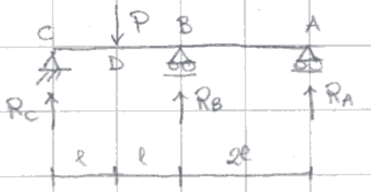
EXERCISE 1

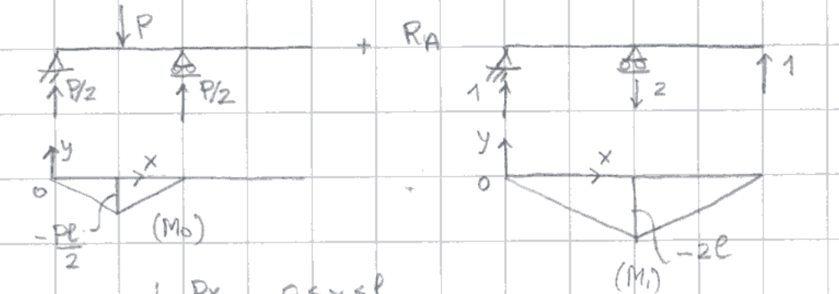


The structure is statically indeterminate, i.e. equilibrium eqns are not sufficient to determine the reactions:

$$(\#1)\quad \begin{cases} R_A + R_B + R_C = P \\ R_B\, 2\cancel{\ell} + R_A\, 4\cancel{\ell} = P\cancel{\ell} \end{cases}$$

Two eqns but three unknowns!

To calculate the reaction, we adopt the following strategy. We apply the principle of superposition:



$$M_0(x) = \begin{cases} -\dfrac{Px}{2}, & 0 \le x \le \ell \\ \dfrac{P}{2}(x-2\ell), & \ell \le x \le 2\ell \\ 0, & 2\ell \le x \le 4\ell \end{cases} \qquad M_1(x) = \begin{cases} -x, & 0 \le x \le 2\ell \\ (x-4\ell), & 2\ell \le x \le 4\ell \end{cases}$$

By superposition: $M(x) = M_0(x) + R_A M_1(x)$

The deformation energy takes the form:

$$E_d = \frac{1}{2EI}\int_0^{4\ell} M^2(x)\,dx = \frac{1}{2EI}\int_0^{4\ell} \left(M_0(x) + R_A M_1(x)\right)^2 dx \qquad (\#2)$$

In view of Menabrea's thm, R_A is the solution to the equation:

$$0 = \frac{\partial E_d}{\partial R_A} = \frac{1}{\cancel{2}EI}\int_0^{4\ell} \cancel{2}\left(M_0(x) + R_A M_1(x)\right) M_1(x)\,dx$$

or equivalently,

$$\left(\int_0^{4\ell} M_0(x) M_1(x)\,dx\right) + R_A\left(\int_0^{4\ell} M_1^2(x)\,dx\right) = 0$$

$$R_A = -\frac{\left(\int_0^{4\ell} M_0(x) M_1(x)\,dx\right)}{\left(\int_0^{4\ell} M_1^2(x)\,dx\right)} \qquad (\#3)$$

We now calculate the two terms:

$$\int_0^{4\ell} M_0(x)\, M_1(x)\, dx = \int_0^{\ell} \frac{P}{2} x^2 dx + \int_{\ell}^{2\ell} \frac{P}{2}\left(2\ell x - x^2\right) dx$$

$$= \frac{P\ell^3}{6} + \frac{P}{2}\left[\ell x^2 - \frac{x^3}{3}\right]_{\ell}^{2\ell} = \frac{P\ell^3}{6} + \frac{P\ell^3}{2}\left(4 - \frac{8}{3} - 1 + \frac{1}{3}\right)$$

$$= P\ell^3\left(\frac{1}{6} + \frac{3}{2} - \frac{7}{6}\right) = P\ell^3\left(\frac{3}{2} - 1\right) = \frac{P\ell^3}{2}$$

$$\int_0^{4\ell} M_1^2(x)\, dx \underbrace{=}_{\text{by symmetry}} 2\int_0^{2\ell} x^2\, dx = \frac{2}{3}\cdot 8\ell^3 = \frac{16}{3}\ell^3$$

After substituting into (#3), we obtain: $R_A = -\frac{P\cancel{\ell^3}}{2}\cdot\frac{3}{16\cancel{\ell^3}} = -\frac{3}{32}P$ (#4)

