Rational surfaces
(preliminary version)

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Introduction

These are intended for a course at LM level.

CHAPTER 1

Weil divisors, Class Group, Linear systems morphisms and rational maps

1.1. Weil divisors and Class Group

Let us fix an algebraically closed field k, of characteristic 0 (the latter only for my lazyness), $\mathbb{C}$ if you want to have something concrete to test. A polynomial in one variable $p \in k[x]$ is uniquely determined (up to a multiplicative constant) by its roots (with algebraic multiplicity) that is

$$p = \gamma \Pi_{i=1}^{h} (x - \alpha_i)^{m_i}.$$

If we consider $k[x]$ as the affine coordinate ring of $\mathbb{A}^1$ this allows to associate to any set of d points (counted with multiplicity) a polynomial and the other way round.

Thanks to primary decomposition this can be extended to $\mathbb{A}^n$ let $p \in k[x_1, \ldots, x_n]$ be a polynomial then $V(p) = D_1 \cup \ldots \cup D_h$. Where the D_i are irreducible codimension 1 subvarieties. Moreover via Nulltellensatz for any irreducible component D_i there is a unique integer m_i such that $p \in I_{D_i}^{m_i} \setminus I_{D_i}^{m_i+1}$. On the other hand any codimension 1 is the zero locus of some polynomial, therefore (up to a multiplicative constant) this yields the following bijection

$$k[x_1, \ldots, x_n] \leftrightarrow \{\text{non negative linear combination of codimension 1 subvarieties of } \mathbb{A}^n\}$$

In other words we associate to any regular function on $\mathbb{A}^n$ its zero locus counted with multiplicities. We can even do something better and more general.

Definition 1.1.1. Let Y be a smooth quasi affine variety and $Z \subset Y$ an irreducible subvariety. For any $p \in A(Y)$ there is an m_i such that $p \in I_Z^{m_i} \setminus I_Z^{m_i+1}$, let

$$ord_Z(p) = m_i$$

Note that, due to Noetherianity, for any $p \in A(Y)$ there are only finitely many codimension 1 irreducible subvarities on which $ord_Z(p) \neq 0$. Therefore to any regular function $p \in A(Y)$ we may associate its *divisor of zeros* given by

$$(p) = \sum_{\mathrm{cod}_Y Z = 1} ord_Z(p) Z.$$

Equivalently for a rational function $f = p/q$ we may define $(f) = (p) - (q)$.

Remark 1.1.2. *It is easy to check that (f) is independent from the choice of the representative p/q.*

In a similar fashion we may argue on $\mathbb{P}^n$ and then on any smooth projective variety X. This time regular functions are constant and therefore we only consider rational ones. For any $f \in K(X)$ we may consider an affine covering $\{U_i\}$ and define $ord_Z(f)$ on any open set U_i such that $U_i \cap Z \neq \emptyset$. This then extends to define (f) as in the affine case.

DEFINITION 1.1.3. Let X be a smooth variety. The Weil divisor group is $\mathrm{Div}(X)$, the free abelian group generated by irreducible codimension 1 subvarieties of X. The latter are called prime divisors.

REMARK 1.1.4. *For those who are familiar with sheaves and line bundles, since X is smooth, the group* $\mathrm{Div}(X)$ *is equivalent to the group of Cartier divisors or line bundles on X.*

DEFINITION 1.1.5. A divisor $D \in \mathrm{Div}(X)$ is effective if $D = \sum n_i Z_i$ with Z_i prime divisors and $n_i \geq 0$ for any i. The support of an effective divisor D is $\mathrm{Supp}(D) = \cup_{n_i>0} Z_i$. Sometimes we may use $D \geq 0$ for effective divisors.

We say that a divisor is principal if $D = (f)$, for some rational function $f \in K(X)$.

Principal divisors are clearly a subgroup of $\mathrm{Div}(X)$ and we say that D is linearly equivalent to D_1, $\mathbf{D \sim D_1}$, if $D - D_1$ is principal. The class group $Cl(X) := \mathrm{Div}(X)/\sim$

REMARK 1.1.6. *For smooth varieties X the group is also called Picard group of X, $Pic(X)$, and it is the group of line bundles up to linear equivalence.*

EXAMPLE 1.1.7. *Let $X = \mathbb{A}^n$ then any codimension 1 is defined by a single equation and every prime divisor is principal. In other words $Cl(\mathbb{A}^n) = 0$. Note more generally that a Noetherian ring A is UFD if and only if every prime ideal of height 1 is principal. In particular if X is quasi affine (smooth) and $A(X)$ is UFD then $Cl(X) = 0$.*

REMARK 1.1.8. *Note that the notion of order works on an affine variety non singular in codimension 1. But one could loose the bijection between codimension 1 subvarieties and zero locus of regular functions. Think to the quadric cone $Q \subset \mathbb{P}^3$ with $I(Q) = (x_0x_1 - x_2^2)$. The lines of the ruling are codimension 1 subvarieties but cannot be defined (with multiplicity 1) by one single equation. This is because the local ring at $[0,0,0,1]$ is not factorial.*

Let us work it out in greater details. Let $l = (x_0 = x_2 = 0) \subset Q$ be a line and $U = Q \setminus l$. It is clear that any divisor on Q can be written as a divisor in U plus some multiple of l. That is

$$Cl(Q) = Cl(U) + \mathbb{Z} < l > .$$

Next observe that

$$A(U) = A(Q)_{x_0} = k[x_0, x_1, x_2, x_0^{-1}]/(x_0x_1 - x_2^2) \cong k[x_0, x_0^{-1}, x_2].$$

The latter is UFD ($k[x, x^{-1}]$ is UFD because any element is of the form $P = \sum_{-d_1}^{d_2} a_i x^i$ and therefore $x^{d_1}P \in k[x]$ has a unique factorization) therefore $Cl(U) = 0$. Moreover $(x_0) = 2l$ therefore $2l$ is principal. A bit more of algebra shows that l itself is not principal and proves that $Cl(Q) \cong \mathbb{Z}_2$ and generated by l, see also [**Ha**].

EXAMPLE 1.1.9. *Let $X = \mathbb{P}^n$ and $H = (x_0 = 0) \subset \mathbb{P}^n$ be an hyperplane. Let Z be a prime divisor with $Z = V(F)$ for some homogeneous polynomial F of degree d. Then F/x_0^d is a rational function, that is $Z - dH$ is principal. Assume that*

$D=\sum d_iZ_i$ then $D-(\sum d_i \deg Z_i)H$ is principal. This yields the following group morphism

$$\deg : Cl(\mathbb{P}^n)\to\mathbb{Z}$$

sending $[D]$ to its degree. The morphism deg *is clearly surjective. Assume that* $\deg[D]=0$*, let $D=D_+-D_-$ with $D_\pm\geq 0$. Then* $\deg[D]=0$ *forces* $\deg[D_+]=\deg[D_-]$ *and therefore D is principal.*

It is important to summarize the above example for future reference

PROPOSITION 1.1.10. *Let $D\in Cl(\mathbb{P}^n)$ be a divisor of degree d, that is the zero locus of a polynomial of degree d, and H an hyperplane. Then*

- i) *$D\sim dH$*
- ii) *for any non vanishing function $f\in K(\mathbb{P}^n)$,* $\deg(f)=0$
- iii) *$Cl(\mathbb{P}^n)\cong\mathbb{Z}$ via the degree map.*

1.2. The canonical divisor

Let X be a smooth variety of dimension n over the complex numbers (this is necessary only for the treatment we are going to do, not for the results obtained). Then we may consider a meromorphic n-form that is

$$\Omega=fdx_1\wedge\ldots\wedge dx_n,$$

where $(x_1,\ldots,x_n)$ are local coordinates and $f\in K(X)$. The main reason that lead us to consider this it is that if we change coordinates $(w_1,\ldots,w_n)$ then

$$dx_1\wedge\ldots\wedge dx_n=\det(\frac{\partial w_i}{\partial x_j})dw_1\wedge\ldots\wedge dw_n.$$

This shows that Ω is canonically defined, up to a multiplicative nonvanishing function. Moreover if we consider two n-forms Ω and Ω_1 then

$$\frac{\Omega}{\Omega_1}\in K(X).$$

This justifies the following

DEFINITION 1.2.1. The canonical divisor is the divisor associated to Ω

$$K_X=(\Omega).$$

In particular K_X is a well defined element in $Cl(X)$.

Let us compute the canonical divisor in some examples

REMARK 1.2.2. *Let $X=\mathbb{P}^n$ be the projective space. Consider $X=\mathbb{C}^n\cup H_0$ with coordinates $(z_1,\ldots,z_n)$ where $z_i=x_i/x_0$ and $H_0=(x_0=0)$. Then we may consider the n-form*

$$\Omega=dz_1\wedge\ldots\wedge dz_n.$$

The form Ω has neither zero nor poles along $\mathbb{C}^n$ therefore we have only to worry about H_0. Consider the change of coordinates $w_j=x_j/x_1$ that induces the change $z_1=1/w_0$ and $z_i=w_i/w_0$, for $i\neq 1$. With this change we have

$$dz_1=\frac{1}{w_0^2}dw_0,\ dz_i=\frac{1}{w_0}dw_i-\frac{w_i}{w_0^2}dw_0.$$

This yields

$$\Omega=\frac{1}{w_0^{2+(n-1)}}dw_0\wedge dw_2\wedge\ldots\wedge dw_n.$$

That is (Ω) has a pole of order $n+1$ along H_0.

This proves that $K_{\mathbb{P}^n} \sim -(n+1)H$ has degree $-(n+1)$.

Another nice fact about the canonical class is the possibility to compute it in a wide range of cases.

THEOREM 1.2.3. *Let $D \subset X$ be an irreducible smooth divisor on smooth variety. Then $K_D = (K_X + D)_{|D}$.*

IDEA OF PROOF. Let Ω be a n-form on X. Let $f \in K(X)$ be such that $(f) = D+R$ for some divisor R with $\mathrm{Supp}(R) \not\supset D$. Then compute the residue of Ω/f. $\square$

REMARK 1.2.4. *In particular for any smooth hypersurface $Y \subset \mathbb{P}^n$ of degree d we have $K_Y \sim L_Y(d-(n+1))$.*

REMARK 1.2.5. *The canonical class is a fundamental birational invariant and can be interpreted as the determinant of the cotangent bundle.*

1.3. Linear systems

Let X be a smooth projective variety and $D = \sum n_i Z_i$ a Weil divisor on X.

DEFINITION 1.3.1. Let

$$L_X(D) := \{f \in K(X) | (f) + D \geq 0\} \cup \{0\}$$

be the linear system associated to D.

REMARK 1.3.2. *Note that $L_X(D)$ is naturally a vector space and it is the set of rational functions having order $\geq -n_i$ along Z_i (keep in mind that the summation is over any codimension 1). It is easy to note that $\mathbb{P}(L_X(D))$ is the, eventually empty, set of effective divisors linearly equivalent to D. We will always tacitly use this identification between rational functions in $L_X(D)$ and divisors linearly equivalent to D.*

DEFINITION 1.3.3. The complete linear system associated to D is $|D| := \mathbb{P}(L_X(D))$.

REMARK 1.3.4. *If $D \sim D_1$ then $|D| = |D_1|$.*

DEFINITION 1.3.5. A linear system Σ on X is a sublinear space $\Sigma \subset |D|$ for some effective divisor D on X. In other words Σ is the projectivization of a subvector space of $W \subseteq L_X(D)$. The projective dimension of $\Sigma = \dim W - 1$.

A crucial point in the theory of projective varieties is the finite dimensionality of $|D|$ for any $D \in Cl(X)$. This can be proved in greater generality via the identification with regular section of line bundles, [**Ha**, Theorem II.5.19]. Our next aim is to give a self contained proof for smooth projective varieties based on simpler arguments.

DEFINITION 1.3.6. Let $L(d) := L_{\mathbb{P}^n}(dH)$, for some hyperplane H, the linear system of degree d.

REMARK 1.3.7. *Let $H = (x_0 = 0)$ be an hyperplane, then $dH \in L(d)$ and for any polynomial $P \in k[x_0, \ldots, x_n]_d$ we have*

$$f = P/x_0^d \in K(\mathbb{P}^n)$$

in particular $S_P := (P = 0) \sim dH$ *and* $S_P \in L_{\mathbb{P}^n}(dH)$ *and allows to call it the linear system of degree* d *of* $\mathbb{P}^n$. *This also shows that* $L(d) \cong k[x_0, \dots, x_n]_d$ *as vector spaces and* $\dim |L(d)| = \binom{d+n}{n} - 1$.

The elements of the linear system $|L(d)|$ are hypersurfaces of degree d. Let $S \in |L(d)|$ be a hypersuface not containing $X \subset \mathbb{P}^n$. Then $X \cap S = \cup Z_i$, with Z_i divisors on X. Moreover for any i there is an hyperplane $(x_{j_i} = 0)$ that does not contain Z_i. Let $S = (F = 0)$ then we may define $z_i := ord_{Z_i}(F/x_{j_i}^d)$ and define the restricted divisor.

$$S_X = \sum z_i Z_i.$$

Note that $S_X \sim S_{1X}$ if $S, S_1 \in |L(d)|$. We may then define the restricted linear system.

