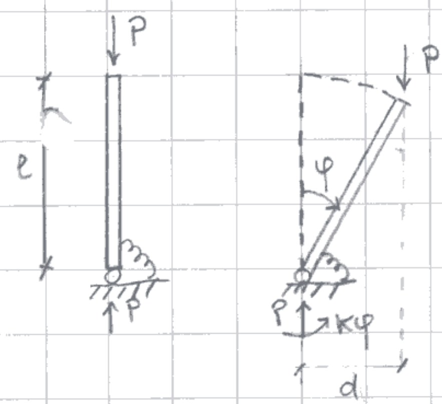
EULER'S INSTABILITY.

Buckling (flambage or flambement) is an instability phenomenon leading to a failure mode. Theoretically, buckling is caused by a bifurcation in the solution to the equations of static equilibrium. At a certain stage under an increasing load, further load can be sustained in one of two or more states of equilibrium: an undeformed state or laterally deformed states.

1. Stabilities of two 1D structures



Consider a rigid beam of length l subject to a compressive dead load of intensity P applied to its free end. The other extremity is constrained by a hinge connected to a rotational spring (torsion spring)(*) of elastic constant K [Nm] obeying a linear law:

$$M = K\varphi.$$

The system has only one degree of freedom, the rotation φ around the hinge. At equilibrium the bending moment of the spring balances the torque of the applied load:

$$\underbrace{P}_{\text{force}}\ \underbrace{l\sin\varphi}_{\substack{\text{moment}\\ \text{arm } d}} = K\varphi$$

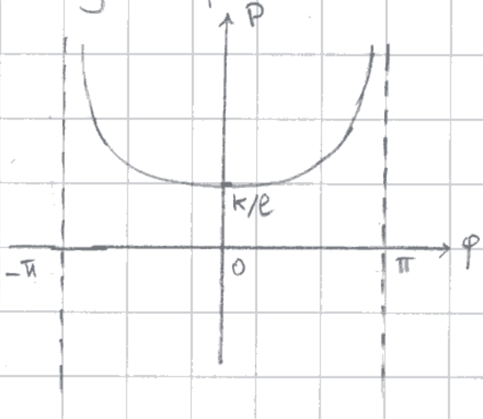
This equation has the following solutions:

$$\varphi = 0 \quad \text{for any } P$$

$$P = \frac{K}{l}\,\frac{\varphi}{\sin\varphi}$$

(*) The type of spring is usually subject to a bending moment; in fact, when the beam rotates around the hinge, the spring tries to push it back to its original position by applying a couple $K\varphi$.

We represent these solutions in the (P,φ)-plane. The first solution is represented by the vertical line through the origin. The second solution by a curve intersecting the first solution at the point. $\varphi=0$, $P=k/\ell$.

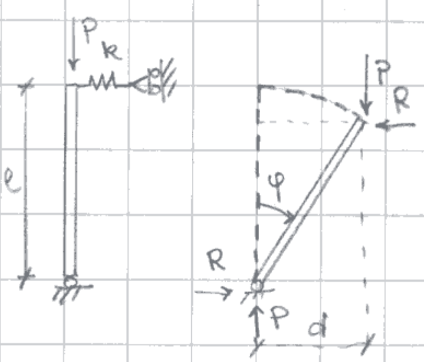


For a load of intensity P increasing from zero, we have

- a unique solution ($\varphi=0$) if $P\le k/\ell$.
- 3 solutions ($\varphi<0$, $\varphi=0$, $\varphi>0$) if $P>k/\ell$.

The curves representing the two solutions intersect at $P=k/\ell$, in other words a BIFURCATION occurs at $P=k/\ell$ or $P=k/\ell$ is a bifurcation point.