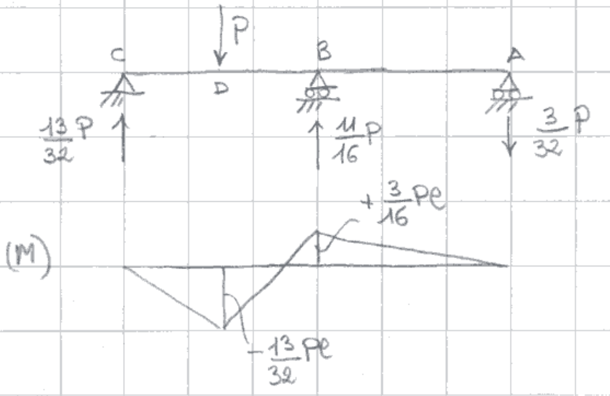
this sign here means that R_A is pointing downwards!

To calculate R_B and R_C, we substitute relation (#4) into (#1):

$$\begin{cases} -\frac{3}{32}P + R_B + R_C = P \\ 2R_B - \cancel{4}\cdot\frac{3}{\cancel{32}\,8}P = P \end{cases} \longrightarrow R_B = \frac{1}{2}\left(\frac{3}{8} + 1\right)P = \frac{11}{16}P$$

$$\hookrightarrow R_C = P + \frac{3}{32}P - \frac{11}{16}P = \frac{(32 + 3 - 22)}{32}P = \frac{13}{32}P$$

At equilibrium, the reactions and the bending moment are as depicted in the fig.



$$M(x) = \begin{cases} -\frac{13}{32}Px, & 0 \le x \le \ell, \\ +\frac{19}{32}P\left(x - \frac{32}{19}\ell\right), & \ell \le x \le 2\ell, \\ -\frac{3}{32}P(x - 4\ell), & 2\ell \le x \le 4\ell. \end{cases} \quad (\#5)$$

To calculate the vertical displacement of the point D, let us still consider the deformation energy given by eqn (#2).

In view of Castigliano's thm, we have:

$$U_D = \frac{\partial E_d}{\partial P} = \frac{1}{\cancel{2}EI}\int_0^{4\ell} \cancel{2}(M_0+R_A M_1)\left(\frac{\partial M_0}{\partial P} + \underbrace{R_A \cancelto{0}{\frac{\partial M_1}{\partial P}}}_{\substack{\text{this term}\\ \text{vanishes because}\\ M_1 \text{ is independent}\\ \text{of } P}} + \frac{\partial R_A}{\partial P} M_1\right) dx$$

$$= \left(\frac{1}{EI}\int_0^{4\ell} \underbrace{(M_0+R_A M_1)}_{\substack{\parallel \\ M}} \frac{\partial M_0}{\partial P}\, dx\right) + \frac{\partial R_A}{\partial P}\underbrace{\left(\frac{1}{EI}\int_0^{4\ell}(M_0+R_A M_1)M_1\, dx\right)}_{\text{this term vanishes because of (\#3).}}$$

Thus, we obtain:

$$U_D = \frac{1}{EI}\int_0^{4\ell} M(x)\,\frac{\partial M_0(x)}{\partial P}\, dx \qquad \frac{\partial M_0(x)}{\partial P} = \begin{cases} -\frac{x}{2}, & 0 \le x \le \ell \\ \frac{1}{2}(x-2\ell), & \ell \le x \le 2\ell \\ 0, & 2\ell \le x \le 4\ell \end{cases}$$

$$= \frac{1}{EI}\int_0^{\ell}\left(-\frac{13}{32}Px\right)\left(-\frac{x}{2}\right)dx + \frac{1}{EI}\int_{\ell}^{2\ell}\left(\frac{19}{32}Px - P\ell\right)\frac{1}{2}(x-2\ell)\, dx \qquad x' = 2\ell - x \;\; \Rightarrow \;\; x = 2\ell - x'$$

$$= \frac{13P}{64EI}\int_0^{\ell} x^2\, dx + \frac{1}{EI}\int_{\ell}^{0}\left[\frac{19}{32}P(2\ell - x') - P\ell\right]\frac{1}{2}(-x')(-dx')$$

$$= \frac{13P}{64EI}\int_0^{\ell} x^2\, dx - \frac{1}{2EI}\int_0^{\ell}\left(\underbrace{\frac{19}{16}P\ell - P\ell}_{\frac{3}{16}P\ell} - \frac{19}{32}Px'\right)x'\, dx'$$