Definition 1.3.8. The restricted linear system is $L_X(d) := \{D_X\}_{D \in |L(d)|}$.

Remark 1.3.9. *Note that in general* $L_X(d)$ *is not complete.* X *is said to be k-projectively normal if* $L_X(k)$ *is complete. A simple example of this phenomenon is given by smooth rational curves in* $\mathbb{P}^3$. *Let* $\Gamma \subset \mathbb{P}^3$ *be a smooth rational curve of degree* $\gamma > 3$. *Then* $\deg L_\Gamma(1) = \deg \Gamma$ *and* $\dim L_\Gamma(1) \leq 3$ *(one could prove it is always 3 via the genus formula for plane curves). On the other hand by Remark 1.3.7, we have* $|L_\Gamma(1)| = |L_{\mathbb{P}^1}(\gamma)| = \gamma + 1$.

Proposition 1.3.10. *Let* $X \subset \mathbb{P}^n$ *be a smooth variety. Then for* $d \gg 0$ *the linear system* $L_X(d)$ *is complete.*

Proof. We may assume that X is non degenerate and let $D \in |L_X(d)|$ be an element. By hypothesis $D \sim dH_i$, for H_i the divisor associated to the equation $x_i = 0$. Then there is $f_i \in K(X)$ such that $(f_i) = D - dH_i$ and this yields

$$(\frac{f_i}{f_j}) = d(H_j - H_i) = (\frac{x_j^d}{x_i^d}).$$

Up to constants this gives

$$\frac{f_i}{f_j} = \frac{x_j^d}{x_i^d},$$

and allows to define $F = x_i^d f_i \in K(A(X))$, the quotient field of $A(X)$ (remember that $K(X)$ is the degree 0 piece of $K(A(X))$, i.e. quotients where the numerator and denominator have the same degree).

The divisor D is effective, hence $(f_i) \geq 0$ on the open $X_i := X \setminus (X \cap H_i)$ and f_i is regular on X_i. Then $f_i = F_i/x_i^{N_i}$, with $\deg F_i = N_i$, and $x_i^M F \in A(X)$ for any $M \geq N_i$. Let $N = \max\{N_i\}$ then

$$x_i^{2N-d} F = x_i^{2N} f_i = x_i^{2N-N_i} F_i \in A(X)$$

Note that $A(X) = k[x_0, \dots, x_n]/I$ with I homogeneus prime ideal, therefore there is a d_0 such that for any $d \geq d_0$ if $F \in K(A(X))$ and $x_i^M F \in A_{M+d}$, for all i, then $F \in A_d$.

This shows that $(F) \geq 0$ on $X \cap (\cap_i H_i)^c = X$, $F \in A(X)_d$ and

$$D = (f_i) + dH_i = (F)$$

that is $D \in L_X(d)$. □

Theorem 1.3.11. *Let* $X \subset \mathbb{P}^n$ *be a smooth variety. Then any linear system is finite dimensional.*

Proof. Let D be a divisor then we may find a $m \gg 0$ and an hypersurface $G \in |L(m)|$ such that $G \cap X = D + D_1$. Then $|D| + D_1 \subseteq L_X(m) \subseteq |L(m)|$ and the latter is finite dimensional by Remark 1.3.7. □

A fundamental result in the theory of linear system is the following theorem of Bertini, to state it we need a definition.

Definition 1.3.12. Let X be a projective variety and Σ a linear system on X. The Base locus of Σ is

$$\mathrm{Bs}(\Sigma) := \{x \in X | x \in \mathrm{Supp}(D), \forall D \in \Sigma\}$$

Theorem 1.3.13 (Bertini's Theorem). *Let Σ be a linear system on a smooth variety, and $D \in \Sigma$ a general element. Then D is smooth outside* $\mathrm{Bs}\,\Sigma$.

Proof. It is enough to prove the statement for a pencil Λ. Let $D \in \Lambda$ a general element. We may assume that locally $D = (f(x_1, \ldots, x_n) + \lambda g(x_1, \ldots, x_n) = 0)$ and $0 \in \mathrm{Supp}(D)$ is a singular point. Suppose that 0 is not in the base locus, that is $f(0) \neq 0$, then also $g(0) \neq 0$ and determine

$$\lambda = -\frac{f(0)}{g(0)}.$$

The divisor D is singular at 0 if and only if

$$\partial_i f(x_1, \ldots, x_n) - \frac{f(0)}{g(0)} \partial_i g(x_1, \ldots, x_n) = 0$$

This shows that

$$\partial_i (\frac{f}{g})(0) = \frac{\partial_i f(0) - \frac{f(0)}{g(0)} \partial_i g(0)}{g(0)} = 0.$$

This shows that $\frac{f}{g}$ is constant on every connected component of the singular locus, say V, outside the base locus. But V is an algebraic variety and therefore has a finite number of connected components. Therefore there are only finitely many divisors that meet V outside the base locus. □

1.4. Morphisms and rational maps

Let X be a projective variety and Σ a linear system on X of dimension r. Note that for any $x \in X \setminus \mathrm{Bs}\,\Sigma$ the sublinear system $\Sigma_x := \{D \in \Sigma | x \in \mathrm{Supp}\, D\}$ is an hyperplane in Σ (the condition to pass through a point is a linear equation in Σ). Fix an isomorphism between $\mathbb{P}^r$ and Σ^* and define the application

$$\varphi_\Sigma : X \dashrightarrow \mathbb{P}^r$$

given by

$$\varphi_\Sigma(x) = [\Sigma_x].$$

There are important remarks before proceeding further.

Remark 1.4.1. *Let $H \subset \mathbb{P}^r$ be an hyperplane. Then $H = (D)^*$, for some $D \in \Sigma$, and by definition $\varphi_\Sigma^{-1}(H) = \{x \in X | [\Sigma_x] \in D^*\}$. Hence duality yields*

$$\varphi_\Sigma^{-1}(H) = \{x \in X | x \in D\}.$$

REMARK 1.4.2. *The map φ_Σ is well defined outside* $\mathrm{Bs}(\Sigma)$. *In particular it is not difficult to prove that if Σ is base point free then φ_Σ is a morphism and not only an application. Moreover the choice of a base in Σ determines homogeneous coordinates on $\mathbb{P}^r$, therefore a linear transformation on $\mathbb{P}^r$ is induced by a change of base in Σ.*

REMARK 1.4.3. *Consider a linear system Σ is $\mathbb{P}(W)$ for some $W \subseteq L_X(D)$. Let $(f_0, \ldots, f_r)$ be a base of W then, up to a linear transformation, $\phi(x) = [f_0(x), \ldots, f_r(x)]$.*

REMARK 1.4.4. *Let Φ_D be the morphism associated to $|D|$ then we may complete $(f_0, \ldots, f_r)$ to a base of $L_X(D)$ adding $(g_{r+1}, \ldots, g_N)$. In this notation $\phi_\Sigma = \Pi \circ \Phi_D$ where Π is the projection of $\mathbb{P}(L_X(D))$ onto the linear space of equation $\{y_{r+1} = \ldots = y_N = 0\}$ from the linear space $\{y_0 = \ldots = y_r = 0\}$.*

Let $\phi : X \to Y \subset \mathbb{P}^N$ be a surjective morphism. Then a divisor $D \in L_Y(1)$ is the restriction of an hyperplane H_D defined as the zero locus of the regular function $f_i = h/y_i$ on the open $U_i := \{y_i \neq 0\}$. Then we may define the pull-back ϕ^*D as divisor locally defined by the regular function $f_i \circ \phi$. In particular $Supp(\phi^*D) = \phi^{-1}(D)$ and clearly $\phi^*D \sim \phi^*D_1$ if $D_1 \in L_Y(1)$. This produces a linear system of dimension N $\Sigma_\phi \subset |\phi^*D|$ as the linear span of the ϕ^*D such that $\phi = \varphi_{\Sigma_\phi}$.

One can define the pull back of an arbitrary prime divisor B on Y considering an hypersurface $S = (P = 0) \subset \mathbb{P}^N$ such that $S_Y = B + D \in L_Y(m)$ with $Supp(D) \not\supset Supp(B)$. Then as above define ϕ^*S_Y as the divisor locally associated to the function $P/y_i^m \circ \phi$. Then $\phi^*B = b_i\phi^{-1}(B)$ with $b_i = ord_{\phi^{-1}B}\phi^*S_Y$. Everything extends by linearity.

Assume that ϕ is generically finite of degree d, then for any effective prime divisor A we may also define the push-forward ϕ_*A as follows

$\phi_*A = 0$ if $\phi(A)$ is not a divisor

$\phi_*A = r\phi(A)$ if $\phi(A)$ is a divisor and $\phi_{|A}$ is generically of degree r, and then extend it by linearity. In particular for any divisor B on Y we have $\phi_*\phi^*B = dB$, and if $A \sim A_1$ then

$$\phi_*A \sim \phi_*A_1 \tag{1}$$

(The latter reduces to prove that $\phi_*(f)$ is principal. It is a bit technical but not difficult to see that $\phi_*(f) = (N(f))$, where $N(r)$ is the norm of r, the determinant of the linear endomorphism given by multiplication by f).

All the above can be extended, with some caution, to rational maps. It is enough to consider everything on the open set where the map is defined. The following proposition is the clue.

PROPOSITION 1.4.5. *Let $\varphi : X \dashrightarrow \mathbb{P}^n$ be a rational map with X smooth. Then the indeterminacy locus of φ is of codimension at least* 2. *In particular*

- *any rational map from a non singular curve is a morphism;*
- *any birational map between smooth curves is an isomorphism;*
- *any smooth rational curve is isomorphic to $\mathbb{P}^1$.*

PROOF. The question is local let $x \in X$ be a point in the indeterminacy locus of φ. We may consider the local ring $\mathcal{O}_{X,x}$ for a point $x \in X$. Since X is smooth $\mathcal{O}_{X,x}$ is regular, hence factorial, and finally UFD. In particular any codimension 1 subvariety is defined by a single equation. Let φ be given by $(g_0, \ldots, g_n)$, with

$g_i \in K(X)$. We may multiply by a common factor $g \in K(X)$ in such a way that $g_i \in \mathcal{O}_{X,x}$ and have no common factor there. Note that this multiplication does not change the map φ. The indeterminacy locus of φ in a nbhd of x is the common zero locus of the g_i. Assume that there is a codimension 1 component Z of indeterminacy passing through x, and let $g \in \mathcal{O}_{X,x}$ be its equation. This forces g_i to have the common factor g and yields the required contradiction. $\square$

REMARK 1.4.6. *From the point of view of linear systems the above proposition states that any rational map can be defined by linear systems without fixed components. That is when considering the map associated to a linear system we may always, freely, either add, or remove a fixed divisor.*

DEFINITION 1.4.7. A rational map $\varphi : X \dashrightarrow Y$ is called birational if it is generically injective and the inverse is an algebraic map. This is equivalent to prove that there are open non empty sets $U_X \subset X$ and $U_Y \subset Y$ such that $\varphi_{|U_X} : U_X \to U_Y$ is an isomorphism.

It is time to introduce one of the main actors of the lecture.

DEFINITION 1.4.8. A projective variety X is called rational if there is a birational map $\varphi : X \dashrightarrow \mathbb{P}^n$.

There is an interesting fact about rational varieties.

PROPOSITION 1.4.9. *Let $X \subset \mathbb{P}^N$ be a rational variety of dimension n. Then there is an integer d and a linear projection $\pi : \mathbb{P}^{\binom{n+d}{n}-1} \dashrightarrow \mathbb{P}^N$. Such that $X = \pi(V_{d,n})$, where $V_{d,n}$ is the Veronese embedding of degree d of $\mathbb{P}^n$. In other words any rational variety is some projection of a Veronese variety.*

PROOF. Let $\phi : \mathbb{P}^n \dashrightarrow X$ be a birational map. Then $\phi^* L_X(1) \subset |L(d)|$ for some d and we conclude by Remark 1.4.4. $\square$

EXAMPLE 1.4.10. *We already described all linear systems on $\mathbb{P}^n$. The morphism associated to $|L(d)|$ are the Veronese embedding of $\mathbb{P}^n$ into $\mathbb{P}^{\binom{n+d}{n}-1}$.*

Let $C \subset \mathbb{P}^n$ be a smooth curve and p, q be two distinct points. Then we may consider p and q as divisors on C. In this terms if $p \sim q$ then we may consider a (sub)linear system of dimension 1 in $L_C(p)$ and define an injective morphism $\varphi : C \to \mathbb{P}^1$. It is easy to see that φ is a bijection, and not too difficult to prove that it is an isomorphism, you may confront the section 1.6 on very ample linear systems. This shows that two points on a curve are linearly equivalent if and only if the curve is $\mathbb{P}^1$. In particular the degree of a linear system is not enough, in general, to characterize the linear system.