Let us now consider the rigid beam represented in the figure below. The beam is hinged at one of the extremities but the other is constrained by a linear spring applying an horizontal force:



$$R = k\,d = k\ell\,\mathrm{sen}\,\varphi .$$

At equilibrium, we have:

$$P\,d = R\ell\cos\varphi ,$$

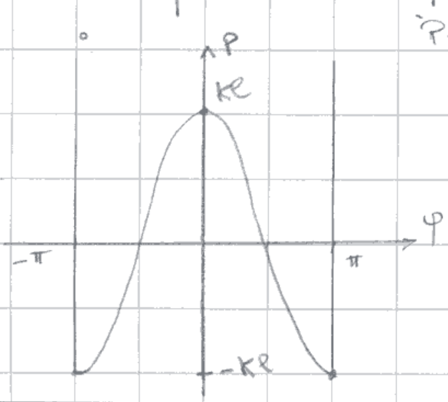
or, equivalently,

$$P\ell\,\mathrm{sen}\,\varphi = k\ell^2\,\mathrm{sen}\,\varphi\cos\varphi .$$

Thus equation admits the following solutions:

∘ $\mathrm{sen}\,\varphi=0$, i.e. $\varphi=0, \pm\pi, \dots$ for any P

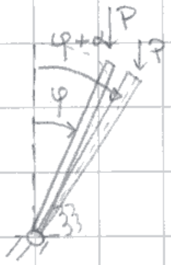
∘ $P=k\ell\cos\varphi$



In this case there are 3 bifurcation points:

$$(P,\varphi)=\begin{cases}(k\ell,0)\\(-k\ell,-\pi)\\(-k\ell,+\pi)\end{cases}$$

Let us now discuss the stability of the equilibrium configurations calculated for the two structures.



Imagine to perturb the equilibrium, i.e. to vary the equilibrium configuration φ solution to (1), of an angle α. In the perturbed configuration, the beam is subject to two actions:

- $Pl\,sen(\varphi+\alpha)$: the torque of the load, which tends to move the beam far from the equilibrium position at φ;
- $K(\varphi+\ )$: the spring torque of the hinge, which tends to restore the equilibrium position.

$$\therefore\ Pl\,sen(\varphi+\alpha) \begin{matrix}>\\=\\<\end{matrix} K(\varphi+\alpha) \Rightarrow \varphi \text{ is } \begin{cases}\text{unstable}\\ \text{neutrally stable}\\ \text{stable}\end{cases}$$

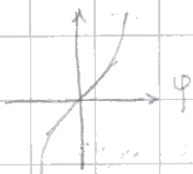
We assume $\alpha <<$ so that $sen\alpha \cong \alpha$, $\cos\alpha \cong 1$:

$$Pl(sen\varphi\cos\alpha + sen\alpha\cos\varphi) \overset{?}{\begin{matrix}>\\=\\<\end{matrix}} K(\varphi+\alpha)$$

$$Pl\,\cancel{sen\varphi} + Pl\alpha\cos\varphi \begin{matrix}>\\=\\<\end{matrix} \cancel{K\varphi} + K\alpha$$

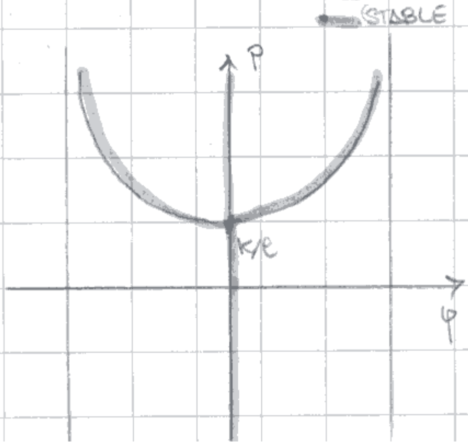
Recall that $P = \dfrac{K\varphi}{l\,sen\varphi}$:

$$\frac{K\cancel{l}}{\cancel{l}}\frac{\varphi}{tg\varphi} \overset{?}{\begin{matrix}>\\=\\<\end{matrix}} K$$



$$\varphi < tg\varphi$$

thus the equilibrium configuration with $P = K\varphi/lsen\varphi$ is stable.