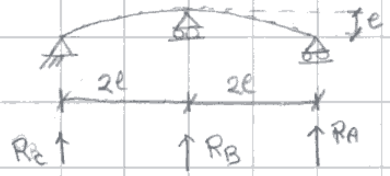
$$= \frac{13P}{64EI}\frac{\ell^3}{3} - \frac{1}{2EI}\int_0^{\ell}\left[\frac{3}{16}P\ell x' - \frac{19}{32}P(x')^2\right]dx'$$

$$= \frac{13P\ell^3}{192EI} - \frac{1}{2EI}\left[\frac{3}{16}P\ell\frac{(x')^2}{2} - \frac{19}{32}P\frac{(x')^3}{3}\right] =$$

$$= \frac{13}{192}\frac{P\ell^3}{EI} + \frac{5}{96}\frac{P\ell^3}{EI} = \frac{23}{192}\frac{P\ell^3}{EI}$$

EXERCISE 2.

Initial undeformed and stress-free configuration

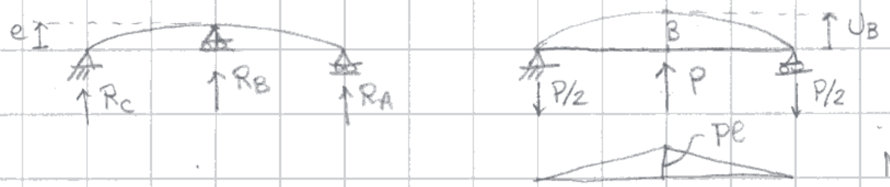


Deformed and stress configuration after defect (for symmetry reasons, we expect $R_A = R_C$!)

Again, equilibrium eqns are not sufficient to determine the reactions:

$$(\#7) \quad \begin{cases} R_A + R_B + R_C = 0 \\ R_B\, 2l + R_A\, 4l = 0 \end{cases}$$ two eqns but three unknowns!

This time, to calculate the reactions, it is convenient to use Betti's thm:



$$M(x) = \begin{cases} \frac{P}{2}x, & 0 \leq x \leq 2l \\ -\frac{P}{2}(x-4l), & 2l \leq x \leq 4l \end{cases}$$

$$Pe = R_B U_B \quad (\#7)$$

We now calculate U_B using Menabrea's thm:

$$E_d = \frac{1}{2EI}\int_0^{4l} M^2(x)\, dx$$

$$\hookrightarrow U_B = \frac{\partial E_d}{\partial P} = \frac{\cancel{2}}{\cancel{2}EI}\int_0^{4l} M(x)\frac{\partial M(x)}{\partial P}\, dx = \frac{\cancel{2}}{EI}\int_0^{2l}\left(\frac{P}{\cancel{2}}x\right)\left(\frac{x}{2}\right) dx \quad (\#8)$$

$$= \frac{P}{2EI}\int_0^{2l} x^2\, dx = \frac{P}{2EI}\left(\frac{8l^3}{3}\right) = \frac{4}{3}\frac{Pl^3}{EI}$$

Substituting (#8) into (#7) gives:

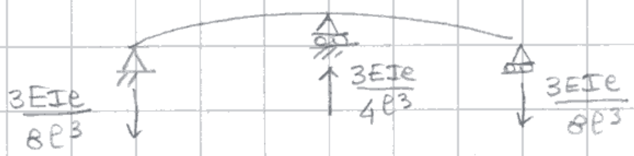
$$R_B = \frac{Pe}{U_B} = \frac{3EI\,\cancel{P}e}{4\cancel{P}l^3} = \frac{3EIe}{4l^3} \quad (\#9)$$

We calculate the remaining reactions by substituting (#9) into (#7):

$$R_A = -\frac{R_B}{2} = -\frac{3EIe}{8\ell^3}$$

$$R_C = -R_A - R_B = -\frac{3EIe}{4\ell^3} + \frac{3EIe}{8\ell^3} = -\frac{3EIe}{8\ell^3}$$

The reactions are as depicted in the following figure.



And the bending moment can be calculated as follows: