Differential Geometry
(preliminary draft)

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Introduction

These notes are intended for an undergraduate level third year. It is a pleasure to thank C. Bisi for a carefull reading.

CHAPTER 1

Manifolds

1.1. Intuitive notions

Recall that a n dimensional topological variety is a Hausdorff topological space locally homeomorphic to $\mathbb{R}^n$ with a countable basis of open sets (the latter condition is not always required but it is useful for what follows).

EXAMPLE 1.1.1. The first examples are clearly $\mathbb{R}^n$ and its open subsets. The circle $S^1 \subset \mathbb{R}^2$ is a 1 dimensional topological variety that is not homeomorphic to any open subset of $\mathbb{R}^1$ (check it as a rust removal exercise). We may also consider $S^n \subset \mathbb{R}^{n+1}$ and even pick a circle in $\mathbb{R}^3$ and let it revolve around a line to produce a torus $S^1 \times S^1$.

In these latter examples the condition required may be checked noticing that in any point of the topological space there is a "normal" vector to our space and we may use it to locally project onto $\mathbb{R}^2$ in a homeomorphic way. Further note that, either intuition or rigor, suggests that the local parametrizations are given by functions that are not only continuous but also differentiable.

More generally let $X \subset \mathbb{R}^N$ be a topological n-variety. We have local parametrizations $f : U \to X \subset \mathbb{R}^N$ that are homeomorphism onto the image. On the other hand f is a continuous function from an open $U \subset \mathbb{R}^n$ to $\mathbb{R}^N$ therefore we may ask whether this function is either differentiable (smooth) or analytic.

All the examples we described, till now, are subsets of some $\mathbb{R}^N$. This is the easiest way to describe and visualize topological spaces. But we may get interesting examples even without this property.

EXAMPLE 1.1.2 (Projective space). The real projective space

$$\mathbb{P}^n_{\mathbb{R}} := \frac{\mathbb{R}^{n-1} \setminus \{0\}}{\sim},$$

where $v \sim w$ if there exists $\lambda \in \mathbb{R}^*$ such that $v = \lambda w$. These spaces are not seen usually, even if they are, as subsets of some $\mathbb{R}^N$ but still are topological varieties.

Remember that $\mathbb{P}^n_{\mathbb{R}}$ can be equivalently thought of as the parameter space of 1-dimensional vector subspaces of $\mathbb{R}^{n+1}$. We could equivalently consider the parameter space of k-dimensional vector spaces of $\mathbb{R}^n$, in a while we will be able to recognize also this as a topological variety and even more.

Another example that deserves attention is the following

EXAMPLE 1.1.3 (Tangent bundle to a sphere). Let $S^2 := \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ be a sphere. For any point $p = (x, y, z) \in S^2$ we may consider the vector (x, y, z) and its orthogonal $T_pS^2 := (x, y, z)^\perp$, the tangent space at the point p. We have not defined the tangent space, yet. But for spheres it is simple to guess it should be the direction of the unique plane passing through p and intersecting the sphere only in p.

Then we define $T(S^2) := \cup_{p \in S^2} T_p S^2$. This is the union of all tangent vectors to the sphere and we may naively define two vectors close if the points p and q are close and the vectors are close. This suggests that $T(S^2)$ could be considered as a topological variety. Note that despite it is defined via $\mathbb{R}^3$ does not admit any embedding there because intuition tells us that it is a 4-dimensional topological space.

The latter examples are defined in an abstract way, not depending on an embedding in some $\mathbb{R}^N$, and for this reason it is more difficult, at first sight, to think about higher regularities of the parametrization functions.

At this point we have two options. The first one is to discard the abstract examples and study only topological spaces that live in some $\mathbb{R}^N$. The second is to develop a theory that allows us to define differentiability also for our abstract construction.

You do know what I am doing to do next! Let us define differentiable manifolds.

Remark 1.1.4. In short we will be able to prove that the two classes are exactly the same. That is we will be able to prove that any (abstract) differentiable manifold admits an embedding in some $\mathbb{R}^N$, cfr. Whitney Theorem 1.7.28.

1.2. Bump functions and differentiable manifolds

Let $\mathbb{R}^n$ and $\mathbb{R}^m$ denote two Euclidean spaces of dimensions n and m, respectively. Let $U \subset \mathbb{R}^n$ and $U_1 \subset \mathbb{R}^m$ be two open subsets and $f : U \to U_1$ a map.