Let $Q \subset \mathbb{P}^3$ be a smooth quadric of equation $(x_0x_3 - x_1x_2 = 0)$. Then $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, via the Segre embedding

$$\phi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$$

given by $\phi([t_0, t_1][s_0, s_1]) = [t_0s_0, t_0s_1, t_1, s_0, t_1, s_1]$. Let $F_1 = \mathbb{P}^1 \times [1, 0]$ and $F_2 = [1, 0] \times \mathbb{P}^1$ be two divisors and $U = \mathbb{P}^1 \times \mathbb{P}^1 \setminus (F_1 \cup F_2)$. Then $U \cong \mathbb{A}^2$ and if we consider a divisor $D \subset Q$ we have $D_{|U} = (f)$ for some rational function on U, cfr Example 1.1.7. Therefore $D \sim (f) + d_1F_1 + d_2F_2$, for some integer d_1 and d_2. This shows that any divisor on Q is uniquely determined by its bidegree (d_1, d_2). Let $C \subset Q$ be a curve of bidegree $(1, 2)$ then $C \cong \mathbb{P}^1$ and $\deg F_{1|C} \neq \deg F_{2|C}$ therefore $F_1 \not\sim F_2$ (the restriction of two linearly equivalent divisors is linearly equivalent).

Note that if we consider Λ, the Linear system generated by F_1 and $\mathbb{P}^1 \times [1,1]$, then ϕ_Λ is the canonical projection onto the second factor, and similarly for F_2.

1.5. The happy life of surfaces: intersection and resolution

In this section we fix a smooth projective surface S. This allows us to give self contained and rather elementary proofs of general results: intersection theory and resolution of the indeterminacy of rational maps. Let us start with intersection and consider $C, C_1 \subset S$ two distinct prime divisors (i.e. two irreducible curves). Our aim is to define the intersection number between C and C_1 and to prove that it is costant in the linear equivalence class.

The definition is easy to get

Definition 1.5.1. Let $C, C_1 \subset S$ be irreducible reduced distinct curves and assume that $x \in C \cap C_1$ and f, respectively f_1 is an equation of C, respectively C_1, in $\mathcal{O}_{S,x}$. The intersection multiplicity at x is

$$m_x(C \cap C_1) := \dim_k \mathcal{O}_{S,x}/(f, f_1).$$

The intersection multiplicity of C and C_1 is just the sum

$$C \cdot C_1 =: \sum_{x \in C \cap C_1} m_x(C \cap C_1)$$

Remark 1.5.2. *By Nulltellensatz $\mathcal{O}_{S,x}/(f, f_1)$ is a finite dimensional k-vector space. Intuitively seems the right definition if both C and C_1 are smooth at x and intersect transversely then f and f_1 generate the maximal ideal and therefore the intersection multiplicity is 1. For those familiar with sheaves one can define a skyscraper sheaf supported on $C \cap C_1$ and such that the global section correspond to the number $C \cdot C_1$.*

Exercise 1.5.3. Let $l \subset \mathbb{P}^2$ be a line and C a plane curve. Check that $\sum_{x \in l \cap C} m_x(l \cap C) = \deg C$.

Note that the intersection defined extends by linearity to any pair of divisors without common components in the support.

Proposition 1.5.4. *The intersection number is invariant in a linear equivalence class(i.e. if $C \sim C_1$ then $C \cdot D = C_1 \cdot D$ for any irreducible divisor D with $D \neq C$ and $D \neq C_1$).*

We prove the result via a reduction technique. First we prove it for the linear system $L_X(1)$. Let $\Gamma \subset S \subset \mathbb{P}^N$ be a prime divisor. It is clear that if $D \in L_S(1)$ is a general element than $D \cdot \Gamma = \deg \Gamma$, the degree of the curve Γ, that is the number of points in common with a general hyperplane. To extend it to any hyperplane we use the following.

Lemma 1.5.5. *Let $\Gamma \subset \mathbb{P}^N$ be a curve. Let $\pi_p : \mathbb{P}^N \dashrightarrow \mathbb{P}^{N-1}$ be the projection from a point $p \in \mathbb{P}^N$ and f_p its restriction to Γ. If there exists a line $l \subset \mathbb{P}^N$ such that f_p is not birational for a general point $p \in l$, then Γ is a plane curve (i.e. $< \Gamma > \cong \mathbb{P}^2$).*

Proof. Let $l \subset \mathbb{P}^N$ be such a line and fix a general point $x \in \Gamma$. Then for a general point $p \in l$ the line $< p, x >$ intersect. $\Gamma \setminus \{x\}$. This shows that $\Gamma \subset < l, x >$. $\square$

Remark 1.5.6. *The Lemma is a special case of a more general result of Segre about the locus of points where the projection of a variety is not birational* [**CC**].

Lemma 1.5.7. *Let $D \in L_S(1)$ be any divisor and Γ an irreducible reduced curve. Assume that $\Gamma \not\subset \operatorname{Supp} D$ then $D\cdot\Gamma = \deg\Gamma$.*

Proof. Let $H \subset \mathbb{P}^N$ be any hyperplane. Since $\Gamma \subset S$ then by definition

$$H\cdot\Gamma = \sum_{x\in H\cap\Gamma} m_x(H_S\cap\Gamma)$$

By Lemma 1.5.5 we may birationally project Γ to $\mathbb{P}^3$ from a general codimension 4 linear space contained in H. Let π be the map, Γ_1, S_1, and H_1 the projections of Γ, S, and H respectively. Then $H_1 = (h_1 = 0)$ is a plane, $S_1 = (f_1 = 0)$ and, by Lemma 1.5.5, Γ_1 is a curve of degree $\deg\Gamma$. Moreover we may assume that $\pi_{|S}$ is an isomorphism in a nbhd of $H\cap\Gamma$ (check this as an exercise). Now let $p \in H_1$ be a general point and $\pi_p : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ the projection from p. Let Γ_2 and H_2 be the projection of Γ_1 and H_1. Then again by Lemma 1.5.5 $\deg\Gamma_2 = \deg\Gamma_1 = \deg\Gamma$ and H_2 is a line. Let $S_2 = (f_2 = 0)$ be the cone over Γ_2 with vertex p. Let $x \in \Gamma_1\cap H_1$ be a point. By definition we have

$$m_x(H_1\cap\Gamma_1)_{S_1} = \dim\mathcal{O}_{S_1,x}/(h_1,f_2) = \dim\mathcal{O}_{S_2,x}/(h_1,f_1) = m_x(H_1\cap\Gamma_1)_{S_2}.$$

On the other hand S_2 is a cone with vertex p and $H_1 \ni p$, therefore

$$m_x(H_1\cap\Gamma_1)_{S_2} = \dim\mathcal{O}_{S_2,x}/(h_1,f_1) = m_{\pi_p(x)}(H_2\cap\Gamma_2).$$

This yields

$$\sum_{x\in H\cap\Gamma} m_x(H\cap\Gamma) = H\cdot\Gamma = H_2\cdot\Gamma_2 = \sum_{x\in H_2\cap\Gamma_2} m_x(H_2\cap\Gamma_2),$$

The intersection $H_2\cdot\Gamma_2$ is given by a line and a plane curve therefore, by Exercise 1.5.3

$$\sum_{x\in H_2\cap\Gamma_2} m_x(H_2\cap\Gamma_2) = \deg\Gamma_2 = \deg\Gamma.$$

□

We are now ready to prove Proposition 1.5.4.

Proof of Proposition 1.5.4. Let $C \sim C_1$ be prime divisors on S and D a fixed divisor. For $m \gg 0$ there exists an effective divisor Δ such that $C+\Delta \in L_S(m)$ and $\operatorname{Supp}\Delta$ is disjoint from $(C\cap D)\cup(C_1\cap D)$. Let $\nu : \mathbb{P}^N \to \mathbb{P}^M$ be the m^{th} Veronese embedding. Then by Lemma 1.5.7 $\nu_*(C+\Delta)\cdot\nu_*(D) = \nu(C_1+\Delta)_*\cdot\nu_*(D)$. This is enough to conclude that $\nu_*(C)\cdot\nu_*(D) = \nu_*(C_1)\cdot\nu_*(D)$, and since ν is an isomorphism, cfr Exercise 1.7 a), yields

$$C\cdot D = C_1\cdot D.$$

□

Thanks to proposition 1.5.4 and the possibility to move the support of a divisor away from any subvariety we may define the intersection in a more general way

Definition 1.5.8. Let D and D_1 be divisors on a smooth surface S. Then the intersection $D\cdot D_1 := C\cdot C_1$, for any pair of divisors $C \sim D$ and $C_1 \sim D_1$ with C and C_1 without common components.

There are special important cases to keep in mind.

COROLLARY 1.5.9. *Let S be a surface and $\varphi : S \to \mathbb{P}^N$ a morphism. Let $D \in \varphi_\Sigma$ be a divisor then*

- *if* $\dim \varphi(S) < 2$ *then* $D^2 := D \cdot D = 0$
- *if φ is generically finite of degree d then* $D^2 = d \deg \varphi(S)$.
- *if D_1 and D_2 are effective divisors without a common component then* $D_1 \cdot D_2 \geq 0$.
- *for any divisor D and $f \in K(X)$, $D \cdot (f) = 0$.*

COROLLARY 1.5.10 (Bezout theorem for plane curves). *Let C and D be two plane curve without a common component of degree γ and δ respectively. Then $C \cdot D = \gamma\delta$.*

It is important to stress a further application of Proposition 1.5.4

COROLLARY 1.5.11. *Let $f : S \to Y$ be a birational morphism between smooth surfaces. Let $C \subset S$ be a curve that is contracted by f, $D \in Cl(S)$, and $A \in Cl(Y)$. Then $C^2 < 0$ and $D \cdot f^*A = f_*D \cdot A$.*

PROOF. As we already observed $f^*A \cdot C = 0$ for any divisor. On the other hand if $\operatorname{Supp} A \ni f(C)$ then $f^*A = A_S + aC + \Delta$ for effective divisors A_S, and Δ not containing C in the support. Therefore

$$0 = f^*A \cdot C = aC^2 + (A_S + \Delta) \cdot C$$

The second addend is positive therefore the first one is negative.

By Proposition 1.5.4 we may choose a divisor $A_1 \sim A$ away from the exceptional locus of f^{-1} and such that $\operatorname{Supp}(f^*A_1)$ does not contain any irreducible component of D. That is we may assume that f is an isomorphism in a neighborhood of A_1. Hence we have

$$D \cdot f^*(A) = D \cdot f^*(A_1) = f_*D \cdot A_1 = f_*D \cdot A.$$

$\square$

REMARK 1.5.12. *For arbitrary varieties X intersection theory is much more complicate but one can define an intersection theory for any pair of subvarieties Z and W such that* $\dim Z + \dim W = \dim X$,[**Fu**]. *The simplified version for surfaces is due to the fact that the only case is that of curves and the intersection number for curves is just the degree of the restricted linear system. All the above could be generalized to the intersection of divisors and curves on an arbitrary smooth varieties with some effort, while the general case is of a totally different magnitude of difficulty.*

The other result that for surfaces is significantly simplified is resolution of the indeterminacy of maps. The main reason is that for surfaces the indeterminacy is given by a bunch of points and therefore it is enough to blow them up sufficiently many times.

Let me briefly recall the blowing up construction. The blow up is a local construction in a nbhd of a subvariety. To simplify everything we assume that $Z \subset \mathbb{A}^n$ is a complete intersection given by the equations $(f_0 = \ldots = f_r = 0)$. Then consider the subvariety

$$Bl_Z(\mathbb{A}^n) := \{([t_0, \ldots, t_r], (x_1, \ldots, x_n)) | f_i t_j = f_j t_i\} \subset \mathbb{P}^r \times \mathbb{A}^n.$$

Consider the natural projections $p_1 : Bl_Z(\mathbb{A}^n) \to \mathbb{A}^n$ and $p_2 : Bl_Z(\mathbb{A}^n) \to \mathbb{P}^r$.

Definition 1.5.13. The variety $Bl_Z(\mathbb{A}^n)$ is called the blow up of $\mathbb{A}^n$ along Z, and $p_1^{-1}(Z)$ is the exceptional divisor of the blow up.

It is easy to see that p_1^{-1} is an isomorphism outside Z (if at least one of the f_i is non zero then the solution is unique), while $p_1^{-1}(z) \cong \mathbb{P}^r$ for any point $z \in Z$.

To get a geometric flavor of the fiber over a point in Z assume $O := (0, \ldots, 0) \in Z$ and assume that $\mathbb{T}_O Z = (x_0 = \ldots = x_r = 0)$ and $f_i = x_i + q_i$, for $i = 0, \ldots r$. Consider a point $p := (a_1, \ldots, a_n) \notin T_O Z$ and the line $l_p :=< p, a >$ with parametric equation $x_i = \lambda a_i$. We may assume that $a_0 \neq 0$ then the preimage $p_1^{-1}(l_p \cap (\mathbb{A}^n \setminus Z))$ has parametric equation $x_i = \lambda a_i$ and

$$t_i = \frac{a_i}{a_0 + \lambda u_i} t_0 + \lambda s_i.$$

Therefore in the limit point for $\lambda = 0$ we have the point $(O, [a_0, a_1, \ldots, a_r])$. This shows that the point in $p_1^{-1}(O)$ may be considered as elements in $(\mathbb{T}_z Z)^{\perp}$, that is the complement to the tangent space at z of Z.

