^+,
\end{aligned}$$

thus we get

$$\lim_{\varepsilon\to 0^+}\log\varepsilon\,\Big(\varphi(\varepsilon)-\varphi(-\varepsilon)\Big) = 2\,\varphi'(0)\,\lim_{\varepsilon\to 0^+}\varepsilon\,\log\varepsilon = 0.$$

In conclusion, we obtained

$$\langle F'_{\log|t|}, \varphi\rangle = \lim_{\varepsilon\to 0^+}\int_{|t|>\varepsilon}\frac{\varphi(t)}{t}\,dt, \qquad \text{for every } \varphi\in\mathcal{S}.$$

By recalling the definition (6.2.1), we conclude the exercise. □

Exercise 6.9.2. *We consider the second order linear differential operator*

$$P(u) = -a\,\frac{d^2u}{dt^2} - i\,b\,\frac{du}{dt} + c\,u, \qquad \textit{for } u\in\mathcal{S}.$$

Let us assume that $a,b,c\in\mathbb{R}$, *with* $a>0$ *and that* $b^2-4\,a\,c<0$. *Find a fundamental solution* $F\in\mathcal{S}'$ *of the operator* P.

Solution. We need to find a tempered distribution $F\in\mathcal{S}'$ such that

$$P(F)=\delta_0, \qquad \text{in } \mathcal{S}'.$$

We take the distributional Fourier transform, i.e.

$$\mathcal{F}[P(F)] = \mathcal{F}[\delta_0] = F_1.$$

By using (6.5.3), we obtain

$$\begin{aligned}
\mathcal{F}[P(F)] &= -a\,\mathcal{F}\left[\frac{d^2}{dt^2}F\right] - i\,b\,\mathcal{F}\left[\frac{d}{dt}F\right] + c\,\mathcal{F}[F] \\
&= \Big[-a\,(i\,\omega)^2 - i\,b\,(i\,\omega) + c\Big]\mathcal{F}[F] \\
&= (a\,\omega^2 + b\,\omega + c)\,\mathcal{F}[F].
\end{aligned}$$

Thus a seeked fundamental solution $F \in \mathcal{S}$ is such that

$$\mathcal{F}[F] = \frac{1}{a\,\omega^2 + b\,\omega + c}\,F_1,$$

i.e. in other words $\mathcal{F}[F]$ is the regular tempered distribution generated by the slowly growing function

$$g_{a,b,c}(\omega) = \frac{1}{a\,\omega^2 + b\,\omega + c}.$$

We now recall from Exercise 5.9.1 that

$$\mathcal{F}[g_{a,b,c}](\omega) = \frac{2\,\pi}{\sqrt{4\,a\,c - b^2}}\,e^{i\,\frac{b}{2\,a}\,\omega}\,e^{-\frac{\sqrt{4\,a\,c-b^2}}{2\,a}\,|\omega|} = h_{a,b,c}(\omega),$$

which implies from the duality formula (5.4.6) that

$$\mathcal{F}[h_{a,b,c}](\omega) = 2\,\pi\,g_{a,b,c}(-\omega) = 2\,\pi\,g_{a,-b,c}(\omega).$$

By using this formula and exchanging b with $-b$, we get

$$\mathcal{F}\left[\frac{1}{2\,\pi}\,h_{a,-b,c}\right](\omega) = g_{a,b,c}(\omega).$$

This in turn implies that

$$\mathcal{F}[F] = \mathcal{F}\left[\frac{1}{2\,\pi}\,F_{h_{a,-b,c}}\right],$$

and thus as a fundamental solution we can take the regular tempered distribution

$$F = \frac{1}{2\,\pi}\,F_{h_{a,-b,c}}, \qquad \text{with } h_{a,-b,c}(t) = \frac{2\,\pi}{\sqrt{4\,a\,c - b^2}}\,e^{-i\,\frac{b}{2\,a}\,t}\,e^{-\frac{\sqrt{4\,a\,c-b^2}}{2\,a}\,|t|}.$$

This concludes the exercise. □

Exercise 6.9.3. *We consider the second order linear differential operator*

$$P(u) = \frac{d^2 u}{dt^2} + \lambda\,u, \qquad \textit{for } u \in \mathcal{S},$$

where $\lambda > 0$. Find a fundamental solution $F \in \mathcal{S}'$ of the operator P.

Solution. As before, we need to find a tempered distribution $F \in \mathcal{S}'$ such that

$$P(F) = \delta_0, \qquad \text{in } \mathcal{S}'.$$

We take the distributional Fourier transform, i.e.

$$\mathcal{F}[P(F)] = \mathcal{F}[\delta_0] = F_1.$$

By using (6.5.3), we obtain

$$\begin{aligned}\mathcal{F}[P(F)] &= \mathcal{F}\left[\frac{d^2}{dt^2}F\right] + \lambda\,\mathcal{F}[F]\\ &= \Big[(i\,\omega)^2 + \lambda\Big]\mathcal{F}[F] = (\lambda - \omega^2)\,\mathcal{F}[F].\end{aligned}$$

From the equation, we formally get

$$\mathcal{F}[F] = \frac{1}{\lambda - \omega^2}\,F_1 = \frac{1}{2\,\sqrt{\lambda}}\left[\frac{1}{\sqrt{\lambda} + \omega} - \frac{1}{\omega - \sqrt{\lambda}}\right]F_1.$$

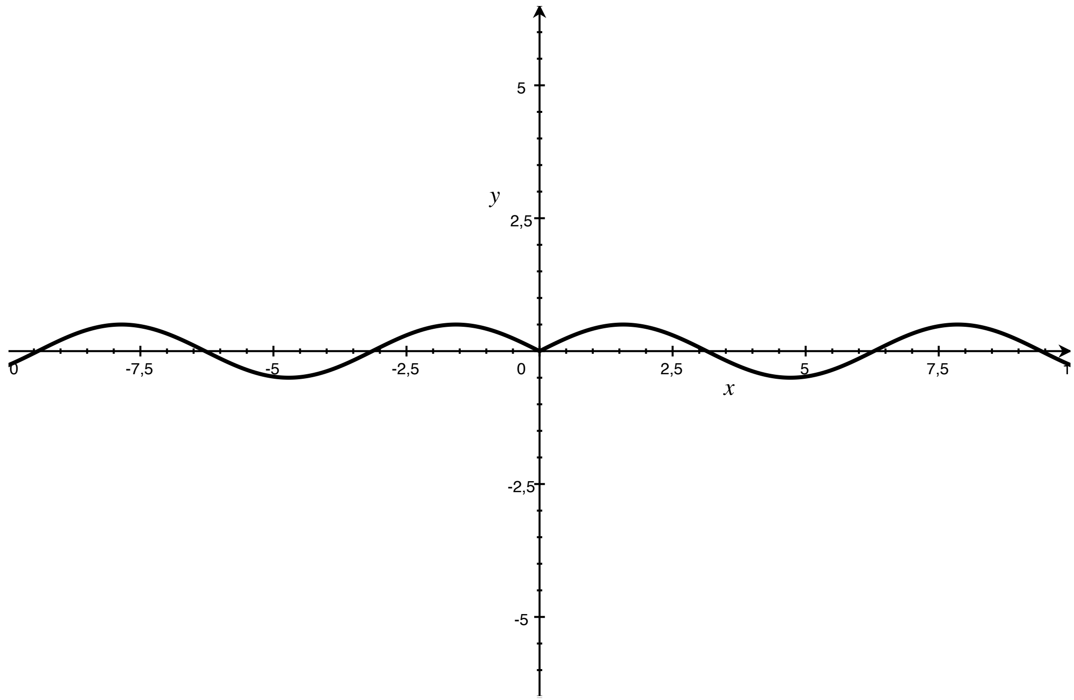

Figure 1. The function f generates a fundamental solution of $P(u) = \frac{d^2u}{dt^2} + \lambda\, u$.

Observe that this computation only holds at a formal level, since the function $1/(\lambda - \omega^2)$ is not a multiplier of the class $\mathcal{S}$. We can interpret the previous formula as

$$\mathcal{F}[F] = -\frac{1}{2\sqrt{\lambda}}\,\mathrm{P.V.}\frac{1}{\omega - \sqrt{\lambda}} + \frac{1}{2\sqrt{\lambda}}\,\mathrm{P.V.}\frac{1}{\omega + \sqrt{\lambda}}.$$

By recalling that

$$\mathcal{F}[F_{\mathrm{sign}}] = -2\,i\,\mathrm{P.V.}\frac{1}{\omega},$$

from formula (6.5.5) we get

$$\mathcal{F}[e^{i\sqrt{\lambda}\,t}F_{\mathrm{sign}}] = -2\,i\,\mathrm{P.V.}\frac{1}{\omega - \sqrt{\lambda}} \qquad \text{and} \qquad \mathcal{F}[e^{-i\sqrt{\lambda}\,t}F_{\mathrm{sign}}] = -2\,i\,\mathrm{P.V.}\frac{1}{\omega + \sqrt{\lambda}}.$$

Thus, if we define (see Figure 1)

$$f(t) = -\frac{i}{4\sqrt{\lambda}}\,\mathrm{sign}(t)\left[e^{i\sqrt{\lambda}\,t} - e^{-i\sqrt{\lambda}\,t}\right] = \frac{\mathrm{sign}(t)}{2\sqrt{\lambda}}\,\sin(\sqrt{\lambda}\,t),$$

we get that

$$\mathcal{F}[F_f] = -\frac{1}{2\sqrt{\lambda}}\,\mathrm{P.V.}\frac{1}{\omega - \sqrt{\lambda}} + \frac{1}{2\sqrt{\lambda}}\,\mathrm{P.V.}\frac{1}{\omega + \sqrt{\lambda}},$$

as desired. Thus the regular tempered distribution generated by the function f is a fundamental solution. □

Exercise 6.9.4. *For every* $n \in \mathbb{N} \setminus \{0\}$, *we set*

$$\mathcal{H}_n(t) = \frac{n}{\pi}\,\mathrm{sinc}\left(\frac{n}{\pi}\,t\right).$$

Prove that if we consider the sequence of regular tempered distributions $\{F_{\mathcal{H}_n}\}_{n\geq 1}\subset \mathcal{S}'$, we have

$$F_{\mathcal{H}_n} \xrightarrow{\mathcal{S}'} \delta_0.$$

Solution. We first observe that $\mathcal{H}_n \in L^\infty(\mathbb{R})$, thus by Proposition 6.2.6 we have that $F_{\mathcal{H}_n}$ is a tempered distribution. We point out that in this case we can not directly apply Proposition 6.3.21, since the function

$$t \mapsto \frac{1}{\pi}\,\mathrm{sinc}\left(\frac{1}{\pi}\,t\right),$$

is not in $L^1(\mathbb{R})$. We take $\varphi \in \mathcal{S}$, we need to show that

$$\lim_{n\to\infty} \langle F_{\mathcal{H}_n}, \varphi\rangle = \varphi(0).$$

By using the definition of $F_{\mathcal{H}_n}$ and the fact that $\mathcal{H}_n$ is even, we have

$$\lim_{n\to\infty} \langle F_{\mathcal{H}_n}, \varphi\rangle = \lim_{n\to\infty}\int_{\mathbb{R}} \mathcal{H}_n(t)\,\varphi(t)\,dt = \lim_{n\to\infty}\int_{\mathbb{R}} \mathcal{H}_n(0-t)\,\varphi(t)\,dt = \lim_{n\to\infty} \varphi * \mathcal{H}_n(0).$$

If we now recall Remark 5.4.3, we get the desired conclusion. □

Exercise 6.9.5. *By using the Poisson's summation formula, compute the sum*

$$\sum_{n\in\mathbb{Z}} \frac{1}{1+n^2}.$$

Solution. We consider the L^1 function

$$f(t) = \frac{1}{1+t^2}.$$

By Exercise 5.8.3, we already know that

$$\mathcal{F}[f](\omega) = \pi\, e^{-|\omega|}.$$

By using Theorem 6.6.3, we then get

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \frac{1}{1+n^2} = \sum_{n\in\mathbb{Z}} f(n) = \sum_{k\in\mathbb{Z}} \mathcal{F}[f](2\,\pi\,k) &= \pi \sum_{k\in\mathbb{Z}} e^{-2\,\pi\,|k|}\\ &= \pi + \pi\sum_{k=1}^{\infty} e^{-2\,\pi\,k} + \pi \sum_{-\infty}^{k=-1} e^{2\,\pi\,k}\\ &= \pi + 2\,\pi \sum_{k=1}^{\infty} e^{-2\,\pi\,k}\\ &= \pi + 2\,\pi\left(\sum_{k=0}^{\infty}\left(e^{-2\,\pi}\right)^k - 1\right)\\ &= \pi + 2\,\pi\left(\frac{1}{1-e^{-2\,\pi}} - 1\right).\end{aligned}$$

In conclusion, we get[5]

$$\sum_{n\in\mathbb{Z}} \frac{1}{1+n^2} = \pi + 2\,\pi\,\frac{e^{-2\,\pi}}{1-e^{-2\,\pi}} = \pi\,\frac{e^{2\,\pi}+1}{e^{2\,\pi}-1} = \frac{\pi}{\tanh \pi}.$$

[5] We recall that

$$\tanh t = \frac{\sinh t}{\cosh t} = \frac{e^t - e^{-t}}{e^t + e^{-t}} = \frac{e^{2\,t}-1}{e^{2\,t}+1}, \qquad t \in \mathbb{R},$$

is the *hyperbolic tangent*.

This concludes the exercise. □

Exercise 6.9.6. *Generalize the previous exercise, by computing the sum*

$$\sum_{n\in\mathbb{Z}} \frac{1}{a^2+n^2},$$

where $a > 0$ is given.

Solution. We can proceed as above, by taking the L^1 function

$$f(t) = \frac{1}{a^2+t^2} = \frac{1}{a^2}\,\frac{1}{1+\left(\dfrac{t}{a}\right)^2}.$$

By using Proposition 5.3.2, we then get

$$\mathcal{F}[f](\omega) = \frac{1}{a^2}\,a\,\pi\,e^{-a\,|\omega|} = \frac{\pi}{a}\,e^{-a\,|\omega|}.$$

We now proceed as in the previous exercise. By using Theorem 6.6.3, we then get

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \frac{1}{a^2+n^2} = \sum_{n\in\mathbb{Z}} f(n) = \sum_{k\in\mathbb{Z}} \mathcal{F}[f](2\,\pi\,k) &= \frac{\pi}{a}\sum_{k\in\mathbb{Z}} e^{-2\,\pi\,a\,|k|}\\
&= \frac{\pi}{a} + \frac{\pi}{a}\sum_{k=1}^{\infty} e^{-2\,\pi\,a\,k} + \frac{\pi}{a}\sum_{-\infty}^{k=-1} e^{2\,\pi\,a\,k}\\
&= \frac{\pi}{a} + 2\,\frac{\pi}{a}\sum_{k=1}^{\infty} e^{-2\,\pi\,a\,k}\\
&= \frac{\pi}{a} + 2\,\frac{\pi}{a}\left(\sum_{k=0}^{\infty}\left(e^{-2\,\pi\,a}\right)^k - 1\right)\\
&= \frac{\pi}{a} + 2\,\frac{\pi}{a}\left(\frac{1}{1-e^{-2\,\pi\,a}} - 1\right).
\end{aligned}$$

In conclusion, we get

$$\sum_{n\in\mathbb{Z}} \frac{1}{a^2+n^2} = \frac{\pi}{a} + 2\,\frac{\pi}{a}\,\frac{e^{-2\,\pi\,a}}{1-e^{-2\,\pi\,a}} = \frac{\pi}{a}\,\frac{e^{2\,\pi\,a}+1}{e^{2\,\pi\,a}-1} = \frac{1}{a^2}\,\frac{\pi\,a}{\tanh(\pi\,a)}.$$

This concludes the exercise. □

Remark 6.9.7. We can use the previous exercise to compute the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Indeed, observe that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^2} &= \lim_{a\to 0^+} \frac{1}{2}\left[\sum_{n\in\mathbb{Z}} \frac{1}{a^2+n^2} - \frac{1}{a^2}\right]\\
&= \frac{1}{2}\lim_{a\to 0^+} \frac{1}{a^2}\left[\frac{\pi\,a}{\tanh(\pi\,a)} - 1\right]\\
&= \frac{1}{2}\lim_{a\to 0^+} \frac{\pi\,a - \tanh(\pi\,a)}{a^2\,\tanh(\pi\,a)}.
\end{aligned}$$

We now use the third order Taylor expansion

$$\tanh(t) = t - \frac{t^3}{3} + o(t^3), \qquad \text{for } t \to 0,$$

which gives

$$\frac{1}{2} \lim_{a \to 0^+} \frac{\pi\, a - \tanh(\pi\, a)}{a^2 \tanh(\pi\, a)} = \frac{1}{2} \lim_{a \to 0^+} \frac{\frac{1}{3}\pi^3 a^3 + o(a^3)}{\pi\, a^3 + o(a^3)} = \frac{\pi^2}{6}.$$

In conclusion, we obtained

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Exercise 6.9.8. *Show that the Hilbert transform of* rect *is the regular tempered distribution generated by the function*

$$t \mapsto \log \left| \frac{t + \frac{1}{2}}{t - \frac{1}{2}} \right|.$$

Appendix A

Limit superior and limit inferior

1. Suprema and infima

Let $E \subset \mathbb{R}$ be a non-empty set. We say that $m \in \mathbb{R}$ is a *lower bound* for E if

$$m \leq x, \qquad \text{for every } x \in E.$$

We say that $M \in \mathbb{R}$ is an *upper bound* for E if

$$x \leq M, \qquad \text{for every } x \in E.$$

Definition A.1.1. Let $E \subset \mathbb{R}$ be a non-empty set. We define its *supremum* as the smallest upper bound for E. We indicate by

$$\sup E,$$

this number, with the convention that $\sup E = +\infty$ if the class of upper bounds is empty.