Definition 1.2.1. The map f is called **differentiable** or **smooth** if

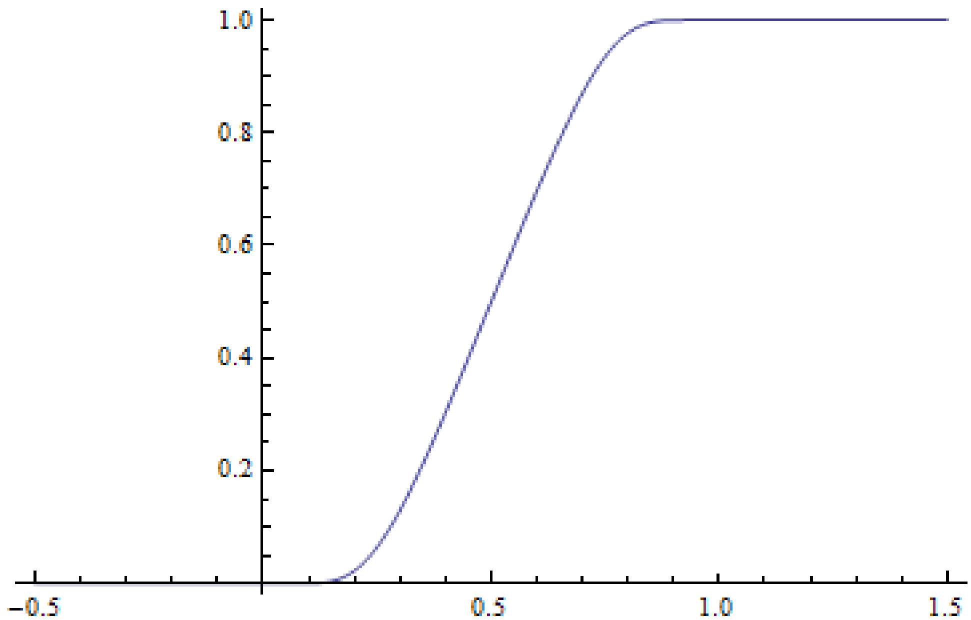
$$f(x_1, \dots, x_n) = (y_1(x_1, \dots, x_n), \dots, y_m(x_1, \dots, x_n))$$

and the $y_i : U \to \mathbb{R}$ are differentiable functions (i.e. infinitely differentiable). The morphism f is called **analytic** if the y_i are analytic functions (i.e. they have a local expansion in power series in a neighborhood of any point $x \in U$).

A differentiable map $f : U \to U_1$ is a **diffeomorphism** if it is bijective and the inverse is a differentiable map.

Let us start with some gymnastic with differentiable functions.

Example 1.2.2. Note that an analytic function that vanishes on an open set is identically zero (the 0 function is already a power series!). For differentiable functions the situation is completely different. There are smooth functions that join the constant function 1 with the constant function 0.



More generally we have the following result.

LEMMA 1.2.3. *Let $A, B \subset \mathbb{R}^n$ be two disjoint subsets. Assume that A is compact and B is closed. Then there is a non negative differentiable function that is identically 1 on A and identically 0 on B. These functions are usually called* **bump functions**.

PROOF. Let $0 < a < b$ be real numbers and define the function

$$f(x) = \begin{cases} exp(\frac{1}{x-b} - \frac{1}{x-a}) & \text{for } a < x < b \\ 0 & \text{otherwise} \end{cases}$$

Then f is differentiable and we may define the function

$$F(x) = \frac{\int_x^b f(t)dt}{\int_a^b f(t)dt},$$

then

$$F(x) = \begin{cases} \frac{\int_x^a f(t)dt}{\int_a^b f(t)dt} + 1 & \text{for } x \leq a \\ -\frac{\int_b^x f(t)dt}{\int_a^b f(t)dt} & \text{for } x \geq b \end{cases}$$

Hence $F(x) = 1$ for $x \leq a$ and $F(x) = 0$ for $x \geq b$. In particular we may define the differentiable function

$$\psi(x_1, \ldots, x_m) = F(x_1^2 + \ldots + x_m^2)$$

that has values 1 on the ball of radius $\sqrt{a}$ and 0 outside the ball of radius $\sqrt{b}$. We may then cover A with k balls in such a way that the union of these balls does not intersect B (remember that A is compact and B is closed) and define ψ_i as above for any ball to produce the differentiable function

$$\varphi = 1 - (1 - \psi_1) \cdots (1 - \psi_k)$$

that realizes the requirements. $\square$

You have to think of these functions as a sort of glue that one can use to patch together functions. It is time to define the main objects of our lectures.

Definition 1.2.4. Let M be a Hausdorff topological space. A (open) **chart** of dimension n on M is a pair (U, φ) where $U \subset M$ is an open subset and $\varphi : U \to V \subset \mathbb{R}^n$ is a homeomorphism on an open subset of $\mathbb{R}^n$.