Analysis for $\varphi = 0$:

$$Pl\,sen\alpha \overset{?}{\gtrless} K\alpha$$

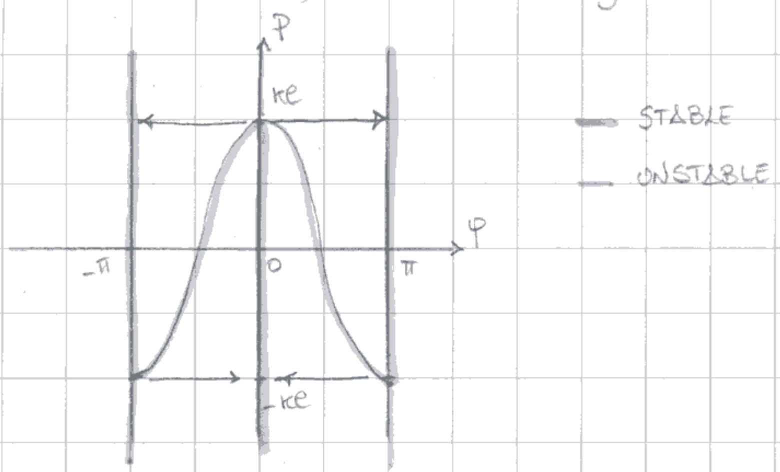
$$Pl\cancel{\alpha} \gtrless K\cancel{\alpha}$$

$$P > K/l$$

$\hookrightarrow$

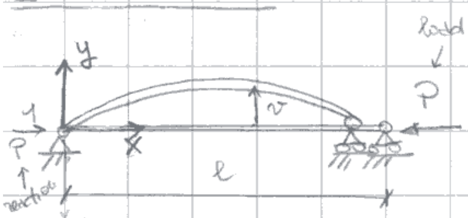
$P < K/l$	$\varphi = 0$	is	stable
$P = K/l$	$\varphi = 0$	is	neutral
$P > K/l$	$\varphi = 0$	is	unstable

For the other structure, a similar analysis shows that:



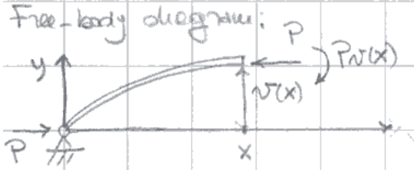
REM: These structures are characterized by the absence of deformation before reaching the critical load, which coincides with bifurcation load. The stability analysis of these structures falls into the class of Eulerian pbs.

2. EULER'S BEAM



- Perfectly straight bar, inextensible.
- uniform cross-section
- EI: flexural stiffness for bending about z
- hinged at extremities
- compressive load P, no other forces

Free-body diagram:



Assume that, for some value of P, the beam can buckle (deform) laterally in the (x-y) plane, and let $v(x)$ be the lateral displacement at point x.

The internal actions at x are:
- a compressive axial force P
- a bending moment $Pv(x)$.

Remark: the presence of $M = Pv(x)$ is due to the lateral deflection $v(x)$!

The moment-curvature relation of the beam is: $M = EIK = EI(-v'')$.

Thus

$$-EIv''(x) = Pv(x)$$

$$v'' + \frac{P}{EI} v(x) = 0$$

the general solution is:

$$v(x) = A\cos(\omega x) + B\,\mathrm{sen}(\omega x), \qquad \omega := \sqrt{\frac{P}{EI}}$$

where A, B are arbitrary constants. We have two boundary conditions to satisfy: at the ends $x = 0, \ell$, $v = 0$. Then

$$\begin{cases} 0 = v(0) = A \\ 0 = v(\ell) = \cancel{A}\cos(\omega\ell) + B\,\mathrm{sen}(\omega\ell) \end{cases}$$