Further note that the coordinate hyperplane $H_i \subset \mathbb{P}^r$ given by $(t_i = 0)$, are pulled back to

$$F_i := p_2^*(H_i) = \overline{p_{1|\mathbb{A}^n \setminus Z}^{-1}((f_i = 0))}.$$

In particular $\cap_{i=0}^r F_i = \emptyset$.

Remark 1.5.14. *The blow up is a much more general construction, one can blow up any ideal in any algebraic variety and this blow up is uniquely determined,*[**Ha**].

Definition 1.5.15. Let $D \subset X$ be a prime divisor and $f : Y \to X$ a birational map. Let $V \subset Y$ and $U \subset X$ be dense open subset where f^{-1} restricts to an isomorphism. The strict transform of D is

$$D_Y := \overline{f^{-1}(U \cap D)}.$$

Remark 1.5.16. *Since a birational map is a morphism outside a codimension 2 the strict transform is a well defined Weil divisor. Note that in general the linear equivalence is not preserved by the strict transform.*

For the moment we are interested in blowing up points on a smooth surface. Let $\mu : Y \to S$ be the blow up of a smooth point x with exceptional divisor $E \cong \mathbb{P}^1$.

Observe that $Y \setminus E \cong S \setminus \{x\}$. Therefore any prime divisor $D \in Cl(Y)$ we have

$$\mu^* D = D_Y + aE,$$

for some integer a. Note further that if D is effective and $\mathrm{Supp}(D) \ni x$ then $a > 0$.

Lemma 1.5.17. *$a = \mathrm{mult}_x D$.*

Proof. it is enough to consider the blow up of $\mathbb{A}^2$ in the origin $(0, 0)$. Let $D = (p = 0)$ with $p = \sum_{i=m}^d p_i$, for $p_i \in k[x_1, x_2]_i$ and $m = \mathrm{mult}_{(0,0)} D$. Let $Y \to \mathbb{A}^2$ be the blow up with $Y \subset \mathbb{A}^2 \times \mathbb{P}^1$ with equations $x_1 t_1 = x_2 t_0$. In particular on the open affine where $t_0 \neq 0$ the local equation of Y is $x_2 = t_1 x_1$, and the local equation of E is $x_1 = 0$. Then a local equation of $\mu^*(D)$ is obtained substituting x_2 with $t_1 x_1$. This gives the required

$$\mu^*(D) = (x_1^m \sum_{i=m}^d \tilde{p}_i(x_1, t_1) = 0)$$

$\square$

EXERCISE 1.5.18. Compute the strict transform of the following curve along the blow up of $(0,0) \in \mathbb{A}^2$:

$C_1 = (x + y^2 = 0)$
$C_2 = (xy + x^3 + y^3 = 0)$
$C_3 = (x^2 - y^3 = 0)$

Arguing as usual we have

$$\mathrm{Cl}(Y) = \mathrm{Cl}(S) \oplus \mathbb{Z} < E > .$$

For any divisor $D \in \mathrm{Div}(S)$ we may associate $D_1 \sim D$ with $Supp(D_1) \not\ni x$, therefore $\mu^* D$ is a divisor not intersecting E. In particular $\mu^* D \cdot E = 0$ and $(\mu^* D)^2 = D^2$. On the other hand if we fix a smooth divisor D passing through x, by Lemma 1.5.17 we have $\mu^* D = D_Y + E$, with D_Y smooth and $D_Y \cdot E = 1$. This yields $E^2 = -1$. This completely determines the divisors on Y and their intersection behaviour.

For the canonical class we have

$$K_Y = \mu^* K_X + eE$$

for some integer e. To determine e note that $E \cong \mathbb{P}^1$ and $E^2 = -1$ therefore adjuntion formula 1.2.3 and the computation 1.2.2 yield

$$-2 = K_Y \cdot E + E \cdot E = K_Y \cdot E - 1,$$

that is $K_Y \cdot E = -1$.

Let us summarize this computations

PROPOSITION 1.5.19. *Let $\mu : X \to S$ be the blow up of a point x in a smooth surface S, with exceptional divisor E. Then*

- $E \cong \mathbb{P}^1$ *and* $E^2 = K_X \cdot E = -1$,
- $K_X = \mu^* K_S + E$,
- $\mu^* D \cdot E = 0$,
- $\mu^* D = D_X + mE$ *where* $m = \mathrm{mult}_x D$ *and* $\mathrm{Supp}(D_X) \not\supset E$.

DEFINITION 1.5.20. Let Σ be a linear system on S and $\epsilon : Y \to S$ be the blow up of a point x with exceptional divisor E. Then the multiplicity of Σ at x, $\mathrm{mult}_x \Sigma$, is the multiplicity of a general element of Σ in x. The strict transform linear system is

$$\Sigma_Y = \epsilon^* \Sigma - (\mathrm{mult}_x \Sigma) E.$$

It is important to stress the notion of equivalent birational maps with a definition.

DEFINITION 1.5.21. Let $f : X \dashrightarrow Y$ and $g : X' \dashrightarrow Y'$ be two birational maps. We say that f is equivalent to g (as birational maps) if there are open dense subsets $U \subset X$ and $V \subset Y$, $U' \subset X'$ and $V' \subset Y'$ and isomorphisms $\psi : U \to U'$ and $\phi : V \to V'$ such that $\phi(f(x)) = g(\psi(x))$ for any $x \in U$ and $f_{|U} : U \to V$ is an isomorphism.

REMARK 1.5.22. *Let Σ be a linear system on S and $\mu : X \to S$ be the blow up of a point $x \in S$ with exceptional divisor E. Let Σ_X be the strict transform linear system. Let φ_Σ and φ_{Σ_X} be the map induced by Σ and Σ_X. Then by definition*

$$\varphi_{\Sigma_X | X \setminus E} = \varphi_{\Sigma | S \setminus \{x\}}.$$

Therefore φ_Σ and φ_{Σ_X} are equivalent birational maps.

We are ready to prove the resolution result we are looking for.

THEOREM 1.5.23. *Let $\varphi : S \dashrightarrow \mathbb{P}^N$ be a rational map then there exists a smooth surface Y and morphisms $p : Y \to S$ and $q : Y \to \mathbb{P}^N$ such that $q = \varphi \circ p$ where φ is defined. Furthermore we may choose p as a sequence of blow ups of smooth points.*

PROOF. Let Z be the base locus of Σ_φ. Then by Proposition 1.4.5 Z is just a bunch of points. Let $D \in \Sigma_\varphi$ be a general element and fix a point $x \in \mathrm{Bs}\,\Sigma_\varphi$, then $m = \mathrm{mult}_x D > 0$. We prove the claim by induction on Σ^2_φ. Let $\mu : Y \to S$ be the blow up with exceptional divisor E. Then $D_Y = \mu^* D - mE$. This yields

$$D^2_Y = D^2 - m^2 < D^2.$$

Since the self intersection of a linear system without fixed components is non negative this concludes the proof. □

REMARK 1.5.24. *A similar result is true for any algebraic variety, but the proof requires more sofisticated techniques.*

A special result for surfaces, a bit beyond our possibilities, is Castelnuovo contraction Theorem

THEOREM 1.5.25. *Let S be a smooth surface and $E \subset S$ a curve with $E^2 = -1$ and $E \cong \mathbb{P}^1$. Then there is a morphism $\varphi : S \to S_1$ that contracts E and φ is the blow up of a smooth point.*

For surfaces, and conjecturally for any variety, it is true a much stronger result than the resolution we proved, called factorization Theorem.

THEOREM 1.5.26. *Let $\varphi : S \dashrightarrow S_1$ be a birational map between smooth surfaces. Then there is a smooth surface Z and two morphisms $p : Z \to S$ and $q : Z \to S_1$ such that $q = \varphi \circ p$ (as birational maps) and both p and q are a succession of blow ups of smooth points.*

PROOF. Thanks to Theorem 1.5.23 it is enough to prove that any birational morphism $q : S \to S_1$ factors via blowing ups. If q^{-1} is a morphism we are done. Assume that this is not the case and let

$$S_1 \xleftarrow{f} Z \xrightarrow{g} S$$

be the resolution of q^{-1}. Let $F \subset Z$ be a curve contracted by f then F is contracted by either g or q. By Theorem 1.5.23 the morphism f is a sequence of smooth blow ups. Therefore all curves contracted by f are $\mathbb{P}^1$ and there exists a curve F, contracted by f with $F^2 = -1$. If F is contracted by g then by Castelnuovo Theorem 1.5.25 we may blow it down and substitute Z with the blow down.

CLAIM 1.5.27. Assume that F is not contracted by g, then g is an isomorphism in a neighborhood of F.

PROOF OF THE CLAIM. By hypothesis $F^2 = -1$ and $g_*(F)$ is a curve contracted by q. Therefore $g(F) \cdot g(F) < 0$. Then $F = g^*(g_*(F)) - \sum e_i E_i$ for $e_i \geq 0$ and E_i exceptional divisors of g. Then

$$-1 = F^2 = F \cdot g^*(g_*(F)) - \sum e_i E_i \cdot F.$$

By Corollary 1.5.11 $F \cdot g^*(g_*(F)) = g_* F \cdot g_* F < 0$ and this yields $E_i \cdot F = 0$ for any i. □

The claim, together with Castelnuovo Theorem yields that q factors with the blow down of $g(F)$, say S'. Therefore we may substitute S with S' and repeat the argument. Since there are only finitely many curves contracted by q this is enough to conclude. □

1.6. Ample and very ample linear systems

We already saw that base point free linear system defines morphisms and viceversa, given a morphism there is a base point free linear system associated to it. The aim question now is to understand if it is possible to characterize linear systems that produce isomorphisms.

Definition 1.6.1. A linear system Σ on X is very ample if it is base point free and the associated morphism $\varphi_\Sigma : X \to \mathbb{P}^N$ is a closed immersion (i.e. induces an isomorphism between X and $\varphi(x)$).

Remark 1.6.2. *A very ample linear system is associated to a morphism that is both bijective and such that the differential has maximal rank at any point. Equivalently the inverse is well defined and algebraic*

There is nice and handy geometric way to characterize very ample linear systems.

Proposition 1.6.3. *A linear system Σ on X is very ample if and only if the following conditions are satisfies:*

a) *for any pair of point $p, q \in X$ there is a divisor $D \in \Sigma_p$ such that $D \not\ni q$ (this is usually said as Σ separates points)*
b) *for any point $p \in X$ and any tangent vector in $v \in \mathbb{T}_pQ$ there is $D \in \Sigma_p$ with $v \notin \mathbb{T}_pD$ (this is called separation of tangents).*

Proof. Condition a) guaranties that φ_Σ is injective, while condition b) guaranties that the differential is of maximal rank, this can be easily seen taking local equation of φ_Σ in a nbhd of a point. □

Definition 1.6.4. A linear system Σ is ample if for some, and hence any, $D \in \Sigma$ there is a positive m such that $|mD|$ is very ample. A linear system Σ is semi-ample if some multiple is base point free.

Remark 1.6.5. *Despite the appearence ample linear system are much easily treated than very ample ones. This is due to the fact that ampleness is "essentially" a numerical property of divisors.*

Semi-ample linear systems, and their opposite, are quite often very important in higher dimensional algebraic geometry. We will not dwell on these here.

It is useful here to recall an ampleness criteria due to Nakai–Moeshizon

Theorem 1.6.6. *Let S be a smooth surface then D is ample if and only if $D^2 > 0$ and $D \cdot C > 0$ for any curve $C \subset S$.*

Remark 1.6.7. *Note that it is not enough the second condition alone. An example is the projective plane blown up in 13 general points. Note that there are no quartics singular at one point and passing through the remaining 12. On the* *other hand if we let $\mu : S \to \mathbb{P}^2$ the blow up of the 13 points, l the pull back of a line and E_i the exceptional divisor of the i^{th} blow up then*

$$4l - 2E_0 - \sum_{i=1}^{12} E_i$$

has positive intersection with any effective curve and self intersection 0.

Later on, maybe, we will introduce the cone of effective curves and get an ampleness criteria based only on intersection with curves.

1.7. Exercises

(1) Complete Remark 1.2.2.
(2) Prove Corollary 1.5.9.
(3) Use the fact that, for X smooth, $Cl(X \times \mathbb{A}^1) \cong Cl(X)$ to prove that for a smooth quadric Q the class group satisfies $Cl(Q) = \mathbb{Z} \oplus \mathbb{Z}$.
(4) Prove that the linear systems $|L(d)|$ on $\mathbb{P}^n$ are very ample for any d.
(5) Let $C \subset \mathbb{P}^2$ be a smooth curve and $p \in C$ a point. Prove that the projection from p is an isomorphism on the image if and only if C is a conic.
(6) Let $Q \subset \mathbb{P}^3$ be a smooth quadric and $p \in Q$ a point. Let $\pi_p : Q \dashrightarrow \mathbb{P}^2$ be the projection from p. Show that π_p is birational and determine the indeterminacy locus of π_p and π_p^{-1}
(7) The previous exercise shows that Q is rational, determine a projection from $V_{2,2}$ as in Proposition 1.4.9.
(8) Show that any irreducible Quadric of any dimension is rational and determine a projection from $V_{2,n}$ for the smooth ones.
(9) Let $A := \{p_1, p_2, p_3\} \subset \mathbb{P}^2$ be three points in general position. Consider the linear system Σ_A of conics through A. Determine the map associated to Σ_A and give explicit equations for the map.
(10) Do the same with 2 points $\{p_0, p_1\}$ and a tangent direction at p_0.
(11) Let $p_1, \ldots, p_6 \subset \mathbb{P}^2$ be points such that no 6 are on a conic and not 3 on a line. Let Σ be the linear system of cubics through the points $p_1, \ldots, p_6$. Show that the map φ_Σ is birational and maps $\mathbb{P}^2$ onto a cubic surface in $\mathbb{P}^3$. Try to determine the linear system giving the inverse map.
(12) Try to understand what happens is we choose either 6 points on a conic or 3 points on a line in exercise 12.
(13) Produce an effective factorization in blows up and down of the rational maps in Exercise 10,11, 12,13.
(14) Prove that there is not a d_0 such that all rational varieties of dimension n are projection of $V_{d_0,n}$.

CHAPTER 2

Rational surfaces

We aim to study and classify rational surfaces.

2.1. Examples of rational surfaces

Let us start with the following useful Lemma.

LEMMA 2.1.1. *Let X be a variety and $D \subset X$ a divisor. Assume that D is locally defined by a single equation and it is smooth. Then X is smooth along D.*

PROOF. Let $x \in \mathrm{Supp}(D)$ be a point and $f \in \mathcal{O}_{X,x}$ a local equation of D. By hypothesis the local ring $\mathcal{O}_{D,x}$ is regular. On the other hand $\mathcal{O}_{D,x} = \mathcal{O}_{X,x}/(f)$. Therefore $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{D,x} + 1$ and $\mathcal{O}_{X,x}$ is regular as well. □

EXAMPLE 2.1.2. *Let $x \in \mathbb{P}^2$ be a point and $\mu : X \to \mathbb{P}^2$ the blow up with exceptional divisor E. Then $Cl(X)$ is generated by $Cl(\mathbb{P}^2)$ and E, that is*

$$Cl(X) = \mathbb{Z} < \mu^* L(1) > \oplus \mathbb{Z} < E > .$$

We may also give a slightly different description of divisors on X. Let $l \subset \mathbb{P}^2$ be a line through x and $F \subset X$ its strict transform on X, then $F = \mu^*(l) - E$, $F^2 = 0$, $F \cdot E = 1$. We already know that $E^2 = -1$ therefore this is enough to prove that $Cl(X)$ is generated by F and E with intersection matrix

$$F^2 = 0, F \cdot E = 1, E^2 = -1.$$

In particular $|F|$ defines a morphism onto $\mathbb{P}^1$, this is the resolution of the projection of $\mathbb{P}^2$ from the point x. Moreover E is the unique irreducible effective divisor with negative self intersection.

The surface X can be realized as a cubic surface in $\mathbb{P}^4$, internal projection of $V_{2,2}$. It is also possible to see X in the following way. Fix a line $r \subset \mathbb{P}^4$ and a conic $\Gamma \subset \mathbb{P}^4$ in general position. Let $\omega : r \to \Gamma$ be a linear automorphism and S the union of lines spanned by the corresponding points y and $\omega(y)$.

Let us prove that $S \cong X$. First we prove that $\deg S = 3$. Let $H \subset \mathbb{P}^4$ be a general hyperplane passing through r. Then $H \cap \Gamma$ is a pair of points, q_1 and q_2. Therefore

$$H \cap S = r \cup l_1 \cup l_2,$$

where l_i is the line passing through the point q_i. This shows, by Lemma 2.1.1, that S is smooth away from r. Arguing similarly with hyperplane contaning the span of the conic Γ we conclude that S is smooth. Moreover this also shows that $l_1 \sim l_2$, $l_i^2 = 0$, and $r^2 = -1$.

Fix a line $L \subset S$, spanned by points on r and Γ. Consider the linear system $\Sigma \subset L_S(1)$ of hyperplane sections containing L. Then Σ has L as a fixed component and the general element $D \in \Sigma$ is

$$D = L + C,$$

for C a smooth conic with $C\cdot r=0$ and $C\cdot C=1$. The base locus of Σ is L and the linear system $\Lambda:=\Sigma-L$ is base point free. Let $\varphi_\Lambda:S\to\mathbb{P}^2$ be the associated morphism. Then $\Lambda^2=1$ and φ_Λ is a birational map that contracts the line r to a smooth point of $p_0\in\mathbb{P}^2$, and maps the lines of S to lines through p_0. This shows that r is the only curve contracted by φ_Λ, and it is enough to conclude

Definition 2.1.3. The surface X has many names. It is the blow up of $\mathbb{P}^2$ in one point, it is called the Segre scroll $S(1,2)$ or the Segre–Hirzebruch surface $\mathbb{F}_1$. The group $Cl(X)$ is generated by F and C_0 with $F^2=0$, $C_0^2=-1$ and $F\cdot C_0=1$. The morphism $\pi:X\to\mathbb{P}^1$ is associated to $|F|$, for any $p\in X$ let $F_p=\pi^{-1}(\pi(p))$.

Remark 2.1.4. *If you are familiar with vector bundles there is also a description of X as follows $\mathbb{F}_1=\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}\oplus\mathcal{O}(-1))$, this also justifies the name $\mathbb{F}_1$, we will see that $\mathbb{F}_e$ is defined for any non negative integer e. All fibers of π are isomorphic to $\mathbb{P}^1$, they are the strict transform of a line in $\mathbb{P}^2$.*

Let $\nu:Y\to\mathbb{F}_1$ be the blow up of a point p outside C_0 with exceptional divisor E. Then Y has still a morphism onto $\mathbb{P}^1$ but there is a reducible fiber composed by two (-1)- curves, E and F_p. Therefore Y contains 3 (-1)-curves and by Castelnuovo we may contract any of them. If we blow down E we go back to $\mathbb{F}_1$. If we blow down C_0 we go onto a surface isomorphic to $\mathbb{F}_1$. If we blow down F_p instead the resulting surface admits two morphisms onto $\mathbb{P}^1$ and it is $\mathbb{P}^1\times\mathbb{P}^1$.

The latter may be seen as follows. Let $Q=\mathbb{P}^1\times\mathbb{P}^1$ be a smooth Quadric and blow up one point p. Then the two fibers of the ruling passing through p becomes (-1) curves and it is easy to see that the resulting surface is Y.

This produces a rational map $\phi:\mathbb{F}_1\dashrightarrow Q$ called elementary transformation. This kind of map can be extended to a wider class of surfaces.

Definition 2.1.5. A surface S is ruled if it admits a morphism $\pi:S\to C$ onto a curve C with all fibers isomorphic to $\mathbb{P}^1$. For any $p\in S$ $F_p=\pi^{-1}(\pi(p))$.

An elementary transformation of a ruled surface S is a birational map $\phi:S\dashrightarrow S_1$ obtained via the blow up of a point $p\in S$ and the blow down of the strict transform of F_p.

Remark 2.1.6. *Let S be a ruled surface and ϕ_p an elementary transformation. Then $\phi_p(S)$ is again a ruled surface. Moreover $Cl(\phi_p(S))$ is generated by the strict transform of generators of $Cl(S)$.*

Definition 2.1.7. Let $\{p_1,\ldots,p_m\}\subset C_0\subset\mathbb{F}_1$ be a collection of m distinct point and Φ_m the composition of the m-th elementary transformation centered in the points p_i. Then

$$\Phi_m(\mathbb{F}_1)=:\mathbb{F}_{m+1}$$

is the Segre–Hirzebruch surface.

Definition 2.1.8. Let r be a line and Γ a rational normal curve of degree a. Fix an linear automorphism ω between r and Γ. Then the Segre scroll $S(1,a)\subset\mathbb{P}^{a+2}$ is the union of lines spanned by corresponding points of r and Γ.

Exercise 2.1.9. Show that $S(1,a)$ is a smooth surface and prove that $S(1,a)\cong\mathbb{F}_{a-1}$. Observe that the internal projection of a rational normal curve $\Gamma_a\subset\mathbb{P}^a$ is the rational normal curve of degree $a-1$ and $S(1,a)$ is covered by rational normal curves of degree a. Then an internal projection, outside the line r, of $S(1,a)$ is

$S(1, a-1)$ and this operation is equivalent to an elementary transformation. Then use Example 2.1.2.

REMARK 2.1.10. *For those familiar with vector bundles* $\mathbb{F}_e \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$.

LEMMA 2.1.11. *The surface* $\mathbb{F}_e$ *is ruled,* $Cl(\mathbb{F}_e)$ *has rank 2 and it is generated by* F *and* C_0 *with* $F^2 = 0$, $F \cdot C_0 = 1$ *and* $C_0^2 = -e$.

PROOF. It is a simple computation that an elementary transformation centered on C_0 decreases the self intersection by 1 and leaves unchanged the other intersection products. $\square$

2.2. Classification of rational surfaces

LEMMA 2.2.1. *Let* C *be a rational curve and* $f : C \to Z$ *a non constant morphism onto a smooth curve. Then* Z *is a rational curve.*

PROOF. Let P_1 and P_2 be points in Z. Fix $Q_i \in f^{-1}(P_i)$ a point. Then $f_* Q_i = P_i$ and by Equation (1) we have $P_1 \sim P_2$. This is enough to conclude by Example 1.4.10. $\square$

REMARK 2.2.2. *If you are familiar with the notion of genus of a curve then Lemma 2.2.1 has a much more general setting. It is true that any, non constant, morphism between curves cannot increase the genus of the curve. This may be seen for instance via Hurwitz formula.*

We are almost ready to study rational surfaces. The final ingredient we need is the so called Tsen's Theorem

THEOREM 2.2.3 (Tsen's theorem). *Let* $F \in k[x][x_1, \ldots, x_n]_m$ *be a polynomial. Assume that* $m < n$. *Then* $F(x_1, \ldots, x_n) = 0$ *has a solution in polynomials* $x_i = p_i(x)$.

PROOF. We are looking for a polynomial solution $x_i = \sum_{j=0}^{h} u_{ij} x^j$ with unknown coefficients. If we plug these into F we obtain a polynomial in x and we have to vanish all coefficients. For $h \gg 0$ the set of equations is asymptotically mh and the number of variables is asymptotically nh. By hypothesis $n > m$ therefore the solution exists by dimensional reason for sufficiently big h. $\square$

PROPOSITION 2.2.4. *Let* S *be a rational ruled surface with a structure morphism* $\pi : S \to C$. *Then* $C \cong \mathbb{P}^1$. *There is section* $\sigma : C \to S$, *that is a curve* $D \subset S$ *with* $D \cdot F = 1$ *and* $Cl(S)$ *is two dimensional. Moreover there is at most a unique irreducible curve with negative self intersection and* $S \cong \mathbb{F}_e$ *for some* e.

PROOF. The surface S is rational, therefore there is a rational curve $\Gamma \subset S$ such that $\pi_{|\Gamma} : \Gamma \to C$ is dominant. Then by Lemma 2.2.1 $C \cong \mathbb{P}^1$.

CLAIM 2.2.5. There is a birational map $\varphi : S \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ that preserves the fibration π.

PROOF. By adjunction formula $-K_{S|F} = L(2)$, then we may consider S as a conic over the non algebraically closed field $K = k(x)$. That is S is birational to a conic

$$(q(x_0, x_1, x_2) = \sum a_{ij}(x) x_i x_j = 0) \subset \mathbb{P}^2 \times \mathbb{A}^1,$$

with $a_{ij}(x) \in k[x]$. Then by Tsen's theorem the conic has a point. And projection from the point shows that S is birational to

$$Y = (x_1 = 0) = \mathbb{P}^1 \times \mathbb{A}^1 \subset \mathbb{P}^2 \times \mathbb{A}^1.$$

This induces the required birational map. $\square$

To conclude it is enough to prove that the map φ can be factored by a chain of elemenatry transformations. Let $\Sigma = \varphi_*^{-1}(A)$ be the pencil in S induced by the canonical projection onto the second factor. Then $\Sigma \cdot F = 1$ and the general element of Σ is a smooth irreducible rational curve, with $\Sigma \cdot F = 1$. Let $y \in \operatorname{Bs}\Sigma$ be a base point and $\mu : Z \to S$ the blow up of y. Let F_{yZ}, the strict transform on Z of the fiber through y. Then F_y is a (-1)-curve and we may contract it to a smooth point via $\nu : Z \to S_1$. Note that $\Sigma_Z \cdot F_{yZ} = 0$ therefore $\nu_*(\Sigma_Z) =: \Sigma_1$ is a linear system without fixed components and $\Sigma_1^2 = \Sigma^2 - 1$. Then after finitely many steps we may assume that Σ_h is a base point free pencil on S_h. This takes us to a smooth surface S_h admitting two distinct morphisms onto $\mathbb{P}^1$ all of whose fibers are smooth rational curves. To conclude do the following exercise.