A **differentiable structure** (DS) of dimension m on M is a collection of (open) charts of dimension m, $\{(U_a, \varphi_a)\}_{a \in A}$ on M such that:

- DS1 $M = \cup_{a \in A} U_a$,
- DS2 for any pair $a, b \in A$ the map $\varphi_a \circ \varphi_b^{-1}$ is a differentiable map of $\varphi_b(U_a \cap U_b)$ onto $\varphi_a(U_a \cap U_b)$,
- DS3 The collection $\{(U_a, \varphi_a)\}_{a \in A}$ is maximal (with respect to inclusion) for all families satisfying DS1 and DS2.

A **differentiable manifold** of dimension m, or m**-manifold**, is a Hausdorff topological space with a countable base and a DS of dimension m.

Remark 1.2.5. Let $p \in U_a$ be a point and (U_a, φ_a) a chart, then $\varphi_a(p) = (x_1, \dots, x_m) \in f_a(U_a) \subset \mathbb{R}^m$. This shows that a chart induces local coordinates on the open U_a. We will frequently identify p with its coordinates in a local chart and use them directly on U_a. In particular we will talk of balls $B_\epsilon(x) \subset V_x$, for $x \in M$ a point and $V_x \subset M$ an open neighborhood of x.

The 0-dimensional differentiable manifold are discrete topological spaces.

Remark 1.2.6. Condition DS3 is not as bad as it seems. Thanks to the axiom of choice the set of differentiable structures satisfying DS1 and DS2 always admits maximal elements.

Remark 1.2.7. We may define analytic structures as well or complex manifold using holomorphic functions and local charts into $\mathbb{C}^n$. These may all be considered as differentiable manifolds and it is sometimes useful to do so thanks to the freedom given by bump functions.

Exercise 1.2.8. Prove that $\mathbb{P}^n_{\mathbb{R}}$ and $S^n \subset \mathbb{R}^{n+1}$ are n-manifolds.

1.3. Differentiable functions

Once defined the differentiable manifolds we need differentiable functions to work on them.

Definition 1.3.1. Let M be a differentiable manifold and $f : M \to \mathbb{R}$ a function. Then f is called **differentiable**, or **smooth**, at a point $p \in M$ if there exists a local chart (U_p, φ) such that $f \circ \varphi^{-1}$ is a differentiable function in $\varphi(p)$. A function is differentiable (smooth) if it is differentiable at each point $p \in M$. Let $\mathcal{F}(M)$ be the set of differentiable functions on M.

It is interesting to note that the set of differentiable functions on a m-manifold M enjoys the following properties:

- DF1 let u be a differentiable function on $\mathbb{R}^r$ and $f_1, \dots, f_r \in \mathcal{F}(M)$ then $u(f_1, \dots, f_r) \in \mathcal{F}(M)$,
- DF2 let $f : M \to \mathbb{R}$ be a function and assume that for any $p \in M$ there is a $g_p \in \mathcal{F}(M)$ and U_p with $f_{|U_p} = g_{p|U_p}$ then $f \in \mathcal{F}(M)$,

DF3 for any $p \in M$ there exist functions $f_1, \dots, f_m \in \mathcal{F}(M)$ and an open neighborhood U_p such that (U_p, φ) is a local chart, where the map $\varphi : U_p \to \mathbb{R}^m$ is given by $\varphi(q) = (f_1(q), \dots, f_m(q))$. In particular any function $g \in \mathcal{F}(M)$ is such that there is a function u_g on $\mathbb{R}^m$ with $g_{|U} = u_g(f_1, \dots, f_m)$.

The first two properties are immediate. To check the third let us play with bump functions.

LEMMA 1.3.2. *Let M be a m-manifold. For any $p \in M$ there exist functions $f_1, \dots, f_m \in \mathcal{F}(M)$ and an open neighborhood U_p such that (U_p, φ) is a local chart, where the map $\varphi : U_p \to \mathbb{R}^m$ is given by $\varphi(q) = (f_1(q), \dots, f_m(q))$. In particular any function $g \in \mathcal{F}(M)$ is such that there is a function u_g on $\mathbb{R}^m$ with $g_{|U} = u_g(f_1, \dots, f_m)$.*

PROOF. Let $\varphi_p : U_p \to B \subset \mathbb{R}^m$ be the local chart and $\pi_i : \mathbb{R}^m \to \mathbb{R}$ the projection on the i^{th} coordinate and define $\psi_i = \pi_i \circ \varphi_p : U_p \to \mathbb{R}$ the composition. Fix a neighborhood $V_{\varphi(p)} \subset B$ and a compact neighborhood $K \subset V_{\varphi(p)}$. Then by Lemma 1.2.3 there is a bump function χ that is identically 1 on K and identically 0 outside of V. This allows to define the smooth function $f_i \in \mathcal{F}(M)$ as

$$f_i(p) = \begin{cases} \psi_i(p)\chi(\varphi(p)) & p \in \varphi^{-1}(V_{\varphi(p)}) \\ 0 & p \in \varphi^{-1}(V_{\varphi(p)})^c \end{cases}$$

$\square$

We can easily extend the definition of differentiable functions to that of smooth map between manifolds.

DEFINITION 1.3.3. Let $F : M \to N$ be a continuos map between manifolds. We say that F is differentiable in a point $p \in M$ if there are local charts (U_p, φ) and $(V_{F(p)}, \psi)$ such that $\psi \circ F \circ \varphi^{-1}$ is differentiable at $\varphi(p)$. If F is differentiable at p then the rank of F at p is the rank of the Jacobian of $\psi \circ F \circ \varphi^{-1}$ at $\varphi(p)$. The map F is called differentiable if it is differentiable at any point.

REMARK 1.3.4. It is easy to check that the definition is well posed. That is it does not depend on the choice of local charts, see the exercises.

The Lemma 1.3.2 shows how useful are bump function to define global objects starting from local constructions.

REMARK 1.3.5. It is not difficult to prove that any collection of functions enjoying properties DF1, DF2, and DF3 are associated to a unique differentiable structure and are $\mathcal{F}(M)$ for this differentiable structure.

DEFINITION 1.3.6. The set $\mathcal{F}(M)$ has a natural structure of real algebra given by the punctual operations $f * g(p) = f(p) * g(p)$. When we think of it as an algebra we call it $C^\infty(M)$.

The following is a useful generalization of bump functions.

LEMMA 1.3.7. *Let $C \subset M$ be a compact subset of a manifold M and $V \supset C$ an open subset. Then there exists an element $f \in C^\infty(M)$ which is identically 1 on C and identically 0 outside of V.*

PROOF. Let $\{U_a\}_{a \in A}$ be a covering of V with local charts. Let $p \in C$ then there is a neighborhood $V_{a,p} \subset U_a$ with $\overline{V_{a,p}} \subset U_a$ a compact neighborhood. Then $\{V_{a,p}\}_{a \in A, p \in C}$ is an open covering of C and admits a finite subcovering. This yields finitely many subsets K_i with

- $K_i \subset U_{a_i}$ compact in a local chart
- $\cup_1^k K_i \subset V$
- $C \subset \cup_1^k K_i$

Then, by Lemmata 1.3.2 1.2.3, for each index i we may define a bump function $f_i : M \to \mathbb{R}$ that is identically 1 on K_i and identically zero outside a neighborhood of K_i contained in U_{a_i} (keep in mind that K_i is contained in a local chart U_a). Then the required function is obtained with the usual trick

$$\psi = 1 - (1 - f_1) \cdots (1 - f_k).$$

□

Definition 1.3.8 (Open submanifold). Let $U \subset M$ be an open subset of an m-manifold with differentiable structure $\{(U_a, \varphi_a)\}_{a \in A}$. Then we may induce on U a differentiable structure of dimension m by restriction. In this way we call U an **open submanifold**.

Remark 1.3.9. Note that in general closed subsets of manifolds do not inherit a differentiable structure. Think to the example of $[0, 1] \subset \mathbb{R}$. One of our goals is to understand what is a reasonable notion of submanifold, see section 1.6.

The following is an easy, but useful, consequence of Lemma 1.3.7.

Lemma 1.3.10. *Let M be a m-manifold and $U \subset M$ an open submanifold. Let $f \in C^\infty(U)$ be a function and $p \in U$ a point. Then there is an open $V_p \subset U$ and a function $\overline{f} \in C^\infty(M)$. such that $\overline{f}_{|V_p} = f_{|V_p}$.*

Proof. Let $B_1(p) \subset U$ be a local neighborhood. Then by Lemma 1.3.7 there is a function $g \in C^\infty(U)$ such that $g_{|B_{1/2}(p)} = 1$ and g is identically zero outside $B_{2/3}(p)$. Therefore we may define the function $\overline{f} : M \to \mathbb{R}$ as follows

$$\overline{f}(x) = \begin{cases} g(x)f(x) & x \in U \\ 0 & x \in U^c \end{cases}$$

□

We conclude this gymnastic with a technical tool that is quite useful in global differential geometry. Recall the following facts from topology:

- the support of a function $f : M \to \mathbb{R}$ is $supp(f) = \overline{\{x \in M | f(x) \neq 0\}}$, f is called with compact support if the support is contained in a compact set,
- a family of subsets $\{B_a\}_{a \in A}$ is called locally finite if for any $x \in X$ there exists a neighborhood U_x such that $B_a \cap U_x \neq \emptyset$ only for finitely many indexes,
- a topological space is paracompact if any open covering has an open refinement that is locally finite,
- a locally compact Hausdorff space, with a countable base, is paracompact. In particular manifolds are paracompact.