$$\begin{cases} A = 0 \\ B\,\mathrm{sen}(\omega\ell) = 0 \end{cases}$$

- If $\mathrm{sen}(\omega\ell) \neq 0$, then $B = 0$ and obviously $v(x) = 0 \to$ undeflected beam
- If $\mathrm{sen}(\omega\ell) = 0$, then B is undetermined and the beam may assume the form:

$$v(x) = B\,\mathrm{sen}(\omega x) \qquad \text{buckled condition}$$

But $\mathrm{sen}(\omega\ell) = 0$ implies that:

$$\omega\ell = n\pi, \qquad n = 0, 1, 2, \dots$$

We do not consider the case $n = 0$ which implies $\omega = 0$ $(P \neq 0!)$.

Therefore, we obtain the following bifurcations points:

$$\omega\ell = n\pi, \qquad n = 1, 2, \dots$$

$$P = \left(\frac{n\pi}{\ell}\right)^2 EI, \qquad n = 1, 2, \dots$$

There are infinite number of values of P for instability. They correspond to various modes of buckling

$n = 1$ $\quad v(x) = B\,\mathrm{sen}\left(\frac{\pi}{\ell}x\right)$, $\quad P_1 = \frac{\pi^2}{\ell^2}EI$ $\Big\}$ SINGLE LONGITUDINAL HALF-WAVE

$n = 2$ $\quad v(x) = B\,\mathrm{sen}\left(\frac{2\pi x}{\ell}\right)$, $\quad P_2 = \frac{4\pi^2}{\ell^2}EI$ $\Big\}$ TWO HALF-WAVES

...

then n indicates the number of longitudinal half-waves!

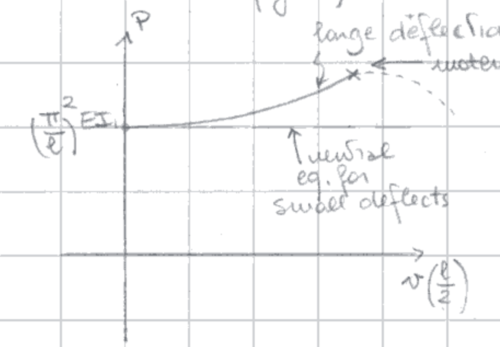
the CRITICAL LOAD corresponds to $n = 1$: $\quad P_{cr} = \frac{\pi^2 EI}{\ell^2}$.

In practice the critical load P_{cr} is never exceeded because high stresses develop at this load and the structure collapses. We are not therefore concerned with buckling loads higher than P_{cr}.

Rem : $P = P_{cr}$ corresponds to a state of NEUTRAL EQUILIBRIUM (see previous examples).

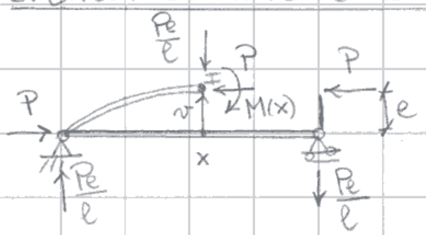
Rem : the condition of neutral equilibrium is attained only for small lateral displacements. When these displacements become large, the moment-curvature relation is no longer valid and the problem becomes more complicated.

The effect of large lateral displacement is to increase the flexural stiffness, and, provided that the material remains elastic, load greater that P_{cr} are attainable (see figure).



When the proportionality limit has been reached, the flexural stiffness falls off rapidly and unstable equilibrium develops.