Remark A.1.2. If $M = \sup E < +\infty$, then it has the following properties:

- $x \leq M$ for every $x \in E$;
- for every $\varepsilon > 0$, there exists $x_\varepsilon \in E$ such that

$$M - \varepsilon < x_\varepsilon.$$

Definition A.1.3. Let $E \subset \mathbb{R}$ be a non-empty set. We define its *infimum* as the greatest lower bound for E. We indicate by

$$\inf E,$$

this number, with the convention that $\inf E = -\infty$ if the class of lower bounds is empty.

Remark A.1.4. If $m = \inf E > -\infty$, then it has the following properties:

- $x \geq m$ for every $x \in E$;
- for every $\varepsilon > 0$, there exists $x_\varepsilon \in E$ such that

$$m + \varepsilon > x_\varepsilon.$$

Given a sequence $\{x_n\}_{n\in\mathbb{N}}$, we will use the notations

$$\sup_{n\in\mathbb{N}} x_n = \sup\{x_n : n \in \mathbb{N}\},$$

and more generally

$$\sup_{n\geq k} x_n = \sup\{x_n : n \geq k\}.$$

We will use a similar notation for the infima of sequences.

2. Limit superior and limit inferior

Definition A.2.1. Let $\{b_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. Its *limit superior* is defined by

$$\inf_{k\in\mathbb{N}} \sup_{n\geq k} b_n.$$

We use the notation

$$\limsup_{n\to\infty} b_n,$$

to denote this quantity.

Remark A.2.2. Observe that the new sequence

$$B_k = \sup_{n\geq k} b_n, \qquad \text{for every } k \in \mathbb{N},$$

is monotone decreasing by construction. Then we have

$$\inf_{k\in\mathbb{N}} B_k = \lim_{k\to\infty} B_k.$$

Definition A.2.3. Let $\{b_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. Its *limit inferior* is defined by

$$\sup_{k\in\mathbb{N}} \inf_{n\geq k} b_n.$$

We use the notation

$$\liminf_{n\to\infty} b_n,$$

to denote this quantity.

Remark A.2.4. Observe that the new sequence

$$B_k = \inf_{n\geq k} b_n, \qquad \text{for every } k \in \mathbb{N},$$

is monotone increasing by construction. Then we have

$$\sup_{k\in\mathbb{N}} B_k = \lim_{k\to\infty} B_k.$$

Example A.2.5. By taking the sequence $b_n = (-1)^n$, it is not difficult to see that

$$\liminf_{n\to\infty} (-1)^n = -1 \qquad \text{and} \qquad \limsup_{n\to\infty} (-1)^n = 1.$$

Example A.2.6. Let us consider the sequence

$$b_n = \begin{cases} \dfrac{n}{n+1}, & \text{if } n \text{ even}, \\[2ex] \dfrac{n+1}{n}, & \text{if } n \text{ odd}. \end{cases}$$

We observe that for every $k \in \mathbb{N}$

$$\sup_{n \geq k} b_n = \begin{cases} b_k, & \text{if } k \text{ odd}, \\ b_{k+1}, & \text{if } k \text{ even} \end{cases} = \begin{cases} \dfrac{k+1}{k}, & \text{if } k \text{ odd}, \\[2ex] \dfrac{k+2}{k+1}, & \text{if } k \text{ even}. \end{cases}$$

This implies that

$$\limsup_{n \to \infty} b_n = \lim_{k \to \infty} \begin{cases} \dfrac{k+1}{k}, & \text{if } k \text{ odd}, \\[2ex] \dfrac{k+2}{k+1}, & \text{if } k \text{ even}. \end{cases} = 1.$$

Theorem A.2.7. *Let $\{b_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers. Then the sequence admits a limit if and only if*

$$\limsup_{n \to \infty} b_n = \liminf_{n \to \infty} b_n.$$

In this case, we have

$$\lim_{n \to \infty} b_n = \limsup_{n \to \infty} b_n = \liminf_{n \to \infty} b_n.$$

Appendix B

First order linear differential equations

1. Variable coefficients case

In this section, we briefly recall how to solve an ordinary differential equation of the form

$$y'(t) + a(t)\,y(t) = b(t), \qquad t \in \mathbb{R},$$

where the continuous functions a, b are given. Let A be a C^1 function such that

$$A'(t) = a(t), \qquad \text{for } t \in \mathbb{R},$$

i.e. A is a primitive of a. Then we observe that

$$y'(t) + a(t)\,y(t) = b(t) \quad \Longleftrightarrow \quad e^{A(t)}\left(y'(t) + a(t)\,y(t)\right) = e^{A(t)}\,b(t).$$

With this simple trick, we can now recognize a derivative on the left-hand side, i.e.

$$e^{A(t)}\left(y'(t) + a(t)\,y(t)\right) = \left(e^{A(t)}\,y(t)\right)'.$$

From the previous identity, we thus get that y is a solution of the differential equation if and only if

$$\left(e^{A(t)}\,y(t)\right)' = e^{A(t)}\,b(t),$$

that is if

$$e^{A(t)}\,y(t) \quad \text{is a primitive of} \quad e^{A(t)}\,b(t).$$

We write this with the formula

$$e^{A(t)}\,y(t) = B(t) + c, \qquad \text{with } B'(t) = e^{A(t)}\,b(t) \quad \text{and} \quad c \in \mathbb{R}.$$

Thus finally we get the solutions

$$y(t) = e^{-A(t)}\,B(t) + c\,e^{-A(t)}, \tag{2.1.1}$$

i.e. we have found infinitely many solutions.

Example B.1.1. Let us solve

$$y'(t) + t\,y(t) = 0, \qquad t \in \mathbb{R}.$$

With the notation above, we have $a(t) = t$ and $b = 0$. We can choose the following primitives

$$A(t) = \frac{t^2}{2} \qquad \text{and} \qquad B(t) = 0,$$

and obtain the family of solutions

$$y(t) = c\,e^{-\frac{t^2}{2}},$$

where $c \in \mathbb{R}$ is an arbitrary constant.

2. Separation of variables

When $b = 0$, the equation

$$y'(t) + a(t)\,y(t) = 0, \qquad t \in \mathbb{R},$$

can be solved by the *separation of variables* technique. We rewrite the equation in the form

$$y'(t) = -a(t)\,y(t),$$

and then divide by $y(t)$ both sides (let us suppose that $y(t) > 0$ for every $t \in \mathbb{R}$). We thus get

$$\frac{y'(t)}{y(t)} = -a(t),$$

and the left-hand side is the derivative of $t \mapsto \log y(t)$. In other words, we have

$$(\log y(t))' = -a(t).$$

We introduce as before a primitive A of a, then we obtain

$$\log y(t) = -A(t) + c$$

and $c \in \mathbb{R}$ is an arbitrary constant. By composing with the exponential function on both sides, we thus obtain

$$y(t) = e^{-A(t)}\,e^{c}.$$

By arbitrariness of $c \in \mathbb{R}$, we can rewrite the previous as

$$y(t) = C\,e^{-A(t)}, \qquad C > 0,$$

which is nothing but (2.1.1). Observe that the restriction $C > 0$ is in accordance with the requirement $y(t) > 0$ that we made during the previous discussion.

Appendix C

Fourier series

In this Appendix, we briefly present some definitions and results about Fourier series.

1. Definition and first properties

Let $T > 0$ and let $f : \mathbb{R} \to \mathbb{C}$ be a $T-$periodic measurable function. The theory of Fourier series aims at solving the problem of writing f as a (possibly infinite) superposition of $T-$periodic functions of the form

$$\cos\left(\frac{2\,\pi}{T}\,n\,t\right) \qquad \text{and} \qquad \sin\left(\frac{2\,\pi}{T}\,n\,t\right), \qquad n \in \mathbb{N}.$$

In other words, we want to understand under which conditions on f is it possible to write

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n\,\cos\left(\frac{2\,\pi}{T}\,n\,t\right) + \sum_{n=1}^{\infty} b_n\,\sin\left(\frac{2\,\pi}{T}\,n\,t\right), \tag{3.1.1}$$

for two suitable sequences of coefficients $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$. Moreover, in which sense does the convergence of the series above should be understood?

It is useful to observe that (3.1.1) can be rewritten in a more compact form. Indeed, by recalling that

$$e^{i\,\vartheta} = \cos\vartheta + i\,\sin\vartheta,$$

we have

$$\cos\vartheta = \frac{e^{i\,\vartheta} + e^{-i\,\vartheta}}{2} \qquad \text{and} \qquad \sin\vartheta = \frac{e^{i\,\vartheta} - e^{-i\,\vartheta}}{2\,i}.$$

This implies that

$$
\begin{aligned}
a_0 &+ \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{T}\, n\, t\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi}{T}\, n\, t\right) \\
&= a_0 + \sum_{n=1}^{\infty} a_n \frac{e^{i\,\frac{2\pi}{T}\, n\, t} + e^{-i\,\frac{2\pi}{T}\, n\, t}}{2} + \sum_{n=1}^{\infty} b_n \frac{e^{i\,\frac{2\pi}{T}\, n\, t} - e^{-i\,\frac{2\pi}{T}\, n\, t}}{2\, i} \\
&= a_0 + \sum_{n=1}^{\infty} \left(\frac{a_n - i\, b_n}{2}\right) e^{i\,\frac{2\pi}{T}\, n\, t} + \sum_{n=1}^{\infty} \left(\frac{a_n + i\, b_n}{2}\right) e^{-i\,\frac{2\pi}{T}\, n\, t} \\
&= a_0 + \sum_{n=1}^{\infty} \left(\frac{a_n - i\, b_n}{2}\right) e^{i\,\frac{2\pi}{T}\, n\, t} + \sum_{-\infty}^{m=-1} \left(\frac{a_{-m} + i\, b_{-m}}{2}\right) e^{i\,\frac{2\pi}{T}\, m\, t},
\end{aligned}
$$

where in the last series we made the change of index $m = -n$. If we then set

$$
c_0 = a_0, \qquad c_n = \frac{a_n - i\, b_n}{2}, \qquad \text{for } n \geq 1, \tag{3.1.2}
$$

and

$$
c_n = \frac{a_{-n} + i\, b_{-n}}{2}, \qquad \text{for } n \leq -1, \tag{3.1.3}
$$

then the problem can be reformulated in compact form as follows: under which conditions on f is it possible to write

$$
f(t) = \sum_{n\in\mathbb{Z}} c_n\, e^{i\, n\, \frac{2\pi}{T}\, t}, \tag{3.1.4}
$$

for a suitable sequence of coefficients $\{c_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$? In which sense does the convergence of the series above should be understood?

In order to answer this question, we first observe that the form of the coefficients can be easily guessed: we choose $k \in \mathbb{Z}$, multiply both sides of (3.1.4) by

$$
e^{-i\, k\, \frac{2\pi}{T}\, t},
$$

and integrate over the periodicity interval $[-T/2, T/2]$. By discarding convergence issues and proceeding formally, we get

$$
\begin{aligned}
\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i\, k\, \frac{2\pi}{T}\, t} f(t)\, dt &= \int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{n\in\mathbb{Z}} c_n\, e^{i\,(n-k)\, \frac{2\pi}{T}\, t}\, dt \\
&= \sum_{n\in\mathbb{Z}} c_n \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i\,(n-k)\, \frac{2\pi}{T}\, t}\, dt.
\end{aligned}
$$

We observe that for $n \neq k$ we have

$$
\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i\,(n-k)\, \frac{2\pi}{T}\, t}\, dt = \frac{e^{i\,(n-k)\,\pi} - e^{-i\,(n-k)\,\pi}}{i\,(n-k)\,\frac{2\pi}{T}} = 0,
$$

while for $n = k$

$$
\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i\,(n-k)\, \frac{2\pi}{T}\, t}\, dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} dt = T.
$$

In conclusion, we get the relation

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i\,k\,\frac{2\pi}{T}\,t} f(t)\,dt = T\,c_k,$$

that is, whenever f can be written as (3.1.4), the coefficients c_k must have the form

$$c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i\,k\,\frac{2\pi}{T}\,t} f(t)\,dt.$$

Definition C.1.1. Let $f:\mathbb{R}\to\mathbb{C}$ be a $T-$periodic measurable function. Let us suppose that $f\in L^1([-T/2,T/2])$, then its *Fourier coefficients* are given by

$$\widehat{f}(n) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i\,n\,\frac{2\pi}{T}\,t} f(t)\,dt, \qquad \text{for every } n\in\mathbb{Z}.$$

The formal expression

$$\mathcal{J}[f](t) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\,e^{i\,n\,\frac{2\pi}{T}\,t},$$

is called *Fourier series of f*.

Theorem C.1.2. *Let $f:\mathbb{R}\to\mathbb{C}$ be a $T-$periodic measurable function. Let us suppose that $f\in L^1([-T/2,T/2])$, then the sequence $\{\widehat{f}(n)\}_{n\in\mathbb{Z}}\subset\mathbb{C}$ is bounded and such that*

$$\lim_{|n|\to\infty} |\widehat{f}(n)| = 0. \tag{3.1.5}$$

Proof. The boundedness of $\{\widehat{f}(n)\}_{n\in\mathbb{Z}}$ easily follows from the definition, indeed for every $n\in\mathbb{Z}$ we have

$$|\widehat{f}(n)| = \left|\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i\,n\,\frac{2\pi}{T}\,t} f(t)\,dt\right| \le \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |e^{-i\,n\,\frac{2\pi}{T}\,t}|\,|f(t)|\,dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|\,dt,$$

and the last quantity is finite and independent of n (compare this estimate with (5.2.1)). In order to prove (3.1.5), we observe that

$$e^{-i\,n\,\frac{2\pi}{T}\,t} = -e^{-i\,n\,\frac{2\pi}{T}\,t}\,e^{-i\,\pi},$$

thus we get

$$\begin{aligned}
\widehat{f}(n) &= -\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i\,n\,\frac{2\pi}{T}\,t}\,e^{-i\,\pi} f(t)\,dt\\
&= -\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i\,n\,\frac{2\pi}{T}\,\left(t+\frac{T}{2n}\right)} f(t)\,dt\\
&= -\frac{1}{T}\int_{-\frac{n-1}{n}\,\frac{T}{2}}^{\frac{n+1}{n}\,\frac{T}{2}} e^{-i\,n\,\frac{2\pi}{T}\,\tau} f\left(\tau-\frac{1}{n}\,\frac{T}{2}\right)\,d\tau.
\end{aligned}$$

We observe that the last integral is performed over an interval of length T and the integrated function is $T-$periodic, thus this integral coincides with

$$-\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i\,n\,\frac{2\pi}{T}\,\tau} f\left(\tau-\frac{1}{n}\,\frac{T}{2}\right)\,d\tau.$$

We thus obtained

$$\widehat{f}(n) = -\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i\,n\,\frac{2\pi}{T}\,\tau}\, f\left(\tau - \frac{1}{n}\,\frac{T}{2}\right)\,d\tau.$$

On the other hand, by definition

$$\widehat{f}(n) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i\,n\,\frac{2\pi}{T}\,t}\, f(t)\,dt.$$

By summing up the two identities, we get

$$\widehat{f}(n) = \frac{1}{2\,T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i\,n\,\frac{2\pi}{T}\,t}\left(f(t) - f\left(t - \frac{1}{n}\,\frac{T}{2}\right)\right)dt,$$

and thus

$$|\widehat{f}(n)| \le \frac{1}{2\,T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \left|f(t) - f\left(t - \frac{1}{n}\,\frac{T}{2}\right)\right| dt. \tag{3.1.6}$$

By using the continuity in L^1 norm of translations, i.e. Proposition 3.4.5, we get the conclusion. □

Lemma C.1.3. *Let $f : \mathbb{R} \to \mathbb{C}$ be a $T-$periodic measurable function. Then:*

1. *if f is even, then the Fourier coefficients are even, i.e.*
$$\widehat{f}(n) = \widehat{f}(-n), \qquad \text{for every } n \in \mathbb{Z};$$
2. *if f is odd, then the Fourier coefficients are odd, i.e.*
$$\widehat{f}(n) = -\widehat{f}(-n), \qquad \text{for every } n \in \mathbb{Z};$$
3. *if f is real-valued and even, then the Fourier coefficients $\{\widehat{f}(n)\}_{n\in\mathbb{Z}}$ are real;*
4. *if f is real-valued and odd, then the Fourier coefficients $\{\widehat{f}(n)\}_{n\in\mathbb{Z}}$ are purely imaginary.*

Proof. We prove the points 1. and 3., leaving the other proofs as an exercise. By using the change of variable $s = -t$ and using that f is even, we get

$$\begin{aligned}\widehat{f}(-n) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i\,n\,\frac{2\pi}{T}\,t}\, f(t)\,dt &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i\,n\,\frac{2\pi}{s}\,t}\, f(-s)\,ds\\ &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i\,n\,\frac{2\pi}{s}\,t}\, f(s)\,ds = \widehat{f}(n),\end{aligned}$$

which proves the first point.