Exercise 2.2.6. Prove that:

- $\tilde{S} \cong \mathbb{P}^1 \times \mathbb{P}^1$ (at least 2 different proofs)
- this chain transformations factors φ and realizes S as $\mathbb{F}_e$ for some e.

$\square$

Remark 2.2.7. *The above Proposition is true in a much wider contest, see Noether-Enriques Theorem,* [**Be**]. *It is enough to have one fiber isomorphic to $\mathbb{P}^1$ to get the same statement up to a birational modification.*

To conclude the classification we are aiming at we introduce notations that may seem artificial at a first glance.

Definition 2.2.8. A $\mathbb{Q}$-divisor is an element of $Cl(S)\otimes\mathbb{Q}$. Let D be a $\mathbb{Q}$-divisor we say that D is nef if $D\cdot C \geq 0$ for any curve $C \subset S$. Let $\pi : S \to X$ be a morphism and D_1 and D_2 divisors. Then $D_1 \equiv_\pi D_2$ (D_1 is π-numerically equivalent to D_2) if if for any curve $C \subset S$, such that $\pi(C)$ is a point, $D_1 \cdot C = D_2 \cdot C$. If π is the constant map we simply say that $D_1 \equiv D_2$ (D_1 and D_2 are numerically equivalent).

Remark 2.2.9. *The above notions are at the core of Minimal model program and give us the opportunity to say a few words about it, as usual restricting to the surface case.*

Let S be a smooth surface. Let

$$N^1(S) := Cl(S)/\equiv \otimes\mathbb{R},$$

this is a finite dimensional vector space. Inside this space we have the convex cone of effective divisors (note that when talking about cones we always consider only non negative linear combination), and also the convex cone of ample divisors (check it is a convex cone) clearly the latter is in general not a closed cone and its closure is given by curves that have non negative intersection with any curve (for surfaces this is a consequence of Nakai-Moishezon ampleness criteion in higher dimension see the following Kleiman criterion).

Definition 2.2.10. A divisor D is pseudoeffective if it is in the closure of the cone of effective divisors.

REMARK 2.2.11. *A nef divisor is always pseudo-effective. It is difficult in general understand pseudo effective non nef divisors.*

We may define $N_1(S)$ as the real vector space given by real linear combination of curves (1-cycles) modulo numerical equivalence. The intersection product induces the pairing

$$N_1(S) \times N^1(S) \to \mathbb{R},$$

or if you prefer $N^1(S)$ is the dual vector space of $N_1(S)$, they have the same dimension and any divisor defines a linear form on $N_1(S)$ (in general this may be seen as a consequence of Poincare duality). In particular to any divisor $D \in Cl(S)$ we may associate the hyperplane of curves where the intersection product vanishes. There is a natural convex cone also in $N_1(S)$ the cone of effective curves $NE(S)$. This time it less immediate but also this cone is in general not closed and its closure (with respect to the topology induced by a norm) encodes an unbelievable amount of information.

THEOREM 2.2.12 (Kleiman's ampleness criterion). *A divisor D is ample if and only if it is positive on $\overline{NE}(S)$.*

In $N_1(S)$ there is an hyperplane more important than the others. The one associated to the canonical class K_S. Minimal model program is based on the study of curves that are negative with respect to the canonical class. That is they sit in the negative half-space with respect to the hyperplane defined by the canonical class, $K_{S<0}$.

Note that if C is an irreducible curve and $C^2 \leq 0$ then C is on the boundary of $\overline{NE}(S)$. Indeed if C is not on the boundary it means that given any ample divisor A there is a small enough $\epsilon > 0$ such that $C - \epsilon A$ is still in $\overline{NE}(S)$ therefore $(C - \epsilon A) \cdot C \geq 0$ and this forces $C^2 > 0$.

In particular all (-1) curves and all ruled structure are on the boundary of $\overline{NE}(S)$ and contained in the negative half space with respect to the canonical class. All curves on the boundary are called rays and the one in the $K_{S<0}$ are called extremal rays. It can be proved that these are the only negative rays for surfaces different from $\mathbb{P}^2$.

The following is again a definition inherited by Minimal Model Program.

DEFINITION 2.2.13. Let $D = \sum d_i D_i$ be a $\mathbb{Q}$-divisor on X with $d_i \leq 1$. Then the pair (X, D) has canonical singularities if there exists a log resolution $\nu : Y \to X$ such that

$$K_Y + D_Y = \nu^*(K_s + D) + \sum a_i E_i$$

with $a_i \geq 0$ for any i. We say that (X, D) is not canonical at x if, after reordering the indexes of the log resolution, $K_Y + D_Y = \nu^*(K_s + D) + \sum a_i E_i$, with $a_0 < 0$ and $\nu(E_0) = x$.

EXERCISE 2.2.14. Let S be a smooth surface, $x \in S$ a point and D a $\mathbb{Q}$-divisor. Then (S, D) is not canonical at x if and only if $\operatorname{mult}_x D > 1$ (note that if $D = \sum d_i D_i$ then $\operatorname{mult}_x D := \sum d_i \operatorname{mult}_x D_i$).

REMARK 2.2.15. *The idea is that canonical singularities preserves elements of canonical divisors indeed $\nu_*(K_Y + D_Y) = K_S + D$. The presence of a negative term in the rhs forces the lost of canonical sections. We call them singularities because if D is trivial they are related to the singularities of embedding defined by the canonical class.*

Theorem 2.2.16. *Let S be a rational surface then S is either $\mathbb{P}^2$ or there is a morphism $f : S \to \mathbb{F}_e$ for some e.*

Proof. Let S be a rational surface. If there exists a morphism $\pi : S \to \mathbb{P}^2$ then either $S \cong \mathbb{P}^2$ or there is a point $x \in \mathbb{P}^2$ such that π factors with the blow up of x. Therefore S admits a morphism onto $\mathbb{F}_1$.

Let $\chi : S \dashrightarrow X$ be a birational map, with X either $\mathbb{P}^2$ or $\mathbb{F}_e$. Fix a very ample linear system $\mathcal{H}_S$ on S and let $\mathcal{H} = \chi_*\mathcal{H}_S$ its strict transform.

We want to use the ruled structure on $\mathbb{F}_e$ and the special Picard of $\mathbb{P}^2$ to produce relatively trivial $\mathbb{Q}$-divisors on X.

If $X \cong \mathbb{F}_e$ there is an integer a such that the $\mathbb{Q}$-divisor

$$K_X + 2/a\mathcal{H} \equiv_\pi 0$$

with π a ruled structure (a is uniquely determined unless $X = \mathbb{F}_0$). If $X \cong \mathbb{P}^2$ there is a unique iteger a such that $K_X + 3/a\mathcal{H} \equiv 0$.

From now on we fix this notation. The surface X is either $\mathbb{F}_e$ or $\mathbb{P}^2$ and $b \in \mathbb{Q}^+$ is such that $K_X + 1/b\mathcal{H} \equiv_\pi 0$, where π is either a ruled structure or the constant map. Note that for H_S ample enough we may always assume that $b > 1$. Our first task is to determine a "numerical" criterion that forces χ to be a morphism.

Claim 2.2.17. If $K_X + 1/b\mathcal{H}$ is nef and has canonical singularities then χ is a morphism.

Proof of the Claim. Let

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\ S \arrow[rr, dashed, "\chi"] & & X \end{tikzcd}$$

be a resolution of the map χ, and $E_0 \subset Z$ be a p-exceptional (-1)-curve. We may assume that E_0 is not q-exceptional. Then by construction

$$(K_Z + 1/b\mathcal{H}_Z) \cdot E_0 = (q^*(K_X + 1/b\mathcal{H}) + \sum_i a_iE_i) \cdot E_0,$$

on the other hand by hypotheis $q^*(K_X+1/b\mathcal{H})$ is nef and $a_i \geq 0$, therefore we have

$$(K_Z + 1/b\mathcal{H}_Z) \cdot E_0 = (q^*(K_X + 1/b\mathcal{H}) + \sum_i a_iE_i) \cdot E_0 \geq 0.$$

The other side of the triangle gives

$$(K_Z + 1/b\mathcal{H}_Z) \cdot E_0 = (p^*(K_S + 1/b\mathcal{H}_S) + \sum_i b_iE_i) \cdot E_0,$$

for some non negative b_i. Therefore

$$(p^*(K_S + 1/b\mathcal{H}_S) + \sum_i b_iE_i) \cdot E_0 \geq 0.$$

Since E_0 is p-exceptional this forces

$$\sum_i b_iE_i \cdot E_0 \geq 0.$$

To conclude the claim let us derive a contradiction proving that $\sum_i b_iE_i \cdot E_0 < 0$.

The linear system $\mathcal{H}_S$ is base point free, therefore

$$K_Z = p^*(K_S) + \sum b_i E_i,$$

with $b_i > 0$. Up to reordering the indexes we may assume that $E_i \cap E_0 \neq \emptyset$ if and only if $i \in \{0, \ldots, h\}$, for some h. Let $\nu : Z \to Z_1$ be the blow down of E_0 to a point z and $g : Z_1 \to S$ the factorization of p. This yields

$$K_Z = \nu^* K_{Z_1} + E_0$$

and

$$K_{Z_1} = g^*(K_S) + \sum_{i>0} b_i E_i,$$

with $E_i \ni z$ if and only if $i \leq h$. This gives

$$K_Z = p^* K_X + (1 + \sum_{i=1}^{h} b_i) E_0 + \sum_{i>0} b_i E_i,$$

and yields

$$b_0 = (1 + \sum_{i=1}^{h} b_i),$$

and

$$\sum_i b_i E_i \cdot E_0 = \sum_{i=1}^{h} b_i - (1 + \sum_{i=1}^{h} b_i) < 0.$$

$\square$

Assume that $K_X + 1/b\mathcal{H}$ has not canonical singularities. Since the linear system $\mathcal{H}$ has not a fixed component and $b > 1$ there are only finitely many non canonical points. We are assuming that there is a point $x \in X$ with $\operatorname{mult}_x \mathcal{H} > b$. If $X = \mathbb{P}^2$ we blow up the point and land on $\mathbb{F}_1$. If $X = \mathbb{F}_e$ after the elementary transformation centered in x we come up with a linear system $\mathcal{H}'$ over $\mathbb{F}_{e\pm 1}$ with one point less of canonical singularities (check it for exercise). After a finite number of these steps we end up with a linear system with canonical singularities over $\mathbb{F}_e$. Then by the claim either we conclude or we may assume it is not nef. This forces $a = 0, 1$ (check it for exercise). If $a = 0$ we swap the ruling and proceed again with $b_1 < b$. Since $b \in \mathbb{Z}/2$ after finitely many substitutions either we conclude or we land on $\mathbb{F}_1$

Claim 2.2.18. If $K_{\mathbb{F}_1} + 2/a\mathcal{H}$ is not nef then we may substitute $\mathbb{F}_1$ with $\mathbb{P}^2$

Proof. If $K_{\mathbb{F}_1} + 2/a\mathcal{H}$ is not nef then $(K_{\mathbb{F}_1} + 2/a\mathcal{H}) \cdot C_0 < 0$ therefore we may contract C_0 without producing canonical singularities. $\square$

Then the resulting linear system is nef and canonical and by Claim 2.2.17 there is a morphism onto $\mathbb{P}^2$. $\square$

Thanks to Theorem 2.2.16 we are also able to go a bit beyond in MMP. Castelnuovo's Theorem tells us that any (-1) curve is contractible. Let us observe the following.

Proposition 2.2.19. *Let S be a rational surface and $x \in S$ a point. Then there is a rational map $\varphi : S \dashrightarrow \mathbb{F}_e$, such that φ is an isomorphism in a nbhd of x.*

Proof. Ifr $S \cong \mathbb{P}^2$ simply blow up a point different from x. Assume that $S \neq P^2$. By Theorem 2.2.16 there is a morphism $\psi : S \to \mathbb{F}_e$.

Claim 2.2.20. There is a birational map $\chi : S_1 \dashrightarrow S$ that is an isomorphism in a neighborhood of x such that on the surface S_1 is defined a morphism $f : S_1 \to \mathbb{F}_0$.