3. EFFECT OF IMPERFECTIONS : • e : eccentricity of the load



• $Pe = V\ell \rightarrow V = \frac{Pe}{\ell}$: reaction

• $M(x) = Pv(x) - \frac{Pe}{\ell}x$ (about I)

Thus:

$$-EI v''(x) = Pv(x) - \frac{Pe}{\ell}x$$

i.e.

$$v''(x) + \frac{P}{EI_1}v(x) - \frac{Pe}{\ell EI_1}x = 0$$

$$v''(x) + \omega^2 v(x) - \eta\,\omega^2 x = 0 \qquad \eta := \frac{e}{\ell}$$

General solution :

$$v(x) = A\cos(\omega x) + B\sin(\omega x) + \eta x$$

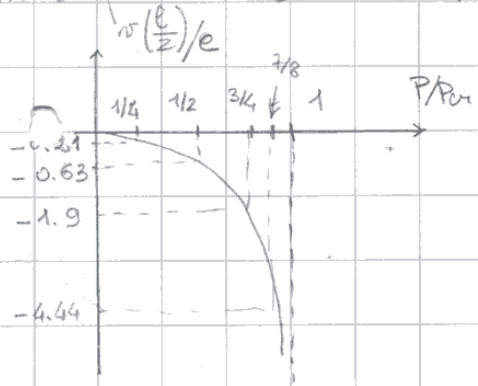
Boundary conditions:

$$\begin{cases} 0 = v(0) = A \\ 0 = v(\ell) = \cancel{A}\cos(\omega\ell) + B\,\mathrm{sen}(\omega\ell) + \eta\ell \end{cases}$$

$$\begin{cases} A = 0 \\ B\,\mathrm{sen}(\omega\ell) + \eta\ell = 0 \end{cases} \qquad \begin{cases} A = 0 \\ B = -\dfrac{\eta\ell}{\mathrm{sen}(\omega\ell)} \end{cases}$$

The lateral displacement takes the form: $\boxed{v(x) = -\dfrac{e}{\mathrm{sen}(\omega\ell)}\,\mathrm{sen}(\omega x) + \dfrac{e}{\ell}x}$

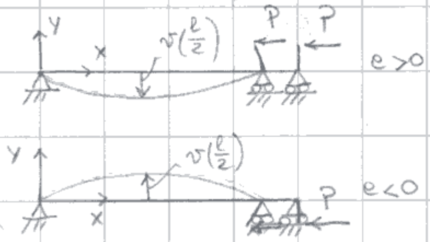
The displacement at the mid-length $x = \ell/2$, is:



$$v\left(\frac{\ell}{2}\right) = -\frac{e}{\mathrm{sen}(\omega\ell)}\,\mathrm{sen}\left(\frac{\omega\ell}{2}\right) + \frac{e}{2}$$

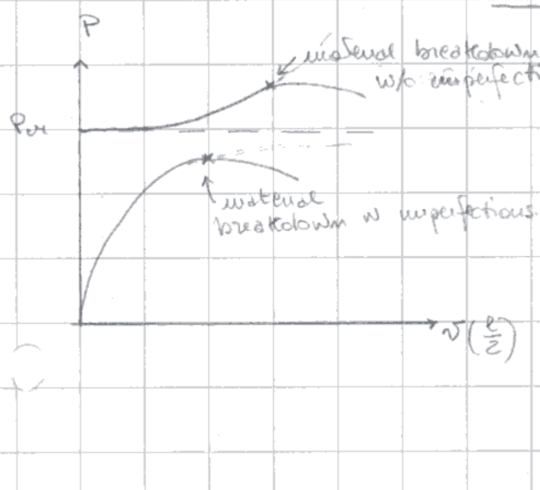
When $P=0$, then $\omega=0$ and $v\left(\frac{\ell}{2}\right)=0$.

As P approaches P_{cr}, $v\left(\frac{\ell}{2}\right) \to -\infty$



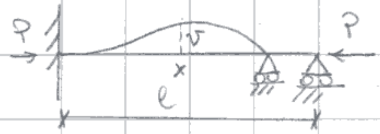
The values of $v\left(\frac{\ell}{2}\right)$ increase non-linearly with increasing P. At $P = P_{cr}$ an infinitely large value of v occurs but material breakdown occurs at some smaller value of P.

↳ sensitivity to imperfections: the critical load is never attained

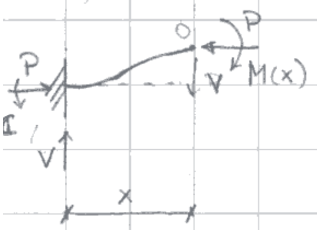


4. EFFECTIVE LENGTH:

So far the ends of the beam were hinged to some foundation. Let us now consider different boundary conditions corresponding to a built-in end at $x = 0$ and a hinged at $x = \ell$.



Free-body diagram:



- V, $\mathbb{C}$ reactions at the built-in end (they do not depend upon x).
- Equilibrium eqn. (Σ torques about $O = \emptyset$):

$$M(x) + Vx - \mathbb{C} - Pv(x) = 0$$

Remark: V, $\mathbb{C}$ are undetermined because the beam is statically indeterminate (or hyperstatic)!

i.e. equilibrium eqns are insufficient for determining the internal forces and reactions.

Let us derive the above equation:

$$M'(x) + V - Pv'(x) = 0$$

$$M''(x) - Pv''(x) = 0$$

$$\left(-EIv''(x)\right)'' - Pv''(x) = 0$$

$$v^{IV}(x) + \omega^2 v''(x) = 0$$

This is a fourth-order o.d.e. General solution:

$$v(x) = A\cos(\omega x) + B\,\text{sen}(\omega x) + Cx + D$$

$$v'(x) = -A\omega\,\text{sen}(\omega x) + B\omega\cos(\omega x) + C$$

$$v''(x) = -A\omega^2\cos(\omega x) - B\omega^2\,\text{sen}(\omega x)$$

B.cs:

$$\begin{cases} 0 = v(0) & \text{null deflection at } x = 0 \\ 0 = v'(0) & \text{null rotation at } x = 0 \\ 0 = v(\ell) & \text{null deflection at } x = \ell \\ 0 = M(\ell) = -EIv''(\ell) & \text{null bending moment at } x = \ell \text{ (no external torques acting)} \end{cases}$$

$$\begin{cases} 0 = A + D \\ 0 = B\omega + C \\ 0 = A\cos(\omega\ell) + B\,\text{sen}(\omega\ell) + C\ell + D \\ 0 = A\cos(\omega\ell) + B\,\text{sen}(\omega\ell) \end{cases}$$

Homogeneous linear algebraic system. It admits the null solution $A = B = C = D = 0$ if $\det \neq 0$. If $\det = 0$, other solutions are possible.

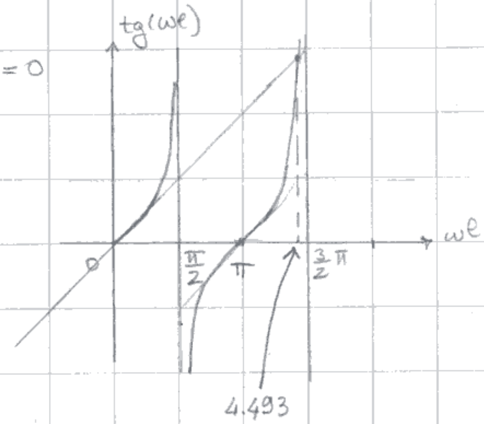
$$\begin{cases} A + D = 0 \\ B\omega + C = 0 \\ A\cos(\omega\ell) + B\,\mathrm{sen}(\omega\ell) = 0 \\ C\ell + D = 0 \end{cases}$$

$$\begin{array}{ccccc} & A & B & C & D \\ \det & \multicolumn{4}{c}{\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & \omega & 1 & 0 \\ \cos & \mathrm{sen} & 0 & 0 \\ 0 & 0 & \ell & 1 \end{bmatrix}} \end{array} =$$