Let us now further assume that f is real-valued, in order to prove that the Fourier coefficients are real, we can prove that

$$\widehat{f}(n) = \widehat{f}(n)^*, \qquad \text{for every } n \in \mathbb{Z}.$$

By definition, we have

$$\widehat{f}(n)^* = \frac{1}{T}\left(\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i\,n\,\frac{2\pi}{T}\,t}\, f(t)\,dt\right)^* = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \left(e^{-i\,n\,\frac{2\pi}{T}\,t}\right)^* (f(t))^*\,dt.$$

Since f is real-valued, we have that $(f(t))^* = f(t)$, while

$$\left(e^{-i\,n\,\frac{2\pi}{T}\,t}\right)^* = e^{i\,n\,\frac{2\pi}{T}\,t}.$$

We thus obtain

$$\widehat{f}(n)^* = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i\,n\,\frac{2\pi}{T}\,t}\, f(t)\,dt = \widehat{f}(-n).$$

By point 1., the last coefficient is equal to $\widehat{f}(n)$. We thus achieved the desired conclusion. $\square$

Remark C.1.4. By recalling the relations (3.1.2) and (3.1.3), we get that if f is even, then its Fourier series can be written as

$$\mathcal{J}[f](t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{T}\,n\,t\right),$$

i.e. it only contains the cosine functions. Indeed, by the previous Lemma we have

$$\frac{a_n - i\,b_n}{2} = \widehat{f}(n) = \widehat{f}(-n) = \frac{a_n + i\,b_n}{2}, \qquad \text{for every } n \in \mathbb{N},$$

which implies that $b_n = 0$, for every $n \in \mathbb{N}$.

Similarly, if f is odd, its Fourier series can be written as

$$\mathcal{J}[f](t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi}{T}\,n\,t\right).$$

Definition C.1.5. Let $f : \mathbb{R} \to \mathbb{C}$ be a measurable $T-$periodic function, such that

$$f \in L^1\left(\left[-\frac{T}{2},\frac{T}{2}\right]\right).$$

Let us suppose that $g \in L^1_{\mathrm{loc}}(\mathbb{R})$. We define their *convolution* to be the function

$$f * g(t) = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t-\tau)\,g(\tau)\,d\tau.$$

Lemma C.1.6. *Under the previous assumption, the functions $f * g$ is still $T-$periodic and*

$$f * g \in L^1\left(\left[-\frac{T}{2},\frac{T}{2}\right]\right),$$

with

$$\|f * g\|_{L^1([-\frac{T}{2},\frac{T}{2}])} \le \|f\|_{L^1([-\frac{T}{2},\frac{T}{2}])}\,\|g\|_{L^1([-\frac{T}{2},\frac{T}{2}])}.$$

Proof. By using the $T-$periodicity of f, we easily get.

$$f * g(t+T) = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t+T-\tau)\,g(\tau)\,d\tau = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t-\tau)\,g(\tau)\,d\tau = f * g(t).$$

In order to prove that $f * g$ is in L^1, we proceed as in the proof of Proposition 3.5.4. We have

$$\begin{aligned}
\int_{-\frac{T}{2}}^{\frac{T}{2}} |f * g(t)|\,dt &= \int_{-\frac{T}{2}}^{\frac{T}{2}} \left|\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t-\tau)\,g(\tau)\,d\tau\right| dt \\
&\le \int_{-\frac{T}{2}}^{\frac{T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t-\tau)|\,|g(\tau)|\,d\tau\,dt \\
&= \int_{-\frac{T}{2}}^{\frac{T}{2}} |g(\tau)| \left(\int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t-\tau)|\,dt\right) d\tau,
\end{aligned}$$

thanks to Fubini's and Tonelli's Theorem. By using a change of variable and the $T-$periodicity of f, we get

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t-\tau)|\,dt = \int_{-\frac{T}{2}-\tau}^{\frac{T}{2}-\tau} |f(s)|\,ds = \int_{-\frac{T}{2}}^{\frac{T}{2}} |f(s)|\,ds,$$

and thus the conclusion follows. $\square$

2. Convergence

We now discuss under which conditions the Fourier series

$$\mathcal{J}[f](t) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\,e^{i\,n\,\frac{2\,\pi}{T}\,t},$$

converges to the original periodic signal f. Moreover, we want to clarify in which sense this convergence must be understood. Observe that whenever we have

$$\mathcal{J}[f](t) = f(t),$$

this can be read as an *inversion formula* for the Fourier series, in analogy with the inversion formulas for the Laplace and Fourier transforms.

In this spirit, the following result can be seen as the natural counterpart of Theorem 5.4.2.

Theorem C.2.1. *Let $f:\mathbb{R}\to\mathbb{C}$ be a $T-$periodic measurable function. Let us assume that f is a piecewise C^1 function on $[-T/2,T/2]$, i.e. f and f' have only jump discontinuities at*

$$-\frac{T}{2} \le t_1 < t_2 < \cdots < t_\ell \le \frac{T}{2}.$$

Then for every $t\in[-T/2,T/2]$, we have

$$\mathcal{J}[f](t) = \frac{f(t^+)+f(t^-)}{2}.$$

Proof. Let us fix $t\in[-T/2,T/2]$, for every $k\in\mathbb{N}$ we define the $k-$*th partial Fourier sum*

$$\mathcal{J}_k[f](t) = \sum_{n=-k}^{k} \widehat{f}(n)\,e^{\frac{2\,\pi\,n}{T}\,i\,t}. \tag{3.2.7}$$

Then we need to show that

$$\lim_{k\to\infty} \mathcal{J}_k[f](t) = \frac{f(t^+)+f(t^-)}{2}.$$

By using the definition of Fourier coefficient, we have

$$\begin{aligned}\mathcal{J}_k[f[(t) &= \sum_{n=-k}^{k} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(\tau)\,e^{-\frac{2\,\pi\,n}{T}\,i\,(\tau-t)}\,d\tau \\ &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(\tau)\left(\sum_{n=-k}^{k} e^{-\frac{2\,\pi\,n}{T}\,i\,(\tau-t)}\right)d\tau.\end{aligned}$$

We now perform the change of index $m = n + k$ in the sum above, so to obtain

$$\begin{aligned}\sum_{n=-k}^{k} e^{-\frac{2\pi n}{T}\, i\,(\tau-t)} &= \sum_{m=0}^{2k} e^{-\frac{2\pi\,(m-k)}{T}\, i\,(\tau-t)} \\ &= e^{\frac{2\pi k}{T}\, i\,(\tau-t)} \sum_{m=0}^{2k} e^{-\frac{2\pi m}{T}\, i\,(\tau-t)} = e^{\frac{2\pi k}{T}\, i\,(\tau-t)} \sum_{m=0}^{2k} \left(e^{-\frac{2\pi}{T}\, i\,(\tau-t)}\right)^m.\end{aligned}$$

We can recognize that the last sum is a partial sum of a geometric series. By recalling that

$$\sum_{m=0}^{2k} \alpha^m = \begin{cases} \dfrac{\alpha^{2k+1}-1}{\alpha-1}, & \text{if } \alpha \neq 1, \\ \\ 2k+1, & \text{if } \alpha = 1, \end{cases} \tag{3.2.8}$$

we get that:

- if $(\tau - t)/T \in \mathbb{R} \setminus \mathbb{Z}$, then
$$e^{-\frac{2\pi}{T}\, i\,(\tau-t)} \neq 1,$$
and thus by (3.2.8) with
$$\alpha = e^{-\frac{2\pi}{T}\, i\,(\tau-t)},$$
we get
$$\begin{aligned} e^{\frac{2\pi k}{T}\, i\,(\tau-t)} \sum_{m=0}^{2k} \left(e^{-\frac{2\pi}{T}\, i\,(\tau-t)}\right)^m &= e^{\frac{2\pi k}{T}\, i\,(\tau-t)}\, \frac{e^{-\frac{2\pi\,(2k+1)}{T}\, i\,(\tau-t)} - 1}{e^{-\frac{2\pi}{T}\, i\,(\tau-t)} - 1} \\ &= \frac{e^{-\frac{2\pi\,(k+1)}{T}\, i\,(\tau-t)} - e^{\frac{2\pi k}{T}\, i\,(\tau-t)}}{e^{-\frac{2\pi}{T}\, i\,(\tau-t)} - 1} \\ &= \frac{e^{-\frac{2\pi\,(k+1)}{T}\, i\,(\tau-t)} - e^{\frac{2\pi k}{T}\, i\,(\tau-t)}}{e^{-\frac{\pi}{T}\, i\,(\tau-t)}\left(e^{-\frac{\pi}{T}\, i\,(\tau-t)} - e^{\frac{\pi}{T}\, i\,(\tau-t)}\right)} \\ &= \frac{e^{-\frac{2\pi\,(k+\frac{1}{2})}{T}\, i\,(\tau-t)} - e^{\frac{2\pi\,(k+\frac{1}{2})}{T}\, i\,(\tau-t)}}{e^{-\frac{\pi}{T}\, i\,(\tau-t)} - e^{\frac{\pi}{T}\, i\,(\tau-t)}} \\ &= \frac{\sin\left(\dfrac{\pi}{T}\,(2k+1)\,(\tau-t)\right)}{\sin\left(\dfrac{\pi}{T}\,(\tau-t)\right)}; \end{aligned}$$
- if $(\tau - t)/T \in \mathbb{Z}$, then
$$e^{-\frac{2\pi}{T}\, i\,(\tau-t)} = 1,$$
and thus
$$e^{\frac{2\pi k}{T}\, i\,(\tau-t)} \sum_{m=0}^{2k} \left(e^{-\frac{2\pi}{T}\, i\,(\tau-t)}\right)^m = 2k+1.$$

By introducing the *Dirichlet kernel*

$$D_k(t) = \begin{cases} \dfrac{\sin\left((2k+1)\,t\right)}{\sin t}, & \text{if } t \in \mathbb{R} \setminus \{\pi\, m\,:\, m \in \mathbb{Z}\}, \\ \\ 2k+1, & \text{if } t \in \{\pi\, m\,:\, m \in \mathbb{Z}\}. \end{cases}$$

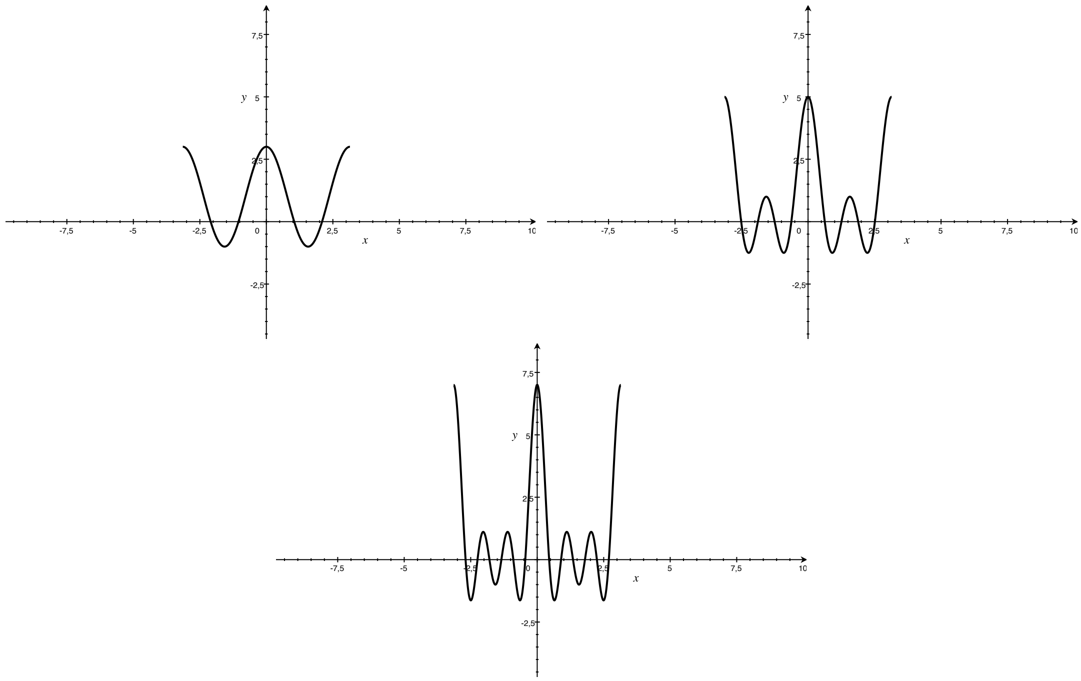

Figure 1. The Dirichlet kernel D_k for $k = 1, 2, 3$.

we can summarize the previous discussion by saying that

$$e^{\frac{2\pi k}{T} i (\tau - t)} \sum_{m=0}^{2k} \left(e^{-\frac{2\pi}{T} i (\tau - t)} \right)^m = D_k \left(\frac{\pi}{T} (\tau - t) \right).$$

We thus obtained

$$\mathcal{J}_k[f](t) = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(\tau)\, D_k \left(\frac{\pi}{T} (\tau - t) \right) d\tau. \tag{3.2.9}$$

Before going on, we manipulate a bit the last integral. Observe that by definition D_k is even, thus

$$D_k \left(\frac{\pi}{T} (\tau - t) \right) = D_k \left(\frac{\pi}{T} (t - \tau) \right).$$

In this way we can recognize the expression of a convolution, in the right-hand side of (3.2.9). Moreover, D_k is $\pi-$periodic, thus we have

$$\begin{aligned} \mathcal{J}_k[f](t) &= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(\tau)\, D_k \left(\frac{\pi}{T} (\tau - t) \right) d\tau \\ &= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(\tau)\, D_k \left(\frac{\pi}{T} (t - \tau) \right) d\tau \\ &= \frac{1}{T} \int_{t-\frac{T}{2}}^{t+\frac{T}{2}} f(t-s)\, D_k \left(\frac{\pi}{T} s \right) ds = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t-s)\, D_k \left(\frac{\pi}{T} s \right) ds. \end{aligned} \tag{3.2.10}$$

In the last identity we used that the integrand is $T-$periodic. In order to conclude, we need to show that

$$\lim_{k\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t-s)\, D_k\left(\frac{\pi}{T}\, s\right)\, ds = \frac{f(t^+)+f(t^-)}{2}.$$

For this, it is sufficient to prove that

$$\lim_{k\to\infty} \frac{1}{T} \int_{0}^{\frac{T}{2}} f(t-s)\, D_k\left(\frac{\pi}{T}\, s\right)\, ds = \frac{f(t^-)}{2}, \tag{3.2.11}$$

and

$$\lim_{k\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{0} f(t-s)\, D_k\left(\frac{\pi}{T}\, s\right)\, ds = \frac{f(t^+)}{2}.$$

The proof of these two facts is quite similar to that of Theorem 5.4.2. We focus on proving (3.2.11), the proof of the other fact being equal. Observe that by using formula (3.2.10) with $f(t)=1$, we have[1]

$$1 = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} D_k\left(\frac{\pi}{T}\,(t-s)\right)\, ds.$$

As already observed, D_k is an even function, thus from the previous computation we also get

$$\frac{1}{T} \int_{0}^{\frac{T}{2}} D_k\left(\frac{\pi}{T}\,(t-s)\right)\, ds = \frac{1}{2}.$$

Thus (3.2.11) can be rewritten as

$$\lim_{k\to\infty} \frac{1}{T} \int_{0}^{\frac{T}{2}} \Big(f(t-s)-f(t^-)\Big)\, D_k\left(\frac{\pi}{T}\, s\right)\, ds = 0.$$

By recalling the definition of Dirichlet kernel, this is turn is equivalent to

$$\lim_{k\to\infty} \frac{1}{T} \int_{0}^{\frac{T}{2}} \frac{\Big(f(t-s)-f(t^-)\Big)}{\sin\left(\frac{\pi}{T}\, s\right)}\, \sin\left(\frac{\pi}{T}\,(2\,k+1)\, s\right)\, ds = 0. \tag{3.2.12}$$

If we can prove that the function

$$s \mapsto \frac{f(t-s)-f(t^-)}{\sin\left(\frac{\pi}{T}\, s\right)},$$

is in $L^1([0,T/2])$, then the desired conclusion (3.2.12) would follow from Lemma 5.4.1. For this, it is sufficient to observe that thanks to the assumption on f we have

$$\frac{f(t-s)-f(t^-)}{\sin\left(\frac{\pi}{T}\, s\right)} \sim \frac{f'(t^-)\, s}{\frac{\pi}{T}\, s} = \frac{f'(t^-)}{\frac{\pi}{T}}, \qquad \text{for } s\to 0^+.$$

We leave the final details to the reader. $\square$

[1] For the costant function $f(t)=1$, we have

$$\widehat{f}(n) = 0, \qquad \text{for every } n\neq 0$$

and $\widehat{f}(0)=1$.

Proposition C.2.2. *Let $f : \mathbb{R} \to \mathbb{C}$ be a $T-$periodic measurable function. Let us suppose that there exists $C > 0$ and $\beta > 1$ such that*

$$|\widehat{f}(n)| \le \frac{C}{1+|n|^\beta}, \qquad \text{for every } n \in \mathbb{Z}.$$

Then the Fourier series of f is totally converging on $\mathbb{R}$.

Proof. We need to prove that

$$\sum_{n\in\mathbb{Z}} \left(\sup_{t\in\mathbb{R}} \left| \widehat{f}(n)\, e^{i\,n\,\frac{2\pi}{T}\,t} \right| \right) < +\infty.$$

By the properties of the complex exponential and the assumption on f, we have

$$\left(\sup_{t\in\mathbb{R}} \left| \widehat{f}(n)\, e^{i\,n\,\frac{2\pi}{T}\,t} \right| \right) \le \frac{C}{1+|n|^\beta}, \qquad \text{for every } n \in \mathbb{Z}.$$

By recalling that (here it is needed $\beta > 1$)

$$\sum_{n=0}^{\infty} \frac{1}{1+|n|^\beta} < +\infty,$$

we get the desired conclusion. □

Theorem C.2.3 (Smooth periodic signals I)**.** *Let $f : \mathbb{R} \to \mathbb{C}$ be a $T-$periodic measurable function. Let us suppose that $f \in C^0(\mathbb{R}) \cap C^1([-T/2, T/2])$ and that f' is $\alpha-$Hölder continuous, i.e. there exists $C > 0$ and $0 < \alpha \le 1$ such that*

$$|f'(t) - f'(s)| \le C\,|t-s|^\alpha, \qquad \text{for } t, s \in \mathbb{R}.$$

Then the Fourier series of f is totally converging on $\mathbb{R}$ to f.