Proposition 1.3.11 (Partition of unity). *Let M be a m-manifold and $\{U_a\}_{a \in A}$ an open covering of M. Then there is a system of functions $\{\theta_a\}_{a \in A} \subset C^\infty(M)$, called* **partition of unity attached to** *$\{U_a\}_{a \in A}$ such that:*

- *each θ_a has compact support contained in U_a*
- $\theta_a \geq 0$

- $\sum_{a \in A} \theta_a \equiv 1$.

PROOF. For any $x \in M$ let $B_{\epsilon(x)}(x) \subset U_{a(x)}$. The collection $\{B_{\frac{\epsilon(x)}{2}}(x)\}_{x \in M}$ is an open covering. M is paracompact hence there is a locally finite refinement $\{Z_a\}_{a \in A}$ of $\{B_{\frac{\epsilon(x)}{2}}(x)\}_{x \in M}$ (note that we may keep the same index because $B_{\frac{\epsilon(x)}{2}}(x) \subset U_a$, for some a). Via Lemma 1.3.7 we may associate a non negative function $\psi_a \in C^\infty(M)$ with (compact) support in Z_a. The family $\{Z_a\}_{a \in A}$ is locally finite therefore

$$\sum_{a \in A} \psi_a = \theta$$

is a well defined positive function in $C^\infty(M)$. Then the function

$$\theta_a := \frac{\psi_a}{\theta}$$

satisfies all the requirements. □

We are not going to use partition of unity much. We will often restrict our global theorems to the simpler setting of compact manifolds avoiding in such a way the technicalities that requires partition of unity. As a toy application we prove the existence of proper functions on any manifold.

Recall that a morphism is said to be **proper** if the preimage of any compact is compact. When M is a compact manifold any function in $C^\infty(M)$ is proper, see Exercise 1.8.5. On the other hand for non compact manifolds it is not clear if proper functions exists at all. As an application of partitions of unity we will show that any manifold admits proper differentiable functions.

COROLLARY 1.3.12. *Let M be a m-manifold then there exists a proper function $f \in C^\infty(M)$.*

PROOF. Let $\{U_i\}$ be a countable locally finite open covering of M. Let θ_i be a partition of unity attached to $\{U_a\}$. Set

$$\rho := \sum_i i\theta_i,$$

then $\rho \in C^\infty(M)$. To prove it is proper it is enough to show that $\rho^{-1}([-n, n])$ is compact for $n \in \mathbb{N}^{>0}$. Let $x \in \rho^{-1}([-n, n])$ then $\theta_i(x) \neq 0$ for some $i \leq n$. Therefore $\rho^{-1}([-n, n])$ is contained in union of the supports of θ_i for $i \leq n$. The latter is compact the former is closed therefore $\rho^{-1}([-n, n])$ is compact. □

1.4. Inverse Theorem and constant rank Theorem

In this section we recall basic facts from function theory in several variables. Let $F : W \to \mathbb{R}^m$ be a differentiable function, with $W \subset \mathbb{R}^n$ open and $p \in W$ a point. Then the differential at p is the linear application induced by the Jacobian matrix evaluated at p.

$$DF_p = JF(p) = (\frac{\partial f_i}{\partial x_j}(p)),$$

where $F = (f_1, \ldots, f_m)$.

THEOREM 1.4.1. *(Inverse function Theorem/Teorema del Dini) Let $W \subset \mathbb{R}^n$ be an open set and $F : W \to \mathbb{R}^n$ a smooth function. If $a \in W$ and DF_a is invertible, then there exists an open $U_a \subset W$ such that $V = F(U)$ is open and $F_{|U} : U \to V$ is a diffeomorphism. Moreover for $y = F(x) \in V$ we have*

$$DF_y^{-1} = (DF_x)^{-1}$$

More generally we have the so called constant rank theorem.

THEOREM 1.4.2. *Let $W \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open subsets and $F : W \to V$ a smooth function. Assume that for any point $p \in W$ we have $rkDF_p = k$. Then for any $a \in W$ there are open subsets $W_a \subset W$ and $V_{F(a)} \subset V$ and diffeomorphisms $\varphi : W_a \to U \subset \mathbb{R}^n$, $\psi : V_{F(a)} \to \mathbb{R}^m$ such that, for any $(x_1, \dots, x_n) \in U$*

$$\psi \circ F \circ \varphi^{-1}(x_1, \dots, x_n) = (x_1, \dots, x_k, 0, \dots, 0).$$

REMARK 1.4.3. Note that this is an extension of inverse function theorem, but all proofs I know relies on the latter, see also Remark 1.5.4. One can visualize it saying that a map of constant rank k behaves, up to a diffeomorphic change of coordinates, like a projection $\pi : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ composed with the injection $\mathbb{R}^k \to \mathbb{R}^k \times \{0\} \subset \mathbb{R}^k \times \mathbb{R}^{m-k}$.