$$= \det \begin{bmatrix} \omega & 1 & 0 \\ \mathrm{sen} & 0 & 0 \\ 0 & \ell & 1 \end{bmatrix} - \det \begin{bmatrix} 0 & \omega & 1 \\ \cos & \mathrm{sen} & 0 \\ 0 & 0 & \ell \end{bmatrix}$$

$$= -\,\mathrm{sen}(\omega\ell) + \ell\omega\cos(\omega\ell) = 0$$

$$\boxed{tg(\omega\ell) = \omega\ell}$$



$$\omega\ell \simeq 4.493$$

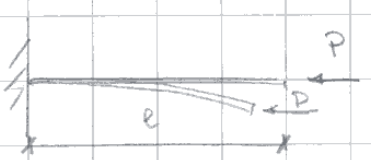
$$\sqrt{\frac{P_{cr}}{EI}} \simeq \frac{4.493}{\ell}$$

$$P_{cr} \simeq \left(\frac{4.493}{\ell}\right)^2 EI = 20.187 \cdot \frac{EI}{\ell^2}$$

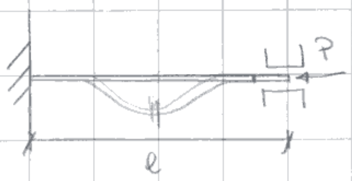
$$\simeq \frac{\pi^2 EI}{(\ell/\sqrt{2})^2}$$

Therefore the critical load of the structure corresponds to the Euler's critical load of a strut of equivalent length $\ell/\sqrt{2}$.

Analogous, one can prove the following results:



$$P_{cr} = \frac{\pi^2 EI}{(2\ell)^2}$$ "half of Euler's strut" equivalent length $L = 2\ell$



$$P_{cr} = \frac{\pi^2 EI}{(\ell/2)^2}$$ "doubled Euler's strut" equivalent length $L = \ell/2$

5. COMPRESSION MEMBER DESIGN

Compression members are structural elements subjected to axial compressive forces. Failure is prevented by designing structures so that the maximum stresses & displ. remain within tolerable limits:

$$P \leq S\sigma_e$$

cross-sectional area ; proportionality limit for axial stress or yield stress

But buckling is another type of failure, therefore

$$P \leq \frac{\pi^2 E I}{L^2} \quad , \quad L : \text{equivalent length}$$

Therefore, safety design requires that

$$P \leq \min \left\{ S\sigma_e \ ; \ \frac{\pi^2 E I}{L^2} \right\}$$

<u>Remark</u> : let us introduce the ratio $\sqrt{\frac{I}{S}} = r$ radius of gyration of the cross-section about z.

Then,

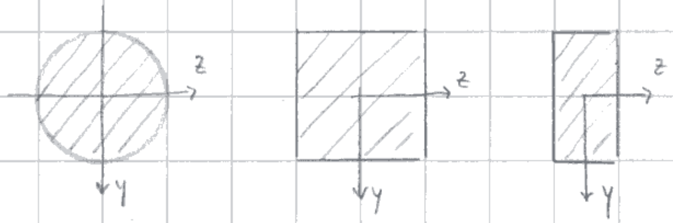
$$P_{cr} = \frac{\pi^2 E}{L^2} r^2 S$$

i.e.

$$\frac{P_{cr}}{S} = \pi^2 E \left(\frac{r}{L}\right)^2 = \frac{\pi^2 E}{\lambda^2} \quad ; \quad \lambda := \frac{L}{r} \ \text{SLENDERNESS RATIO}$$

$P_{cr} \downarrow$ if $\lambda \uparrow$

Thus $P_{cr} \downarrow$ if $r \downarrow$, i.e. the smallest moment should be taken into account to find the critical load.



$I_y = I_z$ $\qquad$ $I_y = I_z$ $\qquad$ $I_y < I_z$!

buckling by bending in ~~x~~-z plane!

Thus $P_{cr} \uparrow$ if $E \uparrow$ (steel quality)