Proof. Under the standing assumptions on f, we already know by Theorem C.2.1 that the Fourier series converges pointwise. In order to infer total convergence, the idea is to apply Proposition C.2.2. By using an integration by parts, we get for $n \in \mathbb{Z} \setminus \{0\}$

$$\begin{aligned}
\widehat{f}(n) = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i\,n\,\frac{2\pi}{T}\,t}\, f(t)\,dt &= \frac{1}{T} \left[\frac{e^{-i\,n\,\frac{2\pi}{T}\,t}}{-i\,n\,2\,\pi}\, f(t) \right]_{-\frac{T}{2}}^{\frac{T}{2}} \\
&\quad - \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{e^{-i\,n\,\frac{2\pi}{T}\,t}}{-i\,n\,2\,\pi}\, f'(t)\,dt \\
&= \frac{1}{T}\, i\, \frac{1}{2\,n\,\pi} \left[e^{-i\,n\,\pi}\, f\left(\frac{T}{2}\right) - e^{i\,n\,\pi}\, f\left(-\frac{T}{2}\right) \right] \\
&\quad - \frac{1}{T}\, i\, \frac{1}{2\,n\,\pi} \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i\,n\,\frac{2\pi}{T}\,t}\, f'(t)\,dt.
\end{aligned}$$

By the hypothesis of continuity of f, we get

$$f\left(\frac{T}{2}\right) = f\left(-\frac{T}{2}\right),$$

and thus

$$|\widehat{f}(n)| = \frac{1}{2\,|n|\,\pi} \left| \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i\,n\,\frac{2\pi}{T}\,t}\, f'(t)\,dt \right|.$$

By hypothesis the derivative f' is continuous on the interval $[-T/2, T/2]$, thus in particular it is bounded. We then obtain

$$|\widehat{f}(n)| = \frac{1}{2\,|n|\,\pi} \left| \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i\,n\,\frac{2\pi}{T}\,t}\, f'(t)\,dt \right| = \frac{1}{2\,|n|\,\pi}\, |\widehat{f'}(n)|.$$

By recalling the estimate (3.1.6) and using the Hölder regularity of f', we obtain

$$|\widehat{f'}(n)| = \frac{1}{2\,T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \left| f'(t) - f'\left(t - \frac{1}{n}\,\frac{T}{2}\right) \right| dt \le \frac{C}{2\,T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \left(\frac{T}{2\,|n|}\right)^{\alpha} dt = \frac{C\,T^{\alpha}}{2^{\alpha+1}\,|n|^{\alpha}}.$$

In conclusion, we obtain for $n \in \mathbb{Z} \setminus \{0\}$

$$|\widehat{f}(n)| = \frac{1}{2\,n\,\pi}\, |\widehat{f'}(n)| \le \frac{C\,T^{\alpha}}{2^{\alpha+2}\,|n|^{1+\alpha}}.$$

By Proposition C.2.2 with $\beta = \alpha + 1$, we obtain the desired conclusion. □

Proposition C.2.4 (Bessel inequality)**.** *Let $f : \mathbb{R} \to \mathbb{C}$ be a $T-$periodic measurable function, such that*

$$f \in L^2\left(\left[-\frac{T}{2}, \frac{T}{2}\right]\right).$$

Then

$$\sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^2 \le \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2\,dt.$$

Proof. We first introduce some notations. For every pair of functions $f, g \in L^2([-T/2, T/2])$, we define the *scalar product*

$$\langle f, g\rangle = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\, g(t)^*\,dt.$$

Observe that this has the following properties:

- $\langle g, f\rangle = (\langle f, g\rangle)^*$;
- $\langle \alpha\, f_1 + \beta\, f_2, g\rangle = \alpha\,\langle f_1, g\rangle + \beta\,\langle f_2, g\rangle$, for every $\alpha, \beta \in \mathbb{C}$;
- $\langle f, f\rangle = \|f\|_{L^2}^2$.

Also observe that by combining the first two properties, we also have for every $\alpha, \beta \in \mathbb{C}$

$$\begin{aligned}\langle f, \alpha\, g_1 + \beta\, g_2\rangle = \Big(\langle \alpha\, g_1 + \beta\, g_2, f\rangle\Big)^* &= \Big(\alpha\,\langle g_1, f\rangle + \beta\,\langle g_2, f\rangle\Big)^* \\ &= \alpha^*\,(\langle g_1, f\rangle)^* + \beta^*\,(\langle g_2, f\rangle)^* \\ &= \alpha^*\,\langle f, g_1\rangle + \beta^*\,\langle f, g_2\rangle.\end{aligned}$$

For every $n \in \mathbb{Z}$, we also set

$$\mathbf{e}_n(t) = \frac{1}{\sqrt{T}}\, e^{i\,n\,\frac{2\pi}{T}\,t},$$

and observe that

$$\langle \mathbf{e}_k, \mathbf{e}_m\rangle = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i\,k\,\frac{2\pi}{T}\,t}\, e^{-i\,m\,\frac{2\pi}{T}\,t}\,dt = \left\{\begin{array}{ll} 1, & \text{if } k = m, \\ 0 & \text{if } k \ne m. \end{array}\right. \tag{3.2.13}$$

We also notice that, with these notations, we have

$$\widehat{f}(n) = \frac{1}{\sqrt{T}} \langle f, \mathbf{e}_n \rangle, \qquad \text{for every } n \in \mathbb{Z}.$$

We now fix $k \in \mathbb{Z}$ and observe that the $k-$th partial Fourier sum of f (3.2.7) ca be rewritten as

$$\mathcal{J}_k[f](t) = \sum_{n=-k}^{k} \langle f, \mathbf{e}_n \rangle \, \mathbf{e}_n(t).$$

Then we decompose f as follows:

$$f(t) = \Big[f(t) - \mathcal{J}_k[f](t)\Big] + \mathcal{J}_k[f](t).$$

We observe that by construction, we have

$$\begin{aligned} \langle f - \mathcal{J}_k[f], \mathcal{J}_k[f] \rangle &= \langle f, \mathcal{J}_k[f] \rangle - \langle \mathcal{J}_k[f], \mathcal{J}_k[f] \rangle \\ &= \left\langle f, \sum_{n=-k}^{k} \langle f, \mathbf{e}_n \rangle \, \mathbf{e}_n \right\rangle - \left\langle \sum_{n=-k}^{k} \langle f, \mathbf{e}_n \rangle \, \mathbf{e}_n, \sum_{n=-k}^{k} \langle f, \mathbf{e}_n \rangle \, \mathbf{e}_n \right\rangle \\ &= \sum_{n=-k}^{k} \langle f, \mathbf{e}_n \rangle \, \langle f, \mathbf{e}_n \rangle^* - \sum_{n=-k}^{k} \langle f, \mathbf{e}_n \rangle \, \langle f, \mathbf{e}_n \rangle^* = 0, \end{aligned} \tag{3.2.14}$$

where we used the properties of the scalar product and the orthogonality relations (3.2.13). From this identity and the properties of the scalar product, we get

$$\langle \mathcal{J}_k[f], f - \mathcal{J}_k[f] \rangle = \Big(\langle f - \mathcal{J}_k[f], \mathcal{J}_k[f] \rangle \Big)^* = 0,$$

as well. By using these facts, we obtain

$$\begin{aligned} \int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2 \, dt &= \langle f, f \rangle \\ &= \Big\langle \Big[f - \mathcal{J}_k[f]\Big] + \mathcal{J}_k[f], \Big[f - \mathcal{J}_k[f]\Big] + \mathcal{J}_k[f] \Big\rangle \\ &= \langle f(t) - \mathcal{J}_k[f], f - \mathcal{J}_k[f] \rangle \\ &+ \langle f(t) - \mathcal{J}_k[f], \mathcal{J}_k[f] \rangle + \langle \mathcal{J}_k[f], f - \mathcal{J}_k[f] \rangle \\ &+ \langle \mathcal{J}_k[f], \mathcal{J}_k[f] \rangle \\ &= \langle f - \mathcal{J}_k[f], f - \mathcal{J}_k[f] \rangle + \langle \mathcal{J}_k[f], \mathcal{J}_k[f] \rangle \\ &= \int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t) - \mathcal{J}_k[f](t)|^2 \, dt + \langle \mathcal{J}_k[f], \mathcal{J}_k[f] \rangle \geq \langle \mathcal{J}_k[f], \mathcal{J}_k[f] \rangle. \end{aligned}$$

By recalling the definition of $\mathcal{J}_k[f]$ and using the orthogonality relations (3.2.13), we obtain

$$\langle \mathcal{J}_k[f], \mathcal{J}_k[f] \rangle = \sum_{n=-k}^{k} \langle f, \mathbf{e}_n \rangle \, \langle f, \mathbf{e}_n \rangle^* = \sum_{n=-k}^{k} |\langle f, \mathbf{e}_n \rangle|^2 = T \sum_{n=-k}^{k} |\widehat{f}(n)|^2. \tag{3.2.15}$$

The previous discussion leads to

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2 \, dt \geq T \sum_{n=-k}^{k} |\widehat{f}(n)|^2.$$

This estimate holds true for every $k \in \mathbb{N}$, thus by taking the limit as k goes to ∞, we get the desired conclusion. $\square$

With the aid of the previous result, we can considerably improve Theorem C.2.3 as follows.

Theorem C.2.5 (Smooth periodic signals II). *Let $f : \mathbb{R} \to \mathbb{C}$ be a $T-$periodic measurable function. Let us suppose that $f \in C^0(\mathbb{R})$ and that f' is piecewise continuous on $[-T/2, T/2]$, i.e. f' have only jump discontinuities at*

$$-\frac{T}{2} \le t_1 < t_2 < \cdots < t_\ell \le \frac{T}{2}.$$

Then the Fourier series $\mathcal{J}[f]$ is totally converging on $\mathbb{R}$ to f.

Proof. Under the standing assumption, we already know by Theorem C.2.1 that $\mathcal{J}[f]$ converges pointwise to f. Thus, we only have to show that the convergence of the Fourier series is actually total, i.e. we need to show that

$$\sum_{n\in\mathbb{Z}} \left(\sup_{t\in\mathbb{R}} \left| \widehat{f}(n)\, e^{i\,n\,\frac{2\pi}{T}\,t} \right| \right) = \sum_{n\in\mathbb{N}} |\widehat{f}(n)| < +\infty.$$

We prove the result by assuming for simplicity that f' only has one discontinuity point $-T/2 < t_0 < T/2$, it is then easy to reproduce the proof in the more general case.

By using an integration by parts, we get for $n \in \mathbb{Z} \setminus \{0\}$

$$\begin{aligned}
\widehat{f}(n) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i\,n\,\frac{2\pi}{T}\,t}\, f(t)\,dt &= \frac{1}{T}\int_{-\frac{T}{2}}^{t_0} e^{-i\,n\,\frac{2\pi}{T}\,t}\, f(t)\,dt + \frac{1}{T}\int_{t_0}^{\frac{T}{2}} e^{-i\,n\,\frac{2\pi}{T}\,t}\, f(t)\,dt \\
&= \frac{1}{T}\left[\frac{e^{-i\,n\,\frac{2\pi}{T}\,t}}{-i\,n\,2\,\pi}\, f(t)\right]_{-\frac{T}{2}}^{t_0} - \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{e^{-i\,n\,\frac{2\pi}{T}\,t}}{-i\,n\,2\,\pi}\, f'(t)\,dt \\
&+ \frac{1}{T}\left[\frac{e^{-i\,n\,\frac{2\pi}{T}\,t}}{-i\,n\,2\,\pi}\, f(t)\right]_{t_0}^{\frac{T}{2}} - \frac{1}{T}\int_{t_0}^{\frac{T}{2}} \frac{e^{-i\,n\,\frac{2\pi}{T}\,t}}{-i\,n\,2\,\pi}\, f'(t)\,dt \\
&= \frac{1}{T}\,i\,\frac{1}{2\,n\,\pi}\left[-e^{-i\,n\,\pi}\, f\left(\frac{T}{2}\right) + e^{i\,n\,\pi}\, f\left(-\frac{T}{2}\right)\right] \\
&+ \frac{1}{T}\,i\,\frac{1}{2\,n\,\pi}\left[\cancel{-e^{-i\,n\,\frac{2\pi}{T}\,t_0}\, f(t_0)} + \cancel{e^{-i\,n\,\frac{2\pi}{T}\,t_0}\, f(t_0)}\right] \\
&- \frac{1}{T}\,i\,\frac{1}{2\,n\,\pi}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i\,n\,\frac{2\pi}{T}\,t}\, f'(t)\,dt.
\end{aligned}$$

We now observe that $e^{-i\,n\,\pi} = e^{i\,n\,\pi}$ and by the hypothesis of continuity of f, we get

$$f\left(\frac{T}{2}\right) = f\left(-\frac{T}{2}\right).$$

Thus from the above computations we get

$$|\widehat{f}(n)| = \frac{1}{2\,|n|\,\pi}\left|\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i\,n\,\frac{2\pi}{T}\,t}\, f'(t)\,dt\right| = \frac{1}{2\,|n|\,\pi}\,|\widehat{f'}(n)|. \tag{3.2.16}$$

By hypothesis the derivative f' is bounded on the interval $[-T/2, T/2]$, thus in particular we have

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} |f'(t)|^2\,dt < +\infty.$$

By applying Bessel inequality to the function f', we thus obtain that

$$\sum_{n\in\mathbb{Z}} |\widehat{f'}(n)|^2 < +\infty.$$

By using (3.2.16), we thus have

$$\begin{aligned}\sum_{n\in\mathbb{Z}} |\widehat{f}(n)| = |\widehat{f}(0)| + \sum_{n\neq 0} |\widehat{f}(n)| &= |\widehat{f}(0)| + \frac{1}{2\,\pi}\sum_{n\neq 0}\frac{1}{|n|}\,|\widehat{f'}(n)| \\ &\leq |\widehat{f}(0)| + \frac{1}{4\,\pi}\sum_{n\neq 0}\frac{1}{|n|^2} + \frac{1}{4\,\pi}\sum_{n\neq 0}|\widehat{f'}(n)|^2 < +\infty,\end{aligned}$$

as desired. Observe that in the last inequality we used Young's inequality (i.e. Lemma 3.3.2 with $p = 2$). □

Remark C.2.6. The statement of the previous result looks quite similar to that of Theorem C.2.1. However, the crucial difference is that in Theorem C.2.5 the signal has to be *globally continuous*, i.e. f does not have jumps.

Theorem C.2.7 (Parseval's formula)**.** *Let $f, g : \mathbb{R} \to \mathbb{C}$ be two $T-$periodic measurable functions, such that*

$$f, g \in L^2\left(\left[-\frac{T}{2}, \frac{T}{2}\right]\right).$$

Then we have Parseval's formula

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,g(t)^*\,dt = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\,\widehat{g}(n)^*.$$

Proof. We perform the proof under the additional assumption that the Fourier series $\mathcal{J}[f]$ and $\mathcal{J}[g]$ both converge uniformly to f and g, respectively. In this case, we have

$$\begin{aligned}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,g(t)^*\,dt &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \mathcal{J}[f](t)\,\mathcal{J}[g](t)^*\,dt \\ &= \lim_{k\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\left(\sum_{n=-k}^{k}\widehat{f}(n)\,e^{-\frac{2\,\pi\,n}{T}\,i\,t}\right)\left(\sum_{m=-k}^{k}\widehat{g}(m)\,e^{-\frac{2\,\pi\,m}{T}\,i\,t}\right)^* dt \\ &= \lim_{k\to\infty}\sum_{n,m=-k}^{k}\widehat{f}(n)\,\widehat{g}(m)^*\,\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-\frac{2\,\pi\,(n-m)}{T}\,i\,t}\,dt.\end{aligned}$$

If we now use the orthogonality relations (3.2.13), we get

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,g(t)^*\,dt = \lim_{k\to\infty}\sum_{n=-k}^{k}\widehat{f}(n)\,\widehat{g}(n)^*.$$

Observe that by Young's inequality (see Lemma 3.3.2 with $p = 2$) we have

$$\begin{aligned}\lim_{k\to\infty}\left|\sum_{n=-k}^{k}\widehat{f}(n)\,\widehat{g}(n)^*\right| &\le \lim_{k\to\infty}\sum_{n=-k}^{k}|\widehat{f}(n)|\,|\widehat{g}(n)^*|\\ &\le \frac{1}{2}\lim_{k\to\infty}\sum_{n=-k}^{k}|\widehat{f}(n)|^2+\frac{1}{2}\lim_{k\to\infty}\sum_{n=-k}^{k}|\widehat{g}(n)^*|^2,\end{aligned}$$

and the last two series are converging, thanks to Bessel inequality. This concludes the proof. □

Remark C.2.8. By choosing $f = g$ in the formula above, we obtain Plancherel's formula

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}|f(t)|^2\,dt=\sum_{n\in\mathbb{Z}}|\widehat{f}(n)|^2.$$

Corollary C.2.9 (Convergence in L^2)**.** *Let $f : \mathbb{R} \to \mathbb{C}$ be a $T-$periodic measurable function, such that*

$$f\in L^2\left(\left[-\frac{T}{2},\frac{T}{2}\right]\right).$$

Then we have

$$f=\mathcal{J}[f]\quad \text{in } L^2\left(\left[-\frac{T}{2},\frac{T}{2}\right]\right).$$

This means that we have

$$\lim_{k\to\infty}\Big\|f-\mathcal{J}_k[f]\Big\|_{L^2([-\frac{T}{2},\frac{T}{2}])}=0,$$

where $\mathcal{J}_k[f]$ is the $k-$th partial Fourier sum, see (3.2.7).