1.5. Constructions of n-manifolds

As we already observed any open subset of $\mathbb{R}^n$ is a n-differentiable manifold. Here we want to provide ways to produce differentiable manifolds.

LEMMA 1.5.1. *Let $U \subset \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}$ a differentiable function. Then the graph of f*

$$\Gamma_f := \{(x, y) \subset \mathbb{R}^n \times \mathbb{R} | y = f(x)\}$$

is a n-manifold.

PROOF. Let $\psi : U \to \mathbb{R}^{n+1}$ be the function $\psi(x) = (x, f(x))$. Then ψ is differentiable and of constant rank n. This is enough to conclude thanks to Theorem 1.4.2. □

EXAMPLE 1.5.2. With Lemma 1.5.1 it is easy to see that $S^n \subset \mathbb{R}^{n+1}$ is a n-manifold, by gluing together two parametrizations.

By inverse function theorem we have the following.

LEMMA 1.5.3. *Let $U \subset \mathbb{R}^n$ be open and $F : U \to \mathbb{R}^s$ a differentiable function given by $F(x) = (f_1(x), \dots, f_s(x))$. Let $p \in U$ and $a = F(p)$. Assume that $n > s$ and DF_p is of maximal rank. Then there is a neighborhood U_p such that $U_p \cap F^{-1}(a)$ is a $(n - s)$-manifold.*

PROOF. Let $H : \mathbb{R}^n \to \mathbb{R}^{s+(n-s)}$ be given by

$$H((x_1, \dots, x_n)) = (f_1(x_1, \dots, x_n), \dots, f_s(x_1, \dots, x_n), x_{s+1}, \dots, x_n).$$

By hypothesis DH_p is invertible and by the inverse function theorem the function H is locally invertible. Let $a = (a_1, \dots, a_s)$ then there are neighborhoods $W_{(a_1,\dots,a_s,x_{s+1},\dots,x_n)}$ and $U_p \subset U$ such that $H^{-1} : W \to U_p$ is a diffeomorphism. Let $L_a := (x_1 = a_1, \dots, x_s = a_s)$ be a linear space, then

$$H^{-1}(L_a \cap W) = U_p \cap F^{-1}(a),$$

and

$$H_{|U_p\cap F^{-1}(a)} : U_p \cap F^{-1}(a) \to L_a \simeq \mathbb{R}^{n-s}$$

is an open chart and shows that $U_p \cap F^{-1}(a)$ is a differentiable manifold of dimension $(n-s)$. □

REMARK 1.5.4. Let me stress that this is "essentially" the way one proves the Rank Theorem from Inverse Function Theorem.

EXAMPLE 1.5.5. When $s = 1$ Lemma 1.5.3 has the following easier translation. If the gradient of a differentiable function $F : \mathbb{R}^n \to \mathbb{R}$ is not identically zero at a point p then $F^{-1}(F(p))$ is an $(n-1)$-manifold around p. This again tells us, via gluing, that S^n is an n-manifold.

DEFINITION 1.5.6 (Product). If M and N are differentiable manifolds of dimension m and n respectively we may consider their product $M \times N$. If $\{(U_a, \varphi_a)\}_{a\in A}$ and $\{(V_b, \varphi_b)\}_{b\in B}$ are their differentiable structures then it is easy to see that $\{(U_a \times V_b, \varphi_a \times \varphi_b)\}_{a\in A, b\in B}$ satisfy DS1 and DS2 for $M \times N$. Then we may extend it to a maximal differentiable structure, keep in mind Remark 1.2.6, and define the **product** as a $(n+m)$-manifold.

1.6. Submanifolds

It is important to have a notion of submanifold of a manifold. Due to the existence of the differentiable structure there are various possibilities. At first one could say, mimicing the construction of topological subspaces, that a submanifold $W = F(N)$ of M is the image in M of an injective map $F : N \to M$ together with the induced topology and DS that makes F a diffeomorphism onto the image. This is somehow the weaker possible definition and as such it presents some difficulties.

EXAMPLE 1.6.1. Let $M = \mathbb{R}^2$ and $F : \mathbb{R} \to \mathbb{R}^2$ constructed as follows. Let $g : \mathbb{R} \to \mathbb{R}$ be a monotone increasing smooth function such that $g(0) = \pi$, $\lim_{t\to-\infty} g(t) = 0$, and $\lim_{t\to\infty} g(t) = 2\pi$. Let

$$F(t) = (2cos(g(t) - \frac{\pi}{2}), sin(2(g(t) - \frac{\pi}{2}))).$$

After a moment thought we realize that the image of F is a figure 8 and F is a 1:1 map onto its image and F is a diffeomorphism onto the image. Therefore $W = F(\mathbb{R})$ is a submanifold with our previous definition. Unfortunately it is not a manifold itself. Indeed it is not even a topological variety if we consider the induced topology of $\mathbb{R}^2$. Indeed removing 0 from $B_\epsilon((0,0)) \cap W$ we get 4 connected component for any $\epsilon \ll 1$.