Proof of the claim. exercise $\square$

By the claim we may assume that S admits a morphism ψ onto $\mathbb{F}_0$. Let $p = \psi(x)$ we may also assume that all the ψ-exceptional divisors are mapped to p. We prove the claim by induction on the dimension of $N_1(S/$
$F_0)$. If $\dim N_1(S/\mathbb{F}_0) = 0$ we have finished. Assume that $\dim N_1(S/\mathbb{F}_0) = m$. Let $\nu : Z \to \mathbb{F}_0$ be the blow up of p. Then $\psi = \nu \circ \chi$ factors via ν. Let $q = \chi(x)$ be the image of the point we are interested in. Then $\dim N_1(S/Z) = m - 1$ and there is at least a (-1)-curve $E \subset Z$ such that $q \notin E$. Let $\mu : Z \to \mathbb{F}_1$ be the blow down of E. Then with an elementary transformation we may substitute S with a surface $\tilde{S}$ that dominates $\mathbb{F}_0$ and such that $\dim N_1(\tilde{S}/\mathbb{F}_0) = \dim N_1(S/Z) = m - 1$. $\square$

Remark 2.2.21. *A variety X is called uniformly rational if for any point $x \in X$ thre is a neighborhood isomorphic to a dense open subset of $\mathbb{P}^n$. The above proposition shows that any rational surface is uniformly rational. In higher dimension it is still not known weather rationality and uniformly rationality are equivalent.*

Proposition 2.2.22. *Let $C \subset S$ be a smooth rational curve with $C^2 = 0$. Assume that S is rational then $\dim |C| > 0$, $K_S \cdot C = -2$ and $|C|$ induces a morphism onto $\mathbb{P}^1$.*

Proof. Let us first observe that it is enough to prove that $\dim |C| > 0$. If this is the case then $|C|$ is base point free (otherwise the self intersection cannot be 0). Therefore $K_S \cdot C = -2$ and the morphism is onto $\mathbb{P}^1$ because S is rational. Let $\nu : Z \to S$ be the blow up of a point on C. Then $C_z^2 = -1$ and we may contract it with a morphism $\mu : Z \to S_1$. Then $(\mu \circ \nu)(C)$ is a smooth point, $x \in S_1$, and by Proposition 2.2.19 there is a morphism $\psi : S_1 \to \mathbb{F}_e$ that is an isomorphism in a neighborhood of x. Then via the elementary transformation $\phi_{\psi(X)}$ we conclude (check the details as an exercise). $\square$

Remark 2.2.23. *The proposition is true in the following much wider contest. S smooth, C rational curve with $C^2 = 0$ then $|mC|$ defines a morphism onto a curve.*

This shows that to all extremal rays we know is associated a "contraction" morphism. This is true for any extremal ray and is the starting point of the Minimal Model Program.

2.3. del Pezzo surfaces

We aim to classify rational surfaces with ample anticanonical class. By Theorem 2.2.16 we know that they are blow-ups of either $\mathbb{P}^2$ or of a surface $\mathbb{F}_e$.

Definition 2.3.1. A del Pezzo surface S of degree d is a smooth surface with ample anticanonical class and $K_S^2 = d$.

Remark 2.3.2. *By either MMP or Castelnuovo criterion such an S is automatically rational.*

Let us start noting the following.

EXERCISE 2.3.3. Let $\mu : X \to S$ be the blow in a point then $K_X^2 = K_S^2 - 1$. Moreover for any e $K_{F_e}^2 = 8$, and $K_{\mathbb{P}^2}^2 = 9$.

By the classification Theorem 2.2.16 we know that either $S \cong \mathbb{P}^2$ or there is a dominant morphism $f : S \to \mathbb{F}_e$, for some e. Hence the exercise, together with the ampleness criteria of Theorem 1.6.6, shows that any surface with ample anticanonical class is either $\mathbb{P}^2$ or the the blow up of $\mathbb{F}_e$ in at most 7 points. Let us further note the following.

LEMMA 2.3.4. *Let $\mu : X \to \mathbb{F}_e$ be a morphism. Then $-K_X$ is ample only if $e = 0, 1$.*

PROOF. First note that by adjunction formula

$$K_{\mathbb{F}_e} \cdot C_0 = -2 + e$$

Then the blow up formula gives

$$K_X \cdot C_{0X} = K_{\mathbb{F}_e} \cdot C_0 + (\sum e_i E_i) \cdot C_{0X}.$$

Therefore $K_X \cdot C_{0X} < 0$ only if $K_{\mathbb{F}_e} \cdot C_0 < 0$, that is for $e = 0, 1$. □

Therefore the surfaces we are looking for are blow ups of $\mathbb{F}_1$ or $\mathbb{F}_0$ in at most 7 points. Something more can be said

EXERCISE 2.3.5. The blow up of $\mathbb{F}_1$ in s is the blow up of $\mathbb{P}^2$ in $s + 1$ points. The blow up of a quadric in s point is the blow up of $\mathbb{P}^2$ in $s + 1$ points.

Let us summarize all this construction

PROPOSITION 2.3.6. *A del Pezzo surface is either $\mathbb{F}_0$ or the blow up of $\mathbb{P}^2$ in at most 8 points.*

To conclude the study we need to understand when this condition is also sufficient.

THEOREM 2.3.7. *A smooth surface is del Pezzo if and only if it is the blow up of $\mathbb{P}^2$ in at most 8 points such that no three are collinear and no 6 lie on a smooth conic and there is not a cubic curve singular in one point and passing through other 7.*

PROOF. Let $\{p_1, \ldots, p_8\}$ be points satysfying the assumptions. Consider the linear system Λ of cubics passing through the points. Let $\mu : S \to \mathbb{P}^2$ be the blow up of the 8 points. Then

$$\Lambda_S = \mu^*\Lambda - \sum_1^8 E_i,$$
$$K_S = \mu^*\mathcal{K}_{\mathbb{P}^2} + \sum E_i$$

this yields

$$\Lambda_S \sim -K_S.$$

The anticanonical class on S is the strict transform of the linear system of cubics passing through the 8 points. By Theorem 1.6.6 we have to check the self intersection and the intersection with other curves. We already know that $K_S^2 = 9 - 8 > 0$.

To conclude let us start observing that any element of Λ is irreducible. Indeed if this is not the case the cubic has to split in a conic (maybe reducible) and a line. But then this violates the general assumption on the points. Next observe that any element of Λ_S is irreducible. Again the possible reducible elements of Λ_S are given by singular cubics passing thorugh 7 points and singular in one point, and this violates the generality assumption.

Let $C \subset S$ be a curve. Note that $\dim |\Lambda_S| \geq 1$, (actually equality is forces by the general assumption but we do not need it). Since any element of Λ_S is irreducible and $\Lambda_S^2 > 0$ then $-K_S \cdot C > 0$ for any curve $C \subset S$.

To conclude observe the following claim.

CLAIM 2.3.8. Let $\mu : X \to Y$ be the blow up of a point. Let $A \in Cl(Y)$ be a divisor and assume that A_X is ample then A is ample.

PROOF OF THE CLAIM. Let $E \subset X$ be the exceptional divisor. Since A_X is ample then

$$A_X = \mu^* A - mE$$

for some positive integer m. This yields

$$0 < A_X^2 < A^2.$$

Let $C \subset Y$ be a curve and C_X its strict transform then

$$0 < A_X \cdot C_X = (\mu^* A - mE) \cdot C_X = A \cdot C - mE \cdot C_X.$$

Then we may conclude by Theorem 1.6.6. □

□

We may say a bit more with 6 or less points.

LEMMA 2.3.9. *Let $Z := \{p_1, \ldots, p_6\} \subset \mathbb{P}^2$ be points in general position (i.e. they do not lie on a conic and no 3 on a line). Let $\Sigma \subset L(3)$ be the linear system of cubics passing through Z, and $\mu : Y \to \mathbb{P}^2$ be the blow up of Z with exceptional divisors $\{E_1, \ldots, E_6\}$. Then Σ_Y is a very ample linear system.*

PROOF. We have to prove that there are sufficently many cubics to separate points and tangents directions. Fix p_j and consider the conic through $Z \setminus \{p_j\}$ and a general line through p_j varying j gives the required separation. □

This shows that any del Pezzo surface of degree $d \geq 3$ is embedded in $\mathbb{P}^d$ by the anticanonical system.

EXERCISE 2.3.10. Prove that a del Pezzo surface of degree 3 contains 27 lines.

EXERCISE 2.3.11. Try to understand the map associated to $|-K_S|$ and $|-2K_S|$ for del Pezzo of degree 1 and 2.

The opposite is also true but it is slightly out of our range. Let $S \subset \mathbb{P}^d$ be a smooth surface of degree d. Then S is a del Pezzo surface. In particular $d \leq 9$.

EXERCISE 2.3.12. Let $S \subset \mathbb{P}^d$ be a smooth surface of degree d. Prove that S is rational.

2.4. Noether–Castelnuovo Theorem

The aim of this section is to give an explicit set of generators of the Cremona group of the plane, that is the group of birational self maps of $\mathbb{P}^2$.

THEOREM 2.4.1 (Noether-Castelnuovo).
The group of birational transformations of the projective plane is generated by linear transformations and the standard Cremona transformation, that is

$$(x_0 : x_1 : x_2) \mapsto (x_1x_2 : x_0x_2 : x_0x_1),$$

where $(x_0 : x_1 : x_2)$ *are the coordinates of* $\mathbb{P}^2$*.*

Let $\chi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map which is not an isomorphism. Our first aim is to factorise it in simpler maps, "elementary links", and then use them to prove the Theorem.

The first step is done with a strategy similar to the one used in Theorem 2.2.16. Let $\mathcal{H} = \chi_*^{-1}\mathcal{O}(1)$, the strict transform of lines in $\mathbb{P}^2$. Somehow the irregularity, "twisting", of the map χ is encoded in the base locus of $\mathcal{H}$. Observe that $\mathcal{H}$ is without fixed components and $\mathcal{H} \subset |\mathcal{O}(n)|$ for some $n > 1$. Therefore the idea is to study the singularities of the log pair $(\mathbb{P}^2, \mathcal{H})$.

THEOREM 2.4.2. *Let* $\mathcal{H}$ *as above then* $(\mathbb{P}^2, (3/n)\mathcal{H})$ *has not canonical singularities.*

PROOF. Take a resolution of χ

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "\chi"] & & \mathbb{P}^2 \end{tikzcd}$$

By construction

$$\begin{aligned} K_W + (3/n)\mathcal{H}_W &= q^*\mathcal{O}_{\mathbb{P}^2}(3(1/n-1)) + \sum_i a_iE_i + \sum_j b_jF_j \\ &= p^*\mathcal{O}_{\mathbb{P}^2} + \sum_i a'_iE_i + \sum_h c_hG_h \end{aligned}$$

where E_i are p and q exceptional divisors, while F_j are q but not p exceptional divisors and G_h are p but not q exceptional divisors. Observe that the a_i's and b_j's are positive integers. Let $l \subset \mathbb{P}^2$ a general line in the right hand side plane, in particular q is an isomorphism on l. That is $E_i \cdot q^*l = F_j \cdot q^*l = 0$ for all i and j. Since $n > 1$ we have

$$(K_W + (3/n)\mathcal{H}_W) \cdot q^*l = (q^*\mathcal{O}_{\mathbb{P}^2}(3(1/n-1)) + \sum_i a_iE_i + \sum_j b_jF_j) \cdot q^*l < 0,$$

we express this intersection number in a different way

$$0 > (K_W + (3/n)\mathcal{H}_W) \cdot q^*l = (p^*\mathcal{O}_{\mathbb{P}^2} + \sum_i a'_iE_i + \sum_h c_hG_h) \cdot q^*l.$$

So that $c_h < 0$ for some h. That is $(\mathbb{P}^2, (3/n)\mathcal{H})$ is not canonical. $\square$

The point of the above result is that the existence of the map χ imposes conditions on linear systems on X. Our first aim is to derive some consequence from it.

Let $\chi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational not biregular map. Let $\chi_*^{-1}(\mathcal{O}(1)) =: \mathcal{H} \subset |\mathcal{O}(n)|$ then $(\mathbb{P}^2, (3/n)\mathcal{H})$ is not canonical. Observe that the linear system $\mathcal{H}$ has not base components. Then, by Theorem 2.4.2, there is a point $x \in \mathbb{P}^2$ such that

$$\operatorname{mult}_x \mathcal{H} > n/3. \tag{2}$$

Our aim is to untwist the map χ. We found a badly singular point x let us blow it up. There is much more than this in the following blow up. Sarkisov theory tell us that whenever there is a non canonical singularity, coming from a birational map χ, then there exists a terminal extraction centered on this singularity. In the surface case this turns out to be always the blow up the maximal ideal of the point x.