Proof. We still use the scalar product

$$\langle f,g\rangle=\int_{-\frac{T}{2}}^{\frac{T}{2}}f(t)\,g(t)^*\,dt,$$

introduced in the proof of Proposition C.2.4. Then we have

$$\begin{aligned}\Big\|f-\mathcal{J}_k[f]\Big\|^2_{L^2([-\frac{T}{2},\frac{T}{2}])}&=\langle f-\mathcal{J}_k[f],f-\mathcal{J}_k[f]\rangle\\ &=\langle f,f\rangle+\langle f,\mathcal{J}_k[f]\rangle-\langle \mathcal{J}_k[f],f\rangle+\langle \mathcal{J}_k[f],\mathcal{J}_k[f]\rangle.\end{aligned}$$

By (3.2.14), we have

$$\langle f,\mathcal{J}_k[f]\rangle=\langle \mathcal{J}_k[f],\mathcal{J}_k[f]\rangle,$$

while by the properties of the scalar product, we get

$$\langle \mathcal{J}_k[f],f\rangle=(\langle f,\mathcal{J}_k[f]\rangle)^*=(\langle \mathcal{J}_k[f],\mathcal{J}_k[f]\rangle)^*=\langle \mathcal{J}_k[f],\mathcal{J}_k[f]\rangle.$$

Thus we obtain

$$\Big\|f-\mathcal{J}_k[f]\Big\|^2_{L^2([-\frac{T}{2},\frac{T}{2}])}=\langle f,f\rangle-\langle \mathcal{J}_k[f],\mathcal{J}_k[f]\rangle.$$

By using that $\langle f, f\rangle$ coincides with the square of the L^2 norm and formula (3.2.15), from the previous identity we get

$$\Big\|f-\mathcal{J}_k[f]\Big\|^2_{L^2([-\frac{T}{2},\frac{T}{2}])}=\|f\|^2_{L^2([-\frac{T}{2},\frac{T}{2}])}-T\sum_{n=-k}^{k}|\widehat{f}(n)|^2.$$

By taking the limit as k goes to ∞ and using Plancherel's formula (see Remark C.2.8), we get the conclusion. $\square$

3. Exercises

Exercise C.3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be the periodic signal*

$$f(t) = \sum_{k\in\mathbb{Z}} \mathrm{rect}(t - 2\,k).$$

Draw the graph of f and compute its Fourier series, by discussing its convergence.

Solution. It is not difficult to see that f is obtained by periodically repeating the rectangular function, extended by 0 to the whole interval $[-1, 1]$. Thus f is 2−periodic. The signal f is piecewise C^1, thus from Theorem C.2.1 we can infer the pointwise convergence of its Fourier series $\mathcal{J}[f]$. More precisely, we have

$$\mathcal{J}[f](t) = \begin{cases} 1, & \text{if } -1/2 < t < 1/2, \\[2mm] \dfrac{1}{2}, & \text{if } t = \pm\dfrac{1}{2}, \\[2mm] 0, & \text{if } t \in \left[-1, -\dfrac{1}{2}\right) \cup \left(\dfrac{1}{2}, 1\right]. \end{cases}$$

We now compute its Fourier coefficients. We have for $n \neq 0$

$$\begin{aligned} \widehat{f}(n) = \frac{1}{2}\int_{-1}^{1} \mathrm{rect}(t)\, e^{-\pi\, n\, i\, t}\, dt = \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-\pi\, n\, i\, t}\, dt &= \frac{1}{2}\left[-\frac{e^{-\pi\, n\, i\, t}}{\pi\, n\, i}\right]_{-\frac{1}{2}}^{\frac{1}{2}} \\ &= \frac{1}{2}\left[\frac{e^{\frac{\pi}{2}\, n\, i}}{\pi\, n\, i} - \frac{e^{-\frac{\pi}{2}\, n\, i}}{\pi\, n\, i}\right] \\ &= \frac{1}{\pi\, n}\,\sin\left(\frac{\pi}{2}\, n\right). \end{aligned}$$

In other words, we obtained

$$\widehat{f}(n) = \frac{1}{\pi\, n}\begin{cases} 0, & \text{if } n \text{ is even}, \\ (-1)^k, & \text{if } n = 2\,k+1. \end{cases}$$

The coefficient $\widehat{f}(0)$ is given by

$$\widehat{f}(0) = \frac{1}{2}\int_{-1}^{1} \mathrm{rect}(t)\, dt = \frac{1}{2}.$$

Finally, the Fourier series is given by

$$\mathcal{J}[f] = \frac{1}{2} + \sum_{k=0}^{\infty} \widehat{f}(2\,k+1)\, e^{\pi\,(2\,k+1)\,i\,t} = \frac{1}{2} + \frac{1}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{2\,k+1}\, e^{\pi\,(2\,k+1)\,i\,t}.$$

We also observe that f is even, thus by Remark C.1.4 we can also rewrite this as a series containing only cosine functions, with coefficients a_k given by (recall (3.1.2))

$$a_0 = \widehat{f}(0), \qquad a_k = 2\,\widehat{f}(k), \quad \text{for } k \geq 1,$$

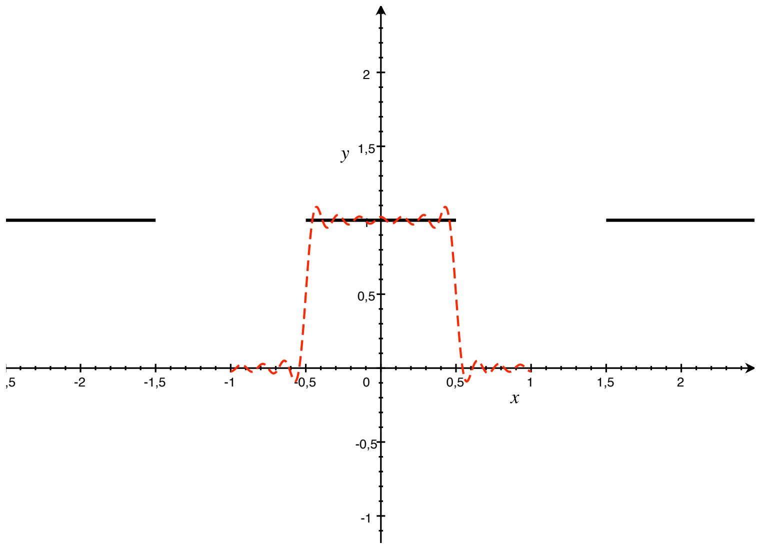

Figure 2. The graph of the signal f of Exercise C.3.1. In red the sum of the first 8 terms of the Fourier expansion.

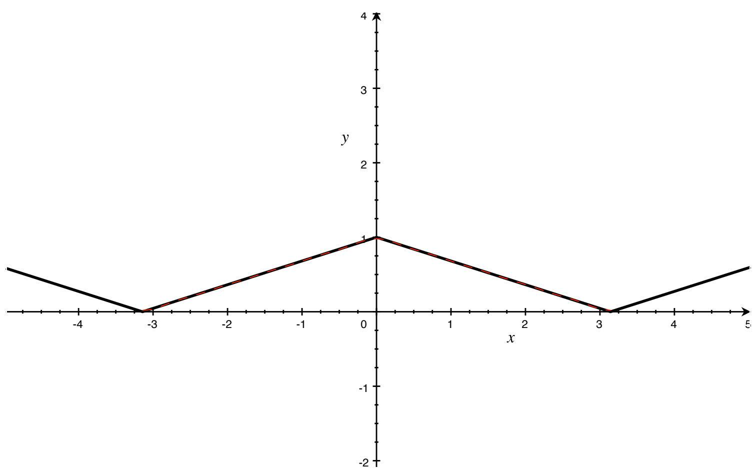

Figure 3. The periodic signal f of Exercise C.3.2.

i.e.

$$\mathcal{J}[f] = \frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2\,k+1} \cos(\pi\,(2\,k+1)\,t).$$

This concludes the exercise. $\square$

Exercise C.3.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be the periodic signal*

$$f(t) = \sum_{k \in \mathbb{Z}} \operatorname{tri}\left(\frac{t}{\pi} - 2\,k\right).$$

Draw the graph of f and compute its Fourier series, by discussing its convergence.

Solution. It is easy to see that f is periodic, with period $T = 2\,\pi$. Moreover, the function f

verifies the assumptions of Theorem C.2.5, thus we have

$$\mathcal{J}[f](t) = f(t), \qquad \text{for every } t \in \left[-\frac{T}{2}, \frac{T}{2}\right],$$

and the convergence of the Fourier series is total.

We first observe that

$$t \mapsto 1 - (|t|/\pi)$$

is a real-valued even function, thus by Lemma C.1.3 we already know that its Fourier coefficients are real and such that $\widehat{f}(-n) = \widehat{f}(n)$. Let us compute them: we have

$$\begin{aligned}
\widehat{f}(0) = \frac{1}{2\,\pi} \int_{-\pi}^{\pi} \left(1 - \frac{|t|}{\pi}\right) dt &= \frac{1}{\pi} \int_0^{\pi} \left(1 - \frac{t}{\pi}\right) dt \\
&= \int_0^1 (1 - s)\, ds = \frac{1}{2}.
\end{aligned}$$

For $n \neq 0$, by using an integration by parts we have

$$\begin{aligned}
\widehat{f}(n) &= \frac{1}{2\,\pi} \int_{-\pi}^{\pi} \left(1 - \frac{|t|}{\pi}\right) e^{-i\,n\,t}\, dt \\
&= \frac{1}{2\,\pi} \int_0^{\pi} \left(1 - \frac{t}{\pi}\right) e^{-i\,n\,t}\, dt\, dt + \frac{1}{2\,\pi} \int_{-\pi}^{0} \left(1 + \frac{t}{\pi}\right) e^{-i\,n\,t}\, dt \\
&= \cancel{\frac{1}{2\,\pi} \left[\left(1 - \frac{t}{\pi}\right) \frac{e^{-i\,n\,t}}{-i\,n}\right]_0^{\pi}} - \frac{1}{2\,\pi^2} \int_0^{\pi} \frac{e^{-i\,n\,t}}{i\,n}\, dt \\
&+ \cancel{\frac{1}{2\,\pi} \left[\left(1 + \frac{t}{\pi}\right) \frac{e^{-i\,n\,t}}{-i\,n}\right]_{-\pi}^{0}} + \frac{1}{2\,\pi^2} \int_{-\pi}^{0} \frac{e^{-i\,n\,t}}{i\,n}\, dt \\
&= -\frac{1}{2\,\pi^2} \int_0^{\pi} \frac{e^{-i\,n\,t}}{i\,n}\, dt + \frac{1}{2\,\pi^2} \int_{-\pi}^{0} \frac{e^{-i\,n\,t}}{i\,n}\, dt \\
&= \frac{1}{2\,\pi^2\,i\,n} \left[\frac{e^{-i\,n\,t}}{i\,n}\right]_0^{\pi} + \frac{1}{2\,\pi^2\,i\,n} \left[\frac{e^{-i\,n\,t}}{-i\,n}\right]_{-\pi}^{0}.
\end{aligned}$$

In conclusion, we get

$$\begin{aligned}
\widehat{f}(n) &= \frac{1}{2\,\pi^2\,i\,n} \left[\frac{e^{-i\,n\,\pi}}{i\,n} - \frac{1}{i\,n}\right] + \frac{1}{2\,\pi^2\,i\,n} \left[\frac{e^{i\,n\,\pi}}{i\,n} - \frac{1}{i\,n}\right] \\
&= -\frac{1}{2\,\pi^2\,n^2}\,[e^{i\,n\,\pi} - 1],
\end{aligned}$$

where we used that $e^{-i\,n\,\pi} = e^{i\,n\,\pi}$. Moreover, we have

$$e^{i\,n\,\pi} = (-1)^n,$$

thus for $n \in \mathbb{Z} \setminus \{0\}$ we finally get

$$\widehat{f}(n) = \frac{1}{n^2\,\pi^2} \begin{cases} 0, & \text{if } n \text{ even}, \\ 2, & \text{if } n \text{ odd}, \end{cases}$$

and thus

$$\mathcal{J}[f](t) = \frac{1}{2} + \frac{2}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{e^{i\,(2\,k+1)\,t}}{(2\,k+1)^2}.$$

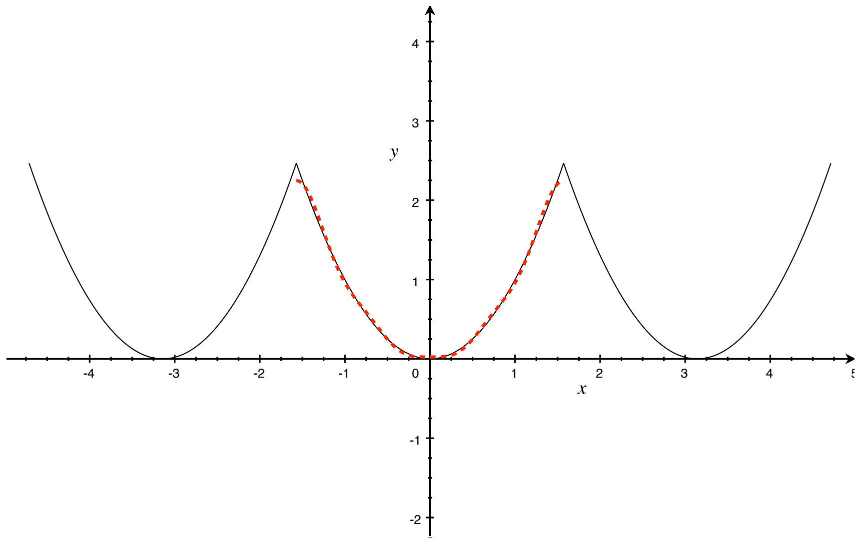

Figure 4. The graph of the signal f of Exercise C.3.3. In red the sum of the first 5 terms of its Fourier series expansion.

As the function f is even, by Remark C.1.4 we can also rewrite this as a series containing only cosine functions, with coefficients a_k given by (recall (3.1.2))

$$a_k = 2\,\widehat{f}(k),$$

i.e.

$$\mathcal{J}[f](t) = \frac{1}{2} + \frac{4}{\pi^2} \sum_{k\in\mathbb{N}} \frac{\cos\Big((2\,k+1)\,t\Big)}{(2\,k+1)^2}.$$

This concludes the exercise. □

Exercise C.3.3. *Let us define*

$$g(t) = t^2 \,\mathrm{rect}\left(\frac{t}{\pi}\right),$$

and

$$f(t) = \sum_{k\in\mathbb{Z}} g(t - k\,\pi).$$

Draw the graph of f and compute its Fourier series, by discussing its convergence.

Solution. It is not difficult to see that the function f is periodic with period $T = \pi$. Moreover, the function f verifies the assumptions of Theorem C.2.5, thus we have

$$\mathcal{J}[f](t) = f(t), \qquad \text{for every } t \in \left[-\frac{T}{2}, \frac{T}{2}\right],$$

and the convergence of the Fourier series is total.