Examples like the one above suggest that a less general definition of submanifold could be useful.

DEFINITION 1.6.2. Let M be a m-manifold and $N \subset M$ a subset. We say that N is a n-**submanifold** (regular submanifold in the literature) of M if for any $p \in N$ there is a local chart (U_p, φ) in the DS of M such that

$$N \cap U_p = \varphi^{-1}(\{x_{n+1} = \ldots = x_m = 0\}).$$

REMARK 1.6.3. Note that if N is a n-submanifold of M then it is for free a n-manifold. Let $\pi : \mathbb{R}^m \to \mathbb{R}^n$ be the projection onto the first n coordinates, then

$\{U_p \cap N, \pi \circ \varphi)\}$ gives a DS structure enjoying properties DS1 and DS2 and therefore defines a DS.

There is a third notion, between the two already presented, of submanifold on which we will not dwell here, the imbedded submanifold.

We opted for a stricter notion of submanifold. This will help us to prove result but is more difficult to produce them. To do this observe the following.

Definition 1.6.4. An **embedding** (**imbedding** in some British books) is an injective map $F : N \to M$ of constant rank, such that U is open in N if and only if $F(U) = V \cap F(N)$ for some open V in M. In other words the topology on N is the induced topology on $F(N)$.

Proposition 1.6.5. *Let $F : N \to M$ be an embedding of manifolds of dimension n and m respectively. Then $F(N)$ is a submanifold and F is a diffeomorphism on the image.*

Proof. By the constant Rank Theorem 1.4.2, and the definition of embedding for any $F(x) = p \in F(N)$ we may choose an open local chart (V_p, φ) in M, with $F^{-1}(V_p \cap F(N)) = U_x$ and such that $F_{|U_x} : U_x \to V_p$ is given (in local coordinates) by $(x_{n+1} = \ldots = x_m = 0)$. This gives the first part of the proof for the latter it is enough to compose with the projection on the first n variables. □

Corollary 1.6.6. *Let $F : N \to M$ be an injective smooth map of constant rank. Assume that N is compact, then F is an embedding and $F(N)$ is a submanifold.*

Proof. By Proposition 1.6.5 it is enough to prove that $F : N \to F(N)$ is open. Since F is injective this is equivalent to prove that it is closed. The latter is immediate since N is compact and M is Hausdorff. □

The following extends and improves Lemma 1.5.3.

Proposition 1.6.7. *Let $F : N \to M$ be a map of constant rank k, and $q \in F(N)$ a point. Then $F^{-1}(q)$ is a closed submanifold of dimension $n - k$.*

Proof. The subset $F^{-1}(q)$ is closed by the continuity of F. Fix $y \in F^{-1}(q)$ then by the Rank Theorem 1.4.2 we may choose coordinate neighborhoods U_y and V_q such that (in local coordinates) $F(U_y) = (x_{k+1} = \ldots = x_m = 0)$, with $y = (0, \ldots, 0)$ and $q = (0, \ldots, 0)$. Therefore $F^{-1}(q) \cap U_y = (x_1 = \ldots = x_k = 0)$ and we conclude. □

Remark 1.6.8. A particular, but useful, case of the above proposition is the following. Let $F : N \to M$ be a smooth function, with $n > m$. Assume that for some $a \in M$ we have $rkDF_x = m$ for any $x \in F^{-1}(a)$. Then $F^{-1}(a)$ is a submanifold. This is particularly useful when $F \in C^\infty(N)$ is a differentiable function.

It is time to get some gratification from all the work we did. Recall that we are still wandering whether abstract manifolds and embedded ones are the same class of objects. We have developed enough theory to study embeddings into $\mathbb{R}^N$. Here the main result is Whitney embedding theorem. We are not able to prove it in full generality. For the time being we are pleased to prove a light version of Whitney theorem. We will show that any compact manifold admits an embedding in some $\mathbb{R}^N$, see Theorem 1.7.28 for a refinement and also Remark 1.7.30.