Let $\nu : \mathbb{F}_1 \to \mathbb{P}^2$ be the blow up of x, with exceptional divisor C_0. Let $\chi' = \chi \circ \nu : \mathbb{F}_1 \dashrightarrow \mathbb{P}^2$ and $\mathcal{H}' = (\chi')_*^{-1}\mathcal{O}(1)$. Let $n' = n - mult_x\mathcal{H}$, then

$$K_{\mathbb{F}_1} + (2/n')\mathcal{H}' \equiv_{\pi_1} 0.$$

We are in the conditions to apply an Claim 2.2.17. Let us first notice that $K_{\mathbb{F}_1} + (2/n')\mathcal{H}'$ is nef. Let $f \subset \mathbb{F}_1$ a generic fiber of the ruled structure. Then

$$K_{\mathbb{F}_1} + (2/n')\mathcal{H}' \cdot f = 0,$$

by definition. We have to check C_0, here is were Sakisov construction plays the main role,

$$\begin{aligned}(K_{\mathbb{F}_1} + (2/n')\mathcal{H}') \cdot C_0 &= -1 + (2/n')mult_x\mathcal{H} \\ &= \frac{-n + 3mult_x\mathcal{H}}{n - mult_x\mathcal{H}} > 0\end{aligned}$$

where the last inequality comes directly from equation (2). That is the existence of non canonical singularities for $(\mathbb{P}^2, (3/n)\mathcal{H})$. This is again another important step of Sarkisov theory.

Then by Claim 2.2.17 $K_{\mathbb{F}_1} + 2/n'\mathcal{H}'$ is not canonical and the linear system $\mathcal{H}'$ admits a point $x' \in \mathbb{F}_1$ with "high multiplicity".

Let us go on with the same game: blow up x'

$$\psi : Z \supset E \to \mathbb{F}_1 \ni x'.$$

This time Z is not a Mori space, but the fiber of $\mathbb{F}_1$ containing x' is now a (-1)-curve and we can contract it $\varphi : Z \to S$.

$$\begin{tikzcd} & Z \arrow[dl, "\psi"'] \arrow[dr, "\varphi"] & \\ \mathbb{F}_1 \arrow[rr, dashed] & & S \end{tikzcd}$$

We already know that S is either a quadric, $\mathbb{F}_0$, or $\mathbb{F}_2$.

Let $x_2 \subset S$ be the exceptional locus of φ^{-1} and $\mathcal{H}_2$ the strict transform of $\mathcal{H}'$. Observe the following two facts:

i) $(K_{\mathbb{F}_1} + (2/n')\mathcal{H}_2) \cdot f = 0$, where, by abuse of notation, f is the strict transform of $f \subset \mathbb{F}_1$,
ii) since $mult_{x'}\mathcal{H}' > \frac{H' \cdot f}{2}$, then $(S, (2/n')\mathcal{H}_2)$ has canonical singularities at x_2.

Then by i) $(S,(2/n')\mathcal{H}_2)$ is the next log pair we use. By ii) after finitely many elementary transformations we reach a not nef pair $(\mathbb{F}_k,\mathcal{H}_r)$ with canonical singularities such that

$$K_{\mathbb{F}_k}+(2/n')\mathcal{H}_r\equiv_{\pi_k}0.$$

Observe that $\overline{NE}(\mathbb{F}_k)$ is a two dimensional cone. In particular it has only two rays. One is spanned by f, a fiber of π_k. Let Z an effective irreducible curve in the other ray. Then

$$(K_{\mathbb{F}_k}+(2/n')\mathcal{H}_r)\cdot Z<0. \tag{3}$$

Since $\mathcal{H}_r$ has not fixed components then F_k is a del Pezzo surface and the only possibilities are therefore $k=0,1$.

In case $k=1$ then what is left is to simply blow down the exceptional curve $\nu:\mathbb{F}_1\to\mathbb{P}^2$, and reach $\mathbb{P}^2$ together with a linear system $\nu_*\mathcal{H}_2=:\tilde{\mathcal{H}}\subset|\mathcal{O}(j)|$. Note that in this case, by equation (3),

$$K_{\mathbb{F}_1}+(2/n')\mathcal{H}_r=\nu^*(K_{\mathbb{P}^2}+(2/n')\tilde{\mathcal{H}})+\delta C_0,$$

for some positive δ. Therefore $K_{\mathbb{P}^2}+(2/n')\tilde{\mathcal{H}}$ is not nef. In other terms

$$(2/n')j<3,$$

and

$$j<\frac{3(n-mult_x\mathcal{H})}{2}<n.$$

This strict inequality allow to iterate the above argument and after finitely many steps we untwist the map χ.

In case $k=0$ observe that $\mathbb{F}_0\cong\mathbb{Q}^2$ is a Mori space for two different fibrations, let f_0 and f_1 two general fibers. Moreover by equation (3)

$$(K_{\mathbb{F}_0}+(2/n')\mathcal{H}_r)\cdot f_1<0.$$

That is there exists an

$$n_1<n' \tag{4}$$

such that

$$(K_{\mathbb{F}_0}+(2/n_1)\mathcal{H}_r)\cdot f_0>0,$$

and

$$(K_{\mathbb{F}_0}+(2/n_1)\mathcal{H}_r)\cdot f_1=0.$$

Again by NF inequalities this implies that $(\mathbb{F}_0,(2/n_1)\mathcal{H}_r)$ is not canonical and we iterate the procedure. As in the previous case the strict inequality of equation (4) allows to conclude after finitely many steps.

We have factorised any birational, not biregular, self-map of $\mathbb{P}^2$ with a sequence of "elementary links". We now use this information to prove the theorem we are aiming at.

The first step is to interpret a standard Cremona transformation in this new language

Exercise 2.4.3. Prove that a standard Cremona transformation is given by the following links

$$\begin{tikzcd} & \mathbb{F}_1 \arrow[r, dashed] \arrow[dl] & \mathbb{F}_0 \arrow[r, dashed] & \mathbb{F}_1 \arrow[dr] & \\ \mathbb{P}^2 & & & & \mathbb{P}^2 \end{tikzcd}$$

Vice-versa any link of type

$$\begin{tikzcd} & \mathbb{F}_1 \arrow[dl] \arrow[r, dashed] & \mathbb{F}_a \arrow[r, dashed] & \mathbb{F}_1 \arrow[dr] & \\ \mathbb{P}^2 & & & & \mathbb{P}^2 \end{tikzcd}$$

can be factorised by Cremona transformations. hint: A standard Cremona transformation is given by conics trough 3 non collinear points. The link above are possible only for $a = 0, 2$. They represents birational maps given by conics with either 3 base points or 2 base point plus a tangent direction. Try to factorise the following map

$$(x_0 : x_1 : x_2) \to (x_1x_2 : x_0x_2 : x_1x_2 + x_0x_2 + x_0^2).$$

with Cremona transformations.

Let $\chi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ a birational map and

(5)
$$\begin{tikzcd} & \mathbb{F}_1 \arrow[dl, "\nu_1"'] \arrow[r, dashed, "l_0"] & \mathbb{F}_k \arrow[r, dashed, "l_1"] & \dots \arrow[r, dashed] & \mathbb{F}_1 \arrow[dr] & \\ \mathbb{P}^2 & & & & & \mathbb{P}^2 \end{tikzcd}$$

the factorisation in elementary links obtained in the first part of the proof. Let us first make the following observation. If there is a link leading to an $\mathbb{F}_1$ we can break the birational map simply blowing down the (-1)-curve. That is substitute χ with the following two pieces

$$\begin{tikzcd} & \mathbb{F}_1 \arrow[dl, "\nu_1"'] \arrow[r, dashed, "l_0"] & \dots \arrow[r, dashed, "l_i"] & \mathbb{F}_1 \arrow[dr, "\nu_2"] \arrow[rr, "\sim"] & & \mathbb{F}_1 \arrow[dl, "\nu_2"'] \arrow[r, dashed, "l_{i+1}"] & \dots \arrow[r, dashed] & \mathbb{F}_1 \arrow[dr] & \\ \mathbb{P}^2 \arrow[rrrr, dashed, "\chi_1"] & & & & \mathbb{P}^2 \arrow[rrrr, dashed, "\chi_2"] & & & & \mathbb{P}^2 \end{tikzcd}$$

So that we can assume

(6) there are no links leading to $\mathbb{F}_1$ "inside" the factorisation.

Let

$$d(\chi) = \max\{k : \text{there is an } F_k \text{ in the diagram}\}.$$

If $d(\chi) \leq 2$ we can factorise it by exercise 2.4.3.

We now prove the Theorem by induction on $d(\chi)$. Consider the left part of the factorisation (5). Since $d(\chi) > 2$, by assumption (6), then l_0 is of type $\mathbb{F}_1 \dashrightarrow \mathbb{F}_2$ and l_1 is of type $\mathbb{F}_2 \dashrightarrow \mathbb{F}_3$. Then we force Cremona like diagrams in it, at the cost of introducing new singularities. Let

$$\begin{tikzcd} & \mathbb{F}_1 \arrow[dl, "\nu_1"'] \arrow[r, dashed, "\alpha"] & \mathbb{F}_0 \arrow[r, dashed, "l_0"] & \mathbb{F}_1 \arrow[dr, "\nu_2"] & & \mathbb{F}_1 \arrow[dl, "\nu_2"'] \arrow[r, dashed, "\alpha^{-1}"] & \mathbb{F}_0 \arrow[r, dashed, "l_1"] & \dots \mathbb{F}_1 \arrow[dr] & \\ \mathbb{P}^2 & & & & \mathbb{P}^2 \arrow[rrrr, dashed, "\chi'"] & & & & \mathbb{P}^2 \end{tikzcd}$$

where $\alpha : \mathbb{F}_1 \dashrightarrow \mathbb{F}_0$ is an elementary transformation centered at a general point of $\mathbb{F}_1$, and $exc(\alpha^{-1}) = \{y_0\}$. So that $\alpha_*(\mathcal{H}')$ has an ordinary singularity at y_0. Then l_0 is exactly the same modification but leads to an F_1 and ν_2 is the blow down of the exceptional curve of this $\mathbb{F}_1$. Observe that neither α_0 nor ν_2 are links in the Sarkisov category, in general. Nonetheless the first part can be factorised by standard Cremona transformations. Furthermore $d(\chi') < d(\chi)$. Therefore by induction hypothesis also χ' can be factorised by Cremona transformations.

CHAPTER 3

Exam

The exam is a discussion of talks and exercises. Choose from the list either one talk or one starred exercise or two non starred exercises.

TALKS

- Prove [**Be**][Theorems IV.13, IV.16] and the necessary Lemmata.
- Prove [**Be**][Theorem III.4] (need to know cohomology).
- Section 2 of [**Re2**].
- Section 8.3.1 (without Theorem 8.3.6) of [**Do1**].

EXERCISES

. Show that any irreducible rational curve of degree d with a a point of multiplicity $d-1$ can be mapped to a line via a Cremona transformation of $\mathbb{P}^2$

. Let $X \subset \mathbb{P}^n$ be a variety of degree d and dimension k. Show that $d \geq n-k+1$.

* Show that any surface of degree d in P^{d+1} is rational and it is either the Veronese surface in $\mathbb{P}^5$ or $S(1,a)$ for some a.

. Let $S \subset \mathbb{P}^3$ be a quartic surface with 3 double lines meeting in a triple point. Prove that S is a projection of the Veronese surface $V \subset \mathbb{P}^5$.

* Let $S \subset \mathbb{P}^3$ be a cubic surface. Show that there is a set of 12 lines $\{l_1,\ldots,l_6,r_1,\ldots,r_6\}$ such that

$$l_i \cap l_j = r_i \cap r_j = l_i \cap r_j = \emptyset \text{ for } i \neq j,$$

and $l_i \cap r_i$ is a point. Determine how many set of such lines exists on S.

. Let $S \subset \mathbb{P}^3$ be a quartic with a double conic. Show that S is the projection of a del Pezzo surface of degree 4 in $\mathbb{P}^4$.

* Show that any surface with infinitely many (-1)-curves is rational and give an example of such a surface.

. Show that there are irreducible curves $C \subset \mathbb{P}^2$ of degree d with $\frac{(d-1)(d-2)}{2}$ double points.

. Let $S_d \subset \mathbb{P}^n$ be a smooth rational surface of degree d. Prove that $d \geq n-1$. Prove that any surface $S_3 \subset \mathbb{P}^4$ is the blow up of $\mathbb{P}^2$ in a point.

* Prove Castelnuovo Theorem stating that any $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ can be factored by De Jonquieres transformations and linear automorphisms. A De Jonquieres transformation is associated to the linear system of curves of degree d with a point of multiplicity $d-1$ and $2(d-1)$ simple points.

. Let $\mathcal{L}$ be an homaloidal system. Prove the so called Noether equations.

$$\sum m_i = 3(d-3) \sum m_i^2 = d^2 - 1,$$

where m_i are the multiplicities of points in $\operatorname{Bs}\mathcal{L}$. Show that for any rational curve $C_d \subset \mathbb{P}^2$ of degree $d \leq 5$ there exists a birational modification $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\omega(C_d)$ is a line.

* Prove that for any degree $d \geq 6$ there are rational curves of degree d that cannot be mapped onto a line by Cremona modifications.

. Let $\mathcal{L}$ be the linear system of quartics through 10 points in $\mathbb{P}^2$. Show that $\varphi_{\mathcal{L}}(\mathbb{P}^2)$ is a sextic surface (Bordiga surface) that contains 10 lines and 10 disjoint plane cubics such that each line meets a single cubic (this is called a double ten).

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