Its Fourier coefficients are given by

$$\begin{aligned}\widehat{f}(n) = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} t^2\, e^{-2\,i\,n\,t}\,dt &= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} t^2\,\cos(2\,n\,t)\,dt \\ &\quad - \frac{i}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} t^2\,\sin(2\,n\,t)\,dt.\end{aligned}$$

We observe that the function $t \mapsto t^2$ is even, thus we get

$$\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} t^2\,\cos(2\,n\,t)\,dt = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} t^2\,\cos(2\,n\,t)\,dt,$$

and

$$\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} t^2\,\sin(2\,n\,t)\,dt = 0.$$

We now observe that with some integration by parts, we obtain for $n \in \mathbb{Z}\setminus\{0\}$

$$\begin{aligned}
\widehat{f}(n) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} t^2\,\cos(2\,n\,t)\,dt &= \cancel{\left[\frac{2}{\pi}\,\frac{\sin(2\,n\,t)}{2\,n}\,t^2\right]_0^{\frac{\pi}{2}}} \\
&\quad + \frac{2}{n\,\pi}\int_0^{\frac{\pi}{2}} t\,(-\sin(2\,n\,t))\,dt \\
&= \frac{2}{n\,\pi}\left\{\left[t\,\frac{\cos(2\,n\,t)}{2\,n}\right]_0^{\frac{\pi}{2}} - \cancel{\int_0^{\frac{\pi}{2}} \frac{\cos(2\,n\,t)}{2\,n}\,dt}\right\} \\
&= \frac{\cos(n\,\pi)}{2\,n^2} = \frac{(-1)^n}{2\,n^2}.
\end{aligned}$$

As for the coefficient $\widehat{f}(0)$, we have

$$\widehat{f}(0) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} t^2\,dt = \frac{\pi^2}{12}.$$

In conclusion, we get

$$\mathcal{J}[f](t) = \frac{\pi^2}{12} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{(-1)^n}{2\,n^2}\,e^{2\,n\,i\,t},$$

which can also be rewritten as

$$\mathcal{J}[f](t) = \frac{\pi^2}{12} + \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}\,\cos(2\,k\,t).$$

This concludes the exercise. $\square$

Exercise C.3.4 (Square wave)**.** *Let us define*

$$g(t) = \mathrm{rect}\left(t - \frac{1}{2}\right) - \mathrm{rect}\left(t + \frac{1}{2}\right).$$

We consider the square wave signal, *defined by*

$$\square(t) = \sum_{n\in\mathbb{Z}} g(t + 2\,n).$$

Draw the graph of $\square$ *and compute its Fourier series, by discussing its convergence.*

Solution. We first observe that $\square$ is $2-$periodic, since

$$\square(t+2) = \sum_{n\in\mathbb{Z}} g(t+2+2\,n) = \sum_{n\in\mathbb{Z}} g(t+2\,(n+1)) = \sum_{m\in\mathbb{Z}} g(t+2\,m) = \square(t).$$

The square wave signal is piecewise C^1, thus from Theorem C.2.1 we can infer the pointwise convergence of its Fourier series $\mathcal{J}[\square]$. More precisely, we have

$$\mathcal{J}[\square](t) = \square(t), \qquad \text{for } t \in (-1,0)\cup(0,1),$$

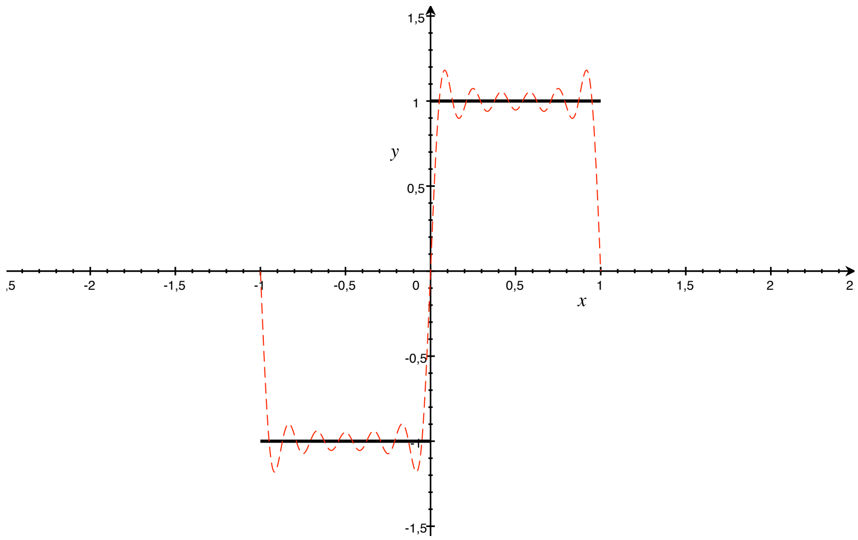

Figure 5. The square wave signal. In red the sum of the first 6 terms of its Fourier expansion.

and

$$\mathcal{J}[\square](0) = \mathcal{J}[\square](-1) = \mathcal{J}[\square](1) = 0.$$

Let us compute its Fourier coefficients: for every $n \in \mathbb{Z} \setminus \{0\}$, we have

$$\begin{aligned}
\widehat{\square}(n) = \frac{1}{2}\int_{-1}^{1} g(t)\, e^{-\pi\, n\, i\, t}\, dt &= \frac{1}{2}\int_{0}^{1} e^{-\pi\, n\, i\, t}\, dt - \frac{1}{2}\int_{-1}^{0} e^{-\pi\, n\, i\, t}\, dt \\
&= \frac{1}{2}\left[\frac{e^{-\pi\, n\, i\, t}}{-\pi\, n\, i}\right]_0^1 - \frac{1}{2}\left[\frac{e^{-\pi\, n\, i\, t}}{-\pi\, n\, i}\right]_{-1}^0 \\
&= \frac{1}{2}\,\frac{1}{\pi\, n\, i} - \frac{1}{2}\,\frac{e^{-\pi\, n\, i}}{\pi\, n\, i} + \frac{1}{2}\,\frac{1}{\pi\, n\, i} - \frac{1}{2}\,\frac{e^{\pi\, n\, i}}{\pi\, n\, i} \\
&= \frac{1}{\pi\, n\, i}\,(1 - e^{n\,\pi\, i}),
\end{aligned}$$

which gives

$$\widehat{\square}(n) = -\frac{i}{\pi\, n}\begin{cases} 0, & \text{if } n \text{ even}, \\ 2, & \text{if } n \text{ odd}. \end{cases}$$

On the other hand, we easily see that

$$\widehat{\square}(0) = 0.$$

Observe that $\square$ is real-valued and odd and what we obtained is perfectly in accordance with Lemma C.1.3. The Fourier series is then given by

$$\mathcal{J}[\square](t) = \sum_{n\in\mathbb{Z}} -\frac{2\, i}{\pi\,(2\, n+1)}\, e^{(2\, n+1)\,\pi\, i\, t}.$$

Since f is real-valued and even, we know by Remark C.1.4 this can be written as a series of sine functions, with coefficients b_n given by (recall (3.1.2))

$$\widehat{f}(n) = \frac{-i\, b_n}{2},$$

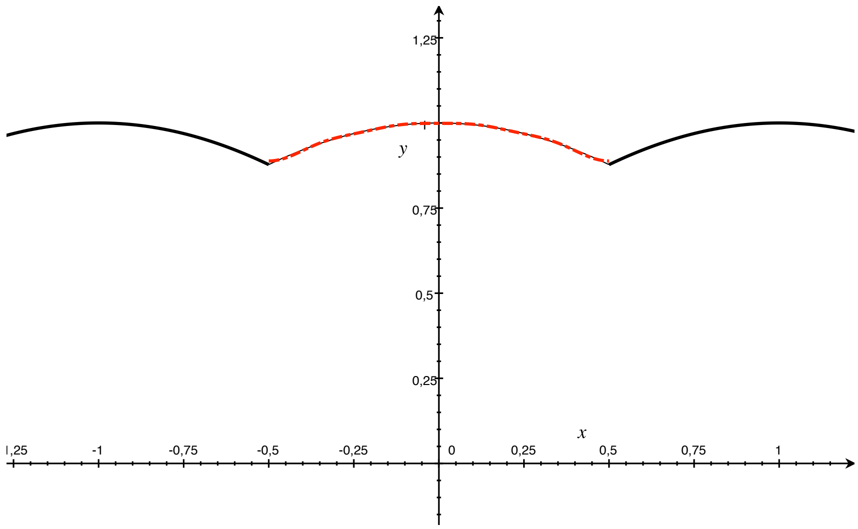

Figure 6. In red, the partial sum of the Fourier series of Exercise C.3.5, corresponding to the first 5 terms.

that is

$$\mathcal{J}[\square](t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin((2\,n+1)\,\pi\,t)}{2\,n+1}.$$

This concludes the exercise. $\square$

Exercise C.3.5. *Let us set*

$$g(t) = \cos(t)\,\mathrm{rect}(t),$$

and consider the periodic signal $f : \mathbb{R} \to \mathbb{C}$ defined by

$$f(t) = \sum_{k\in\mathbb{Z}} g(t-k).$$

Draw the graph of f and compute its Fourier series, by discussing its convergence.

Soluzione. The signal f is $1-$periodic, since we have

$$f(t+1) = \sum_{k\in\mathbb{Z}} g(t+1-k) = \sum_{m\in\mathbb{Z}} g(t-m) = f(t).$$

Moreover, this is piecewise C^1 signal, globally continuous on $\mathbb{R}$, because g is continuous on $[-1/2, 1/2]$ and we have

$$g\left(\frac{1}{2}\right) = g\left(-\frac{1}{2}\right).$$

By Theorem C.2.5 we thus have that the Fourier series $\mathcal{J}[f]$ is totally converging. Let us now compute the Fourier coefficients of f: at this aim, it is useful to observe that

$$\cos t = \frac{e^{i\,t} + e^{-i\,t}}{2}.$$

We thus have for every $n \in \mathbb{Z}$

$$
\begin{aligned}
\widehat{f}(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos t \, e^{-2\pi n i t}\, dt &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{e^{-(2\pi n-1)\, i\, t}}{2}\, dt + \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{e^{-(2\pi n+1)\, i\, t}}{2}\, dt \\
&= \left[-\frac{e^{-(2\pi n-1)\, i\, t}}{2\, i\, (2\pi n-1)}\, dt\right]_{-\frac{1}{2}}^{\frac{1}{2}} + \left[-\frac{e^{-(2\pi n+1)\, i\, t}}{2\, i\, (2\pi n+1)}\, dt\right]_{-\frac{1}{2}}^{\frac{1}{2}} \\
&= \frac{1}{2\pi n-1}\, \frac{e^{(2\pi n-1)\frac{i}{2}} - e^{-(2\pi n-1)\frac{i}{2}}}{2\, i} \\
&+ \frac{1}{2\pi n+1}\, \frac{e^{(2\pi n+1)\frac{i}{2}} - e^{-(2\pi n+1)\frac{i}{2}}}{2\, i}.
\end{aligned}
$$

If we now recall that

$$
\sin t = \frac{e^{i\,t} - e^{-i\,t}}{2\, i},
$$

we get

$$
\widehat{f}(n) = \frac{1}{2\pi n-1}\, \sin\left(\pi n - \frac{1}{2}\right) + \frac{1}{2\pi n+1}\, \sin\left(\pi n + \frac{1}{2}\right).
$$

Observe that by using trigonometric formulas, we have

$$
\sin\left(\pi n - \frac{1}{2}\right) = -\cos(\pi n)\, \sin\left(\frac{1}{2}\right) = -(-1)^n\, \sin\left(\frac{1}{2}\right),
$$

and

$$
\sin\left(\pi n + \frac{1}{2}\right) = \cos(\pi n)\, \sin\left(\frac{1}{2}\right) = (-1)^n\, \sin\left(\frac{1}{2}\right),
$$

which yield

$$
\begin{aligned}
\widehat{f}(n) &= \left[\frac{1}{2\pi n+1} - \frac{1}{2\pi n-1}\right]\, (-1)^n\, \sin\left(\frac{1}{2}\right) \\
&= \frac{2\cdot(-1)^{n+1}}{4\pi^2 n^2 - 1}\, \sin\left(\frac{1}{2}\right).
\end{aligned}
$$

In conclusion, we get

$$
\mathcal{J}[f](t) = 2\, \sin\left(\frac{1}{2}\right) \sum_{n\in\mathbb{Z}} \frac{(-1)^{n+1}}{4\pi^2 n^2 - 1}\, e^{2\pi n i t}.
$$

Finally, let us observe that f is even, we can thus rewrite $\mathcal{J}[f]$ as a series of cosines. By recalling the relations

$$
\widehat{f}(0) = a_0, \qquad \widehat{f}(n) = \frac{a_n}{2} \text{ for } n \geq 1, \qquad \widehat{f}(n) = \frac{a_{-n}}{2} \text{ for } n \leq -1,
$$

we get

$$
\mathcal{J}[f](t) = 2\, \sin\left(\frac{1}{2}\right) + 4\, \sin\left(\frac{1}{2}\right) \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{4\pi^2 k^2 - 1}\, \cos(2\pi k t).
$$

This concludes the exercise. $\square$

Appendix D

Harmonic functions in the plane

1. Examples

We have seen in Chapter 1 that a function $u : \Omega \to \mathbb{R}$ of class C^2 on the open set $\Omega \subset \mathbb{R}^2$ is said to be *harmonic* if it verifies

$$u_{xx}(x,y) + u_{yy}(x,y) = 0, \qquad \text{for every } (x,y) \in \Omega.$$

We set

$$\Delta u = \operatorname{div}(\nabla u) = u_{xx} + u_{yy},$$

this differential operator is called *Laplacian*. Then u is harmonic if $\Delta u = 0$.

By Remark 1.4.14, we know that by taking the real or imaginary part of a holomorphic function, we get a harmonic function in the plane. Let us have a look at some explicit examples.

Example D.1.1. Let us take $f(z) = e^z = e^x (\cos y + i \sin y)$, where as usual we write $z = x + i\,y$. Then the functions

$$u(x,y) = \operatorname{Re}(e^z) = e^x \cos y \qquad \text{and} \qquad v(x,y) = \operatorname{Im}(e^z) = e^x \sin y,$$

are harmonic in $\mathbb{R}^2$.

Example D.1.2. Similarly, by considering $f(z) = \operatorname{Log} z = \log|z| + i \operatorname{Arg}(z)$ and recalling that this is holomorphic in $\mathbb{C}^{**}$, we get that

$$u(x,y) = \operatorname{Re}(\operatorname{Log} z) = \log\sqrt{x^2+y^2},$$

is harmonic in $\mathbb{R}^2 \setminus \{(x,0) : x \le 0\}$. More precisely, by direct computation, we can see that u is harmonic in $\mathbb{R}^2 \setminus \{(0,0)\}$.

2. Construction of conjugate pairs

We have seen in Remark 1.4.14 that two harmonic functions $u, v : \Omega \to \mathbb{R}$ on the open set $\Omega \subset \mathbb{R}^2$ are said to be *conjugate* if they satisfy

$$\begin{cases} u_x &= & v_y, \\ u_y &= & -u_x, \end{cases}$$

i.e. the system of *Cauchy-Riemann equations*. It is a remarkable fact that given a harmonic function u on Ω, we can always construct another harmonic function v on Ω such that (u, v) are conjugate, provided the open set Ω is “nice”.

We first need to recall some facts from the 2nd year course in Mathematical Analysis.

Definition D.2.1. Let $\Omega \subset \mathbb{R}^2$ be a non-empty open set. We say that Ω is *starshaped with respect to a point* $(x_0, y_0) \in \Omega$ if for every $(x, y) \in \Omega$ the segment joining (x, y) and (x_0, y_0) is entirely contained in Ω.

Definition D.2.2. Let $\Omega \subset \mathbb{R}^2$ be an open set. Let $\mathbf{F} : \Omega \to \mathbb{R}^2$ be a vector field of class $C^1(\Omega)$. We say that:

- $\mathbf{F}$ is *irrotational* if
$$\frac{\partial \mathbf{F}_2}{\partial x}(x, y) - \frac{\partial \mathbf{F}_1}{\partial y}(x, y) = 0, \qquad \text{for every } (x, y) \in \Omega;$$
- $\mathbf{F}$ is *conservative* if there exists a function $U : \Omega \to \mathbb{R}$ of class $C^2(\Omega)$ such that
$$\mathbf{F}(x, y) = \nabla U(x, y), \qquad \text{for every } (x, y) \in \Omega.$$

We can now state the main result of this section.

Theorem D.2.3. *Let $\Omega \subset \mathbb{R}^2$ be a starshaped set. If $u : \Omega \to \mathbb{R}$ is a harmonic function Ω, then there exists $v : \Omega \to \mathbb{R}$ such that v is harmonic in Ω and*

$$\begin{cases} u_x &= & v_y, \\ u_y &= & -u_x, \end{cases}$$

in Ω.

Proof. We start by defining the vector field in the plane

$$\mathbf{F}(x, y) = (-u_y(x, y), u_x(x, y)).$$

Observe that $\mathbf{F}$ coincides with the anti-cloackwise rotation of ∇u by $\pi/2$. Since u is harmonic in Ω, we get that $\mathbf{F}$ is irrotational in Ω, i.e.

$$\frac{\partial \mathbf{F}_2}{\partial x} - \frac{\partial \mathbf{F}_1}{\partial y} = -u_{yy} - u_{xx} = 0.$$

By recalling that “*on a starshaped open set a vector field of class C^1 is conservative if and only if is irrotational*”, we get that there exists a C^2 function $v : \Omega \to \mathbb{R}$ such that

$$\mathbf{F}(x, y) = \nabla v(x, y), \qquad \text{for every } (x, y) \in \Omega.$$

By recalling the definition of $\mathbf{F}$, this is the same as

$$-u_y = v_x \qquad \text{and} \qquad u_x = v_y.$$

In other words, u and v solve the system of Cauchy-Riemann equations. The fact that v is harmonic now follows as in Remark 1.4.14. □

3. The mean value property

Harmonic functions have the following remarkable property, which is a consequence of Cauchy's integral formula (i.e. Theorem 1.6.14).

Theorem D.3.1. *Let $\Omega \subset \mathbb{R}^2$ be a starshaped open set. Let $u : \Omega \to \mathbb{R}$ be a harmonic function in Ω. For every point $(x_0, y_0) \in \Omega$ and every $r > 0$ such that $B_r((x_0, y_0)) \subset \Omega$, we have*

$$u(x_0, y_0) = \frac{1}{2\,\pi\,r} \int_{\partial B_r((x_0,y_0))} u(x, y)\,d\ell. \tag{4.3.1}$$

In other words, the value of u in a point (x_0, y_0) coincides with the integral mean of u on the boundary of any ball centered at the same point.

Proof. By using Theorem D.2.3, we know that there exists $v : \Omega \to \mathbb{R}$ harmonic such that u and v are conjugate. Thus by Corollary 1.4.9 the function

$$f(z) = f(x + i\,y) = u(x, y) + i\,v(x, y),$$

is holomorphic in Ω. By Theorem 1.6.14, if we set $z_0 = x_0 + i\,y_0$, we have

$$f(z_0) = \frac{1}{2\,\pi\,i} \int_{\gamma_r(z_0)} \frac{f(z)}{z - z_0}\,dz,$$

where

$$\gamma_r(z_0) = r\,e^{i\,t} + z_0, \qquad t \in [0, 2\,\pi].$$

Observe that $\gamma_r(z_0)$ is a smooth parametrization of $\partial B_r((x_0, y_0))$ with positive orientation. We now write explicitly the integral above, in terms of its real and imaginary parts:

$$\begin{aligned}
&u(x_0, y_0) + i\,v(x_0, y_0)\\
&\quad= \frac{1}{2\,\pi} \int_0^{2\,\pi} \frac{u(x_0 + r\,\cos t, y_0 + r\,\sin t) + i\,v(x_0 + r\,\cos t, y_0 + r\,\sin t)}{r\,\cos t + i\,r\,\sin t}\,(r\,\cos t + i\,r\,\sin t)\,dt\\
&\quad= \frac{1}{2\,\pi} \int_0^{2\,\pi} u(x_0 + r\,\cos t, y_0 + r\,\sin t)\,dt\\
&\quad+ \frac{i}{2\,\pi} \int_0^{2\,\pi} v(x_0 + r\,\cos t, y_0 + r\,\sin t)\,dt\\
&\quad= \frac{1}{2\,\pi\,r} \int_{\partial B_r((x_0,y_0))} u(x, y)\,d\ell\\
&\quad+ \frac{i}{2\,\pi\,r} \int_{\partial B_r((x_0,y_0))} v(x, y)\,d\ell.
\end{aligned}$$

Thus we get

$$u(x_0, y_0) = \frac{1}{2\,\pi\,r} \int_{\partial B_r((x_0,y_0))} u(x, y)\,d\ell \qquad \text{and} \qquad v(x_0, y_0) = \frac{1}{2\,\pi\,r} \int_{\partial B_r((x_0,y_0))} v(x, y)\,d\ell.$$

This concludes the proof. □

Corollary D.3.2. *Let $\Omega \subset \mathbb{R}^2$ be a starshaped open set. Let $u : \Omega \to \mathbb{R}$ be a harmonic function in Ω. For every point $(x_0, y_0) \in \Omega$ and every $R > 0$ such that $B_R((x_0, y_0)) \subset \Omega$, we have*

$$u(x_0, y_0) = \frac{1}{\pi\,R^2} \int_{B_R((x_0,y_0))} u(x, y)\,dx\,dy.$$

Proof. We fix $R > 0$ as in the statement and use (4.3.1) for $0 < r \le R$, i.e.

$$2\,\pi\,r\,u(x_0, y_0) = \int_{\partial B_r((x_0,y_0))} u(x, y)\,d\ell.$$

By integrating this formula in r, we get

$$\begin{aligned}
\pi\,R^2\,u(x_0, y_0) &= \int_0^R \left(\int_{\partial B_r((x_0,y_0))} u(x, y)\,d\ell\right) dr \\
&= \int_0^R \left(\int_0^{2\,\pi} u(x_0 + r\,\cos t, y_0 + r\,\sin t)\,r\,dt\right) dr \\
&= \int_0^R \int_0^{2\,\pi} u(x_0 + r\,\cos t, y_0 + r\,\sin t)\,r\,dr\,dt.
\end{aligned}$$

Observe that by using the polar coordinates, we have

$$\int_{B_R((x_0,y_0))} u(x, y)\,dx\,dy = \int_0^{2\,\pi} \int_0^R u(x_0 + r\,\cos t, y_0 + r\,\sin t)\,r\,dr\,dt.$$

This gives the desired conclusion. $\square$

4. Harmonic functions in the disk

We now suppose to work in a disk D of radius $R > 0$, centered for simplicity at the origin $(0, 0)$. In this case, we can introduce the polar coordinates

$$x = \varrho\,\cos\vartheta \qquad y = \varrho\,\sin\vartheta, \qquad 0 \le \varrho \le R, 0 \le \vartheta \le 2\,\pi.$$

Thus, given a function $u : D \to \mathbb{R}$ of class $C^2(D)$, we want to write its Laplacian in terms of the new coordinates ϱ and ϑ.

We first use the chain rule for functions of several variables for the function $u(x, y) = u(\varrho\,\cos\vartheta, \varrho\,\sin\vartheta)$, so to get

$$\frac{\partial u}{\partial \varrho} = \cos\vartheta\,\frac{\partial u}{\partial x} + \sin\vartheta\,\frac{\partial u}{\partial y}, \tag{4.4.1}$$

and

$$\frac{\partial u}{\partial \vartheta} = -\varrho\,\sin\vartheta\,\frac{\partial u}{\partial x} + \varrho\,\cos\vartheta\,\frac{\partial u}{\partial y}. \tag{4.4.2}$$

We now wish to invert these relations and write

$$\frac{\partial u}{\partial x} \qquad \text{and} \qquad \frac{\partial u}{\partial y},$$

in terms of

$$\frac{\partial u}{\partial \varrho} \qquad \text{and} \qquad \frac{\partial u}{\partial \vartheta}.$$

At this aim, we multiply equation (4.4.1) by $\varrho\,\sin\vartheta$, multiply equation (4.4.2) by $\cos\vartheta$ and sum the two relevant equations. We get

$$\begin{aligned}
\varrho\,\sin\vartheta\,\frac{\partial u}{\partial \varrho} + \cos\vartheta\,\frac{\partial u}{\partial \vartheta} &= \cancel{\varrho\,\sin\vartheta\,\cos\vartheta\,\frac{\partial u}{\partial x}} + \varrho\,\sin^2\vartheta\,\frac{\partial u}{\partial y}\,\sin\vartheta \\
&\quad \cancel{-\varrho\,\cos\vartheta\,\sin\vartheta\,\frac{\partial u}{\partial x}} + \varrho\,\cos^2\vartheta\,\frac{\partial u}{\partial y}\,\sin\vartheta.
\end{aligned}$$

By recalling that $\cos^2\vartheta + \sin^2\vartheta = 1$, we obtain

$$\boxed{\frac{\partial u}{\partial y} = \sin\vartheta\,\frac{\partial u}{\partial \varrho} + \frac{\cos\vartheta}{\varrho}\,\frac{\partial u}{\partial \vartheta}} \tag{4.4.3}$$

In order to find $\partial u/\partial x$, we argue in a similar fashion: we multiply equation (4.4.1) by $\varrho\cos\vartheta$, multiply equation (4.4.2) by $-\sin\vartheta$ and the take the sum. We get

$$\begin{aligned}\varrho\cos\vartheta\,\frac{\partial u}{\partial \varrho} - \sin\vartheta\,\frac{\partial u}{\partial \vartheta} &= \varrho\cos^2\vartheta\,\frac{\partial u}{\partial x} + \cancel{\varrho\sin\vartheta\cos\vartheta\,\frac{\partial u}{\partial y}\sin\vartheta} \\ &\quad + \varrho\sin^2\vartheta\,\frac{\partial u}{\partial x} + \cancel{\varrho\cos\vartheta\sin\vartheta\,\frac{\partial u}{\partial y}\sin\vartheta}.\end{aligned}$$

We use again that $\cos^2\vartheta + \sin^2\vartheta = 1$, this yields

$$\boxed{\frac{\partial u}{\partial x} = \cos\vartheta\,\frac{\partial u}{\partial \varrho} - \frac{\sin\vartheta}{\varrho}\,\frac{\partial u}{\partial \vartheta}}. \tag{4.4.4}$$

Equations (4.4.3) and (4.4.3) give the expression of ∇u in terms of the polar coordinates. Let us now proceed to get the expression of the Laplacian: by observing that

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\,\frac{\partial u}{\partial x},$$

and

$$u_{yy} = \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\,\frac{\partial u}{\partial y},$$

we need to iterate (4.4.3) and (4.4.4). Thus we get

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\,\frac{\partial u}{\partial x} &= \left(\cos\vartheta\,\frac{\partial}{\partial \varrho} - \frac{\sin\vartheta}{\varrho}\,\frac{\partial}{\partial \vartheta}\right)\left(\cos\vartheta\,\frac{\partial u}{\partial \varrho} - \frac{\sin\vartheta}{\varrho}\,\frac{\partial u}{\partial \vartheta}\right) \\ &= \cos\vartheta\,\frac{\partial}{\partial \varrho}\left(\cos\vartheta\,\frac{\partial u}{\partial \varrho} - \frac{\sin\vartheta}{\varrho}\,\frac{\partial u}{\partial \vartheta}\right) \\ &\quad - \frac{\sin\vartheta}{\varrho}\,\frac{\partial}{\partial \vartheta}\left(\cos\vartheta\,\frac{\partial u}{\partial \varrho} - \frac{\sin\vartheta}{\varrho}\,\frac{\partial u}{\partial \vartheta}\right) \\ &= \cos^2\vartheta\,\frac{\partial^2 u}{\partial \varrho^2} - \cos\vartheta\,\sin\vartheta\,\frac{\partial}{\partial \varrho}\left(\frac{1}{\varrho}\,\frac{\partial u}{\partial \vartheta}\right) \\ &\quad - \frac{\sin\vartheta}{\varrho}\,\frac{\partial}{\partial \vartheta}\left(\cos\vartheta\,\frac{\partial u}{\partial \varrho}\right) + \frac{\sin\vartheta}{\varrho^2}\,\frac{\partial}{\partial \vartheta}\left(\sin\vartheta\,\frac{\partial u}{\partial \vartheta}\right)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\,\frac{\partial u}{\partial y} &= \left(\sin\vartheta\,\frac{\partial}{\partial \varrho} + \frac{\cos\vartheta}{\varrho}\,\frac{\partial}{\partial \vartheta}\right)\left(\sin\vartheta\,\frac{\partial u}{\partial \varrho} + \frac{\cos\vartheta}{\varrho}\,\frac{\partial u}{\partial \vartheta}\right) \\ &= \sin\vartheta\,\frac{\partial}{\partial \varrho}\left(\sin\vartheta\,\frac{\partial u}{\partial \varrho} + \frac{\cos\vartheta}{\varrho}\,\frac{\partial u}{\partial \vartheta}\right) \\ &\quad + \frac{\cos\vartheta}{\varrho}\,\frac{\partial}{\partial \vartheta}\left(\sin\vartheta\,\frac{\partial u}{\partial \varrho} + \frac{\cos\vartheta}{\varrho}\,\frac{\partial u}{\partial \vartheta}\right) \qquad . \\ &= \sin^2\vartheta\,\frac{\partial^2 u}{\partial \varrho^2} + \cos\vartheta\,\sin\vartheta\,\frac{\partial}{\partial \varrho}\left(\frac{1}{\varrho}\,\frac{\partial u}{\partial \vartheta}\right) \\ &\quad + \frac{\cos\vartheta}{\varrho}\,\frac{\partial}{\partial \vartheta}\left(\sin\vartheta\,\frac{\partial u}{\partial \varrho}\right) + \frac{\cos\vartheta}{\varrho^2}\,\frac{\partial}{\partial \vartheta}\left(\cos\vartheta\,\frac{\partial u}{\partial \vartheta}\right)\end{aligned}$$

When we sum up the last two quantities, we get

$$\begin{aligned}\Delta u = {} & \cos^2\vartheta\,\frac{\partial^2 u}{\partial\varrho^2} \cancel{- \cos\vartheta\,\sin\vartheta\,\frac{\partial}{\partial\varrho}\left(\frac{1}{\varrho}\,\frac{\partial u}{\partial\vartheta}\right)}\\ & - \frac{\sin\vartheta}{\varrho}\,\frac{\partial}{\partial\vartheta}\left(\cos\vartheta\,\frac{\partial u}{\partial\varrho}\right) + \frac{\sin\vartheta}{\varrho^2}\,\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\,\frac{\partial u}{\partial\vartheta}\right)\\ & + \sin^2\vartheta\,\frac{\partial^2 u}{\partial\varrho^2} \cancel{+ \cos\vartheta\,\sin\vartheta\,\frac{\partial}{\partial\varrho}\left(\frac{1}{\varrho}\,\frac{\partial u}{\partial\vartheta}\right)}\\ & + \frac{\cos\vartheta}{\varrho}\,\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\,\frac{\partial u}{\partial\varrho}\right) + \frac{\cos\vartheta}{\varrho^2}\,\frac{\partial}{\partial\vartheta}\left(\cos\vartheta\,\frac{\partial u}{\partial\vartheta}\right),\end{aligned}$$

that is, by using the fundamental trigonometric identity,

$$\begin{aligned}\Delta u = {} & \frac{\partial^2 u}{\partial\varrho^2}\\ & - \frac{\sin\vartheta}{\varrho}\,\frac{\partial}{\partial\vartheta}\left(\cos\vartheta\,\frac{\partial u}{\partial\varrho}\right) + \frac{\sin\vartheta}{\varrho^2}\,\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\,\frac{\partial u}{\partial\vartheta}\right)\\ & + \frac{\cos\vartheta}{\varrho}\,\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\,\frac{\partial u}{\partial\varrho}\right) + \frac{\cos\vartheta}{\varrho^2}\,\frac{\partial}{\partial\vartheta}\left(\cos\vartheta\,\frac{\partial u}{\partial\vartheta}\right),\end{aligned}$$

We are now left to compute the last derivatives: this yields

$$\begin{aligned}\Delta u = {} & \frac{\partial^2 u}{\partial\varrho^2}\\ & + \frac{\sin^2\vartheta}{\varrho}\,\frac{\partial u}{\partial\varrho} - \cancel{\frac{\sin\vartheta\,\cos\vartheta}{\varrho}\,\frac{\partial^2 u}{\partial\varrho\,\partial\vartheta}}\\ & + \cancel{\frac{\sin\vartheta\,\cos\vartheta}{\varrho^2}\,\frac{\partial u}{\partial\vartheta}} + \frac{\sin^2\vartheta}{\varrho^2}\,\frac{\partial^2 u}{\partial\vartheta^2}\\ & + \frac{\cos^2\vartheta}{\varrho}\,\frac{\partial u}{\partial\varrho} + \cancel{\frac{\cos\vartheta\,\sin\vartheta}{\varrho}\,\frac{\partial^2 u}{\partial\varrho\,\partial\vartheta}}\\ & - \cancel{\frac{\cos\vartheta\,\sin\vartheta}{\varrho^2}\,\frac{\partial u}{\partial\vartheta}} + \frac{\cos^2\vartheta}{\varrho^2}\,\frac{\partial^2 u}{\partial\vartheta^2}\\ = {} & \frac{\partial^2 u}{\partial\varrho^2} + \frac{\cos^2\vartheta+\sin^2\vartheta}{\varrho}\,\frac{\partial u}{\partial\varrho} + \frac{\cos^2\vartheta+\sin^2\vartheta}{\varrho^2}\,\frac{\partial^2 u}{\partial\vartheta^2}.\end{aligned}$$

In conclusion, we get

$$\boxed{\Delta u = \frac{\partial^2 u}{\partial\varrho^2} + \frac{1}{\varrho}\,\frac{\partial u}{\partial\varrho} + \frac{1}{\varrho^2}\,\frac{\partial^2 u}{\partial\vartheta^2}} \tag{4.4.5}$$

Example D.4.1 (Spherical harmonics)**.** Let $n \in \mathbb{N}\setminus\{0\}$, we consider the functions in polar coordinates

$$u_n(\varrho,\vartheta) = \varrho^n\,\cos(n\,\vartheta) \qquad \text{and} \qquad v_n(\varrho,\vartheta) = \varrho^n\,\sin(n\,\vartheta).$$

By using formula (4.4.5), it is easy to see that these are harmonic functions. Indeed, we have

$$
\begin{aligned}
\Delta u_n &= \frac{\partial^2}{\partial \varrho^2}(\varrho^2\, \cos(n\,\vartheta)) \\
&+ \frac{1}{\varrho}\,\frac{\partial}{\partial \varrho}(\varrho^n\, \cos(n\,\vartheta)) \\
&+ \frac{1}{\varrho^2}\,\frac{\partial^2}{\partial \vartheta^2}(\varrho^n\, \cos(n\,\vartheta)) \\
&= n\,(n-1)\,\varrho^{n-2}\, \cos(n\,\vartheta) + n\,\varrho^{n-2}\, \cos(n\,\vartheta) - n^2\,\varrho^{n-2}\, \cos(n\,\vartheta) \\
&= (n^2 - n + n - n^2)\,\varrho^{n-2}\, \cos(n\,\vartheta) = 0.
\end{aligned}
$$

Similar computations work for the function v_n. We recall that by using polar coordinates in the complex plane, i.e. $z = \varrho\,(\cos\vartheta + i\,\sin\vartheta)$, then we know that

$$z^n = \varrho^n\,(\cos(n\,\vartheta) + i\,\sin(n\,\vartheta)).$$

Thus we can write the functions above as

$$u_n = \mathrm{Re}(z^n) \qquad \text{and} \qquad v_n = \mathrm{Im}(z^n).$$

This shows that (u_n, v_n) is a conjugate pair, for every $n \in \mathbb{N}\setminus\{0\}$. We also observe that the relation above between u_n (or v_n) and z^n, permits to find u_n and v_n as functions of the standard cartesian variables (x, y). Indeed, we have

$$u_n(x,y) = \mathrm{Re}(z^n) = \mathrm{Re}((x+i\,y)^n) = \mathrm{Re}\left(\sum_{k=0}^{n} \binom{n}{k} x^k\,(i\,y)^{n-k}\right),$$

and

$$v_n(x,y) = \mathrm{Im}(z^n) = \mathrm{Im}((x+i\,y)^n) = \mathrm{Im}\left(\sum_{k=0}^{n} \binom{n}{k} x^k\,(i\,y)^{n-k}\right).$$

For example, for $n = 4$ we have

$$u_4(x,y) = \mathrm{Re}((i\,y)^4 + 4\,x\,(i\,y)^3 + 6\,x^2\,(i\,y)^2 + 4\,x^3\,(i\,y) + x^4) = y^4 - 6\,x^2\,y^2 + x^4,$$

and

$$v_4(x,y) = \mathrm{Im}((i\,y)^4 + 4\,x\,(i\,y)^3 + 6\,x^2\,(i\,y)^2 + 4\,x^3\,(i\,y) + x^4) = -4\,x\,y^3 + 4\,x^3\,y.$$

The functions (u_n, v_n) are called *spherical harmonics of order* n.

5. Exercises

Exercise D.5.1. *Find an explicit solution $u \in C^2(\mathbb{R}_+ \times \mathbb{R})$ of the following two-dimensional boundary value problem*

$$
\begin{cases}
\Delta u(x,y) = 0, & \text{in } (x,y) \in \mathbb{R}_+ \times \mathbb{R}, \\
u(0,y) = \operatorname{sinc}\left(\dfrac{y}{2\,\pi}\right), & y \in \mathbb{R}.
\end{cases}
$$

Solution. We have seen that

$$\operatorname{sinc}\left(\frac{y}{2\,\pi}\right) = \mathcal{F}[\mathrm{rect}](y) = \mathcal{B}[\mathrm{rect}](i\,y),$$

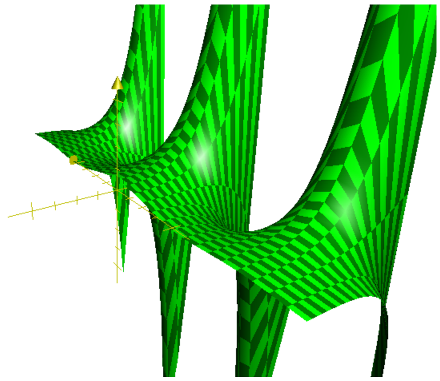

Figure 1. The solution of Exercise D.5.1.

with

$$\mathcal{B}[\mathrm{rect}](z) = \frac{e^{\frac{z}{2}} - e^{-\frac{z}{2}}}{z}, \qquad \text{for } z \in \mathbb{C},$$

see Example 4.8.10. The function $\mathcal{B}[\mathrm{rect}]$ is holomorphic, thus by recalling Remark 1.4.14

$$u(x,y) = \mathrm{Re}\left(\mathcal{B}[\mathrm{rect}](x+i\,y)\right) \qquad \text{and} \qquad v(x,y) = \mathrm{Im}\left(\mathcal{B}[\mathrm{rect}](x+i\,y)\right),$$

are two harmonic functions. Moreover, we have

$$u(0,y) = \mathrm{Re}\left(\mathcal{B}[\mathrm{rect}](i\,y)\right) = \mathrm{sinc}\left(\frac{y}{2\,\pi}\right),$$

thus u is a solution of the boundary value problem. We are only left with computing explicitly the real part of $\mathcal{B}[\mathrm{rect}](x+i\,y)$: we have

$$\begin{aligned}
\mathcal{B}[\mathrm{rect}](z) &= \frac{e^{\frac{x}{2}}\,e^{i\frac{y}{2}} - e^{-\frac{x}{2}}\,e^{-i\,\frac{y}{2}}}{x+i\,y} \\
&= \frac{e^{\frac{x}{2}}\,\cos\left(\frac{y}{2}\right) + i\,e^{\frac{x}{2}}\,\sin\left(\frac{y}{2}\right) - e^{-\frac{x}{2}}\,\cos\left(\frac{y}{2}\right) + i\,e^{-\frac{x}{2}}\,\sin\left(\frac{y}{2}\right)}{x^2+y^2}\,(x-i\,y) \\
&= \frac{2\,\sinh\left(\frac{x}{2}\right)\,\cos\left(\frac{y}{2}\right) + 2\,i\,\sin\left(\frac{y}{2}\right)\,\cosh\left(\frac{x}{2}\right)}{x^2+y^2}\,(x-i\,y) \\
&= 2\,\sinh\left(\frac{x}{2}\right)\,\cos\left(\frac{y}{2}\right)\,\frac{x}{x^2+y^2} + 2\,\sin\left(\frac{y}{2}\right)\,\cosh\left(\frac{x}{2}\right)\,\frac{y}{x^2+y^2} \\
&+ i\left[\sin\left(\frac{y}{2}\right)\,\cosh\left(\frac{x}{2}\right)\,\frac{x}{x^2+y^2} - 2\,\sinh\left(\frac{x}{2}\right)\,\cos\left(\frac{y}{2}\right)\,\frac{y}{x^2+y^2}\right].
\end{aligned}$$

Thus in conclusion we obtain that

$$u(x,y) = 2\,\sinh\left(\frac{x}{2}\right)\,\cos\left(\frac{y}{2}\right)\,\frac{x}{x^2+y^2} + 2\,\sin\left(\frac{y}{2}\right)\,\cosh\left(\frac{x}{2}\right)\,\frac{y}{x^2+y^2},$$

is a solution. $\square$

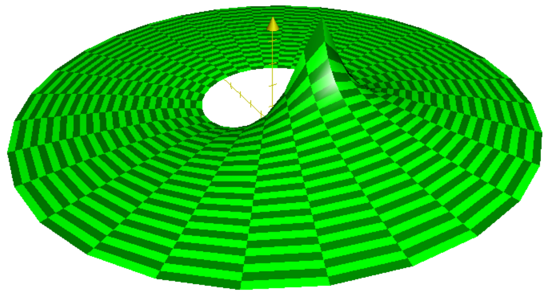

Figure 2. The solution of Exercise D.5.2 for $R = 3/2$.

Exercise D.5.2. *Let $R > 1$, find an explicit solution $u \in C^2(\mathbb{R}^2 \setminus \overline{B_R((0,0))})$ of the following two-dimensional boundary value problem*

$$\begin{cases} \Delta u(x,y) = 0, & \text{in } (x,y) \in \mathbb{R}^2 \setminus \overline{B_R((0,0))}, \\ u(x,y) = \dfrac{R^2 - x}{R^2 + 1 - 2\,x}, & (x,y) \in \partial B_R((0,0)). \end{cases}$$

Solution. We observe that

$$\frac{R^2 - x}{R^2 + 1 - 2\,x} = \frac{x^2 - x + y^2}{x^2 + 1 - 2\,x + y^2} = \frac{x\,(x-1) + y^2}{(x-1)^2 + y^2}, \qquad \text{for } (x,y) \in \partial B_R((0,0)).$$

The last function is the real part of the function,

$$\frac{z}{z-1}$$

which is holomorphic for $z \neq 1$. Thus we can take

$$u(x,y) = \frac{x\,(x-1) + y^2}{(x-1)^2 + y^2},$$

as a solution. □

Exercise D.5.3 (Poisson's kernel for the half-space)**.** *Let us consider the function*

$$P(x,y) = \frac{1}{\pi}\,\frac{x}{x^2 + y^2}, \qquad \text{for } (x,y) \in (0,+\infty) \times \mathbb{R}.$$

Prove that:

(1) *P is harmonic in $(0,+\infty) \times \mathbb{R}$;*

(2) *for every $f \in \mathcal{S}$, the function*

$$U_f(x,y) = \int_{\mathbb{R}} P(x,t)\,f(y-t)\,dt, \qquad (x,y) \in (0,+\infty) \times \mathbb{R},$$

is harmonic;

(3) *for every $f \in \mathcal{S}$, we have*

$$\lim_{x \to 0^+} U_f(x,y) = f(y), \qquad \text{for every } t \in \mathbb{R}.$$

Solution. It is easily seen that

$$\frac{x}{x^2+y^2} = \mathrm{Re}\left(\frac{1}{x+i\,y}\right) = \mathrm{Re}\left(\frac{x-i\,y}{x^2+y^2}\right),$$

thus point (1) follows directly from Remark 1.4.14.

As for point (3), we already know from Exercise 6.8.10 that

$$\lim_{x\to 0^+}\int_{\mathbb{R}} \frac{x}{x^2+t^2}\,\varphi(t)\,dt = \langle \pi\,\delta_0,\varphi\rangle = \pi\,\varphi(0), \qquad \text{for every } \varphi\in\mathcal{S}.$$

By recalling the definition of P, this automatically gives

$$\lim_{x\to 0^+}\int_{\mathbb{R}} P(x,t)\,f(y-t)\,dt = \langle \delta_0, f(y-\cdot)\rangle = f(y), \qquad \text{for every } f\in\mathcal{S}.$$

We are left with proving point (2). We first observe that for every fixed $x>0$, we have

$$U_f(x,y) = P(x,\cdot)*f(y).$$

Thus we can directly claim that U_f can be differentiated infinitely many times in y, thanks to Corollary 3.5.12. Moreover, by the same result we have

$$\frac{\partial^2 U_f}{\partial y^2} = \left(\frac{\partial^2}{\partial y^2}P(x,\cdot)\right)*f = \int_{\mathbb{R}} \frac{\partial^2}{\partial t^2}P(x,t)\,f(y-t)\,dy. \tag{4.5.1}$$

In order to prove differentiability in the x variable, we need to use the Lebesgue Dominated Convergence Theorem (see Theorem 3.2.5). We first observe that for every $x>0$ and $y\in\mathbb{R}$

$$\left|\frac{\partial}{\partial y}P(x,\cdot)\right| = \left|\frac{y^2-x^2}{(x^2+y^2)^2}\right| \le \frac{y^2+x^2}{(y^2+x^2)^2} = \frac{1}{x^2+y^2} \le \frac{1}{x^2}, \tag{4.5.2}$$

and

$$\left|\frac{\partial^2}{\partial y^2}P(x,\cdot)\right| = 2\,x\,\frac{|x^2-3\,y^2|}{(x^2+y^2)^3} \le 6\,x\,\frac{x^2+y^2}{(x^2+y^2)^3} \le 6\,x\,\frac{1}{x^4} = \frac{6}{x^3}. \tag{4.5.3}$$

These will help us to show the differentiability of U_f in x. Indeed, for every $x>0$ and $y\in\mathbb{R}$, we have

$$\lim_{h\to 0}\frac{U_f(x+h,y)-U_f(x,y)}{h} = \lim_{h\to 0}\int_{\mathbb{R}} \frac{P(x+h,t)-P(x,t)}{h}\,f(y-t)\,dt.$$

In order to pass the limit under the integral sign, we observe that

$$\lim_{h\to 0}\frac{P(x+h,t)-P(x,t)}{h} = \frac{\partial}{\partial x}P(x,t),$$

and that for every $|h|<x/2$, we have for a point $\xi_{x,h}$ such that $|\xi_{x,h}-x|\le |h|$

$$\left|\frac{P(x+h,t)-P(x,t)}{h}\,f(y-t)\right| = \left|\frac{\partial}{\partial x}P(\xi_{x,h},t)\right|\,|f(t-y)| \le \frac{1}{(\xi_{x,h})^2}\,|f(t-y)|.$$

By observing that by construction

$$\xi_{x,h} \ge x-|h| \ge \frac{x}{2} > 0,$$

the last estimate gives the summable upper bound independent of h, needed to apply the Dominated Convergence Theorem. Thus we get

$$\begin{aligned}\frac{\partial}{\partial x}U_f(x,y) = \lim_{h\to 0}\frac{U_f(x+h,y)-U_f(x,y)}{h} &= \lim_{h\to 0}\int_{\mathbb{R}}\frac{P(x+h,t)-P(x,t)}{h}\,f(y-t)\,dt \\ &= \int_{\mathbb{R}}\frac{\partial}{\partial x}P(x,t)\,f(y-t)\,dt.\end{aligned}$$

In a similar way, we prove that

$$\frac{\partial^2}{\partial x^2}U_f(x,y) = \int_{\mathbb{R}}\frac{\partial^2}{\partial x^2}P(x,t)\,f(y-t)\,dt. \tag{4.5.4}$$

By putting together (4.5.4) and (4.5.1) and using that P is harmonic by point (1), we finally get that U_f is harmonic, as well. $\square$

Appendix E

Tables of transforms

1. $Z-$transforms

Sequence	Transform	Radius of convergence
δ_0	1	0
1	$\dfrac{z}{z-1}$	1
$a^n\ (a \in \mathbb{C}^*)$	$\dfrac{z}{z-a}$	$\lvert a \rvert$
n	$\dfrac{z}{(z-1)^2}$	1
n^2	$\dfrac{z\,(1+z)}{(z-1)^3}$	1
$\cos(n\,\tau)\ (\tau > 0)$	$\dfrac{z\,(z-\cos\tau)}{z^2 - 2\,z\,\cos\tau + 1}$	1
$\sin(n\,\tau)\quad(\tau > 0)$	$\dfrac{z\,\sin\tau}{z^2 - 2\,z\,\cos\tau + 1}$	1

Sequence	Transform	Radius of convergence
$\dfrac{1}{n}$	$-\mathrm{Log}\left(1-\dfrac{1}{z}\right)$	1
$\dfrac{1}{n!}$	$e^{\frac{1}{z}}$	0

2. Laplace transforms

Causal signal	Transform	Abscissa of convergence
$H(t)$	$\dfrac{1}{z}$	0
$R(t)$	$\dfrac{1}{z^2}$	0
$t^k\, e^{a\,t}\, H(t)$	$\dfrac{k!}{(z-a)^{k+1}}$	$\mathrm{Re}(a)$
$\cos(t\,\tau)\ (\tau > 0)$	$\dfrac{z}{z^2+\tau^2}$	0
$\sin(t\,\tau)\ \ (\tau > 0)$	$\dfrac{\tau}{z^2+\tau^2}$	0
$SW(t)$	$\dfrac{e^z-1-z}{z^2\,(e^z-1)}$	0

3. Bilateral Laplace transforms

Signal	Transform	Abscissa of convergence	Upper abscissa of convergence
$\mathrm{rect}(t)$	$\dfrac{e^{\frac{z}{2}}-e^{-\frac{z}{2}}}{z}$	$-\infty$	$+\infty$
$e^{-\lvert t\rvert}$	$\dfrac{2}{1-z^2}$	-1	1

4. Mellin transforms

Causal Signal	Transform	Abscissa of convergence	Upper abscissa of convergence
$1_{[0,1]}(t)$	$\dfrac{1}{z}$	0	$+\infty$
$e^{-t}\,H(t)$	$\Gamma(z)$	0	$+\infty$

5. Hilbert transforms

Signal	Transform
$\mathrm{rect}(t)$	$\log\left\lvert\dfrac{\omega+\frac{1}{2}}{\omega-\frac{1}{2}}\right\rvert$

6. Fourier transforms

Signal	Transform	Notes
$\operatorname{rect}(t)$	$\operatorname{sinc}\left(\frac{\omega}{2\,\pi}\right)$	
$\operatorname{tri}(t)$	$\left(\operatorname{sinc}\left(\frac{\omega}{2\,\pi}\right)\right)^2$	
$e^{-\lvert t\rvert}$	$\frac{2}{1+\omega^2}$	
$\frac{1}{1+t^2}$	$\pi\, e^{-\lvert\omega\rvert}$	
$\frac{1}{a\,t^2+b\,t+c}$ (with $a>0$, $b^2-4\,a\,c<0$)	$\frac{2\,\pi}{\sqrt{4\,a\,c-b^2}}\, e^{i\,\frac{b}{2\,a}\,\omega}\, e^{-\frac{\sqrt{4\,a\,c-b^2}}{2\,a}\,\lvert\omega\rvert}$	
$\frac{1}{1+t^4}$	$\pi\, e^{-\frac{\sqrt{2}}{2}\,\lvert\omega\rvert}\, \sin\left(\frac{\sqrt{2}}{2}\,\lvert\omega\rvert+\frac{\pi}{4}\right)$	
$1_{[a,b]}(t)$	$(b-a)\, e^{-\frac{a+b}{2}\,i\,\omega}\, \operatorname{sinc}\left(\frac{b-a}{2\,\pi}\,\omega\right)$	
$\operatorname{sinc}(t)$	$1_{[-\pi,\pi]}(\omega)$	in L^2 or $\mathcal{S}'$
$\operatorname{sinc}^2(t)$	$\operatorname{tri}\left(\frac{\omega}{2\,\pi}\right)$	

Signal	**Transform**	**Notes**
e^{-t^2}	$\sqrt{\pi}\, e^{-\frac{\omega^2}{4}}$	
$e^{-a\,(t-t_0)^2}\ \ (a>0)$	$\sqrt{\dfrac{\pi}{a}}\, e^{-\frac{\omega^2}{4\,a}}\, \Big(\cos(t_0\,\omega) - i\ \sin(t_0\,\omega)\Big)$	
δ_{t_0}	$e^{-i\,t_0\,\omega}$	in $\mathcal{S}'$
1	$2\,\pi\,\delta_0$	in $\mathcal{S}'$
$e^{i\,t_0\,t}$	$2\,\pi\,\delta_{t_0}$	in $\mathcal{S}'$
$H(t)$	$\pi\,\delta_0 + \dfrac{1}{i}\,\mathrm{P.V.}\dfrac{1}{\omega}$	in $\mathcal{S}'$
$\mathrm{P.V.}\dfrac{1}{t}$	$-\pi\,i\,\mathrm{sign}(\omega)$	in $\mathcal{S}'$
$\cos t$	$\pi\,\Big(\delta_1 + \delta_{-1}\Big)$	in $\mathcal{S}'$
$\sin t$	$\pi\,i\,\Big(\delta_{-1} - \delta_1\Big)$	in $\mathcal{S}'$
P_τ	$\dfrac{2\,\pi}{\tau}\,P_{\frac{2\,\pi}{\tau}}$	in $\mathcal{S}'$

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