PROPOSITION 1.6.9. *Let N be a compact n-manifold. Then there is an embedding $\varphi : N \to \mathbb{R}^a$, for some a.*

PROOF. The manifold N is compact, hence we may choose a finite collection of local charts

$$\{(U_i, \varphi_i)\}_{i=1,\ldots,s}$$

and a finite open covering $\{V_i\}_{i=1,\ldots,s}$ such that $\overline{V}_i \subset U_i$, if you are wandering why go back to the proof of Proposition 1.3.11. By Lemma 1.3.7 there are smooth functions $f_i : N \to \mathbb{R}$ such that $f_i(x) = 1$ for any $x \in V_i$ and $f_{i|U_i^c} \equiv 0$. Let

$$\sigma_i(x) = \begin{cases} f_i(x)\varphi_i(x) & x \in U_i \\ 0 & x \notin U_i \end{cases} \tag{1}$$

The functions σ_i are smooth and we may define

$$F : N \to \mathbb{R}^{m(s+1)}$$

as

$$F(x) = (\sigma_1(x), \ldots, \sigma_s(x), \sigma_1(x)\varphi_1(x), \ldots, \sigma_s(x)\varphi_s(x)).$$

By construction $\sigma_{i|U_i^c} \equiv 0$ then the function F is well defined and smooth. It is then easy to see that F is injective, keep in mind that (U_i, φ_i) are local charts. The manifold N is compact, therefore by Corollary 1.6.6 it is enough to prove that the map is of constant rank n. Let $p \in U_i \subset N$ be a point, then in a neighborhood of p we have

$$rkJ(F \circ \varphi_i^{-1})_{\varphi_i(p)} \geq rkJ((\sigma_i\varphi_i) \circ \varphi_i^{-1})_{\varphi_i(p)}.$$

By construction we may assume that $\sigma_i \equiv 1$ in a neighborhood of p therefore

$$rkJ((\sigma_i\varphi_i) \circ \varphi_i^{-1})_{\varphi_i(p)} = rkJ(\varphi_i \circ \varphi_i^{-1}) = n.$$

□

REMARK 1.6.10. Whitney theorem applies to arbitrary manifold. The extension to non compact manifold it is not really hard. One shows that any manifold can be written as a countable union of increasing compact sets and plays a bit with these and partitions of unity to produce the required functions.

1.7. Tangent space and tangent bundle

To take the full advantage of differentiable functions we need to introduce an equivalent of the Jacobian of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$. Let us start stressing that the Jacobian is defined at any point p as the linear map $Jf_p : \mathbb{R}^n \to \mathbb{R}^m$

$$Jf_p(v) := (\partial_i f_j(p))(v)$$

that evaluates the partial derivatives of f at the point p.

Here we used v and not x as point of $\mathbb{R}^n$ because, in this occasion, it is useful to think of $\mathbb{R}^n$ as a vector space rather than a differentiable manifold. To get a similar construction for a differentiable manifold we need to associate a vector space at any point. This is what is called the tangent space.

We will start with an embedded m-manifold $M \subset \mathbb{R}^s$, i.e. a submanifold of $\mathbb{R}^s$. Fix a point $p \in M$ and let $J \subset \mathbb{R}$ be an interval containing 0 and $f : J \to C \subset M \subset \mathbb{R}^s$ be an embedding with $f(0) = p$. In other words C is a differentiable curve (1-manifold) in a neighborhood of p. Then we may associate the **tangent vector** in $\mathbb{R}^s$ to the curve C at the point p simply considering $f'(0) \in \mathbb{R}^s$. This may seem not really satisfactory since we would like to land in some vector space of dimension m. To do this let (U_p, φ) be a local chart, then $f = \varphi^{-1} \circ \varphi \circ f$, in other words

$$f'(0) = Df_0(1) \subset Im(D\varphi^{-1})_{\varphi(p)} \subset \mathbb{R}^m.$$

At first you may wonder why this is well defined and does not depend on the choice of the local chart. To convince yourself consider two charts (U_p, φ), (U_p, ψ), and the following commutative diagrams, the second one induced by the differentials,

$$\begin{tikzcd} U_p & \mathbb{R}^m \arrow[l, "\varphi^{-1}"'] \\ \mathbb{R}^m \arrow[u, "\psi^{-1}"] \arrow[ur, "\varphi\circ\psi^{-1}"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathbb{R}^s & \mathbb{R}^m \arrow[l, "D(\varphi^{-1})_{\varphi(p)}"'] \\ \mathbb{R}^m \arrow[u, "D(\psi^{-1})_{\psi(p)}"] \arrow[ur, "\simeq", "D(\varphi\circ\psi^{-1})_{\psi(0)}"'] & \end{tikzcd}.$$

We may consider the union of all tangent vectors at p in M.

Definition 1.7.1. (**Geometric point of view on Tangent space**) Let $p \in M \subset \mathbb{R}^s$ be a point. The tangent space of M at the point p is the set of all tangent vectors at p

$$T_pM =: \{\text{tangent vectors at } p \text{ in } M\} = Im(D\varphi^{-1})_{\varphi(p)} \subset \mathbb{R}^s,$$

where $\varphi : U_p \to \mathbb{R}^m$ is a local parametrization.

This gives us a quick way to compute T_pM for an embedded manifold M.

Corollary 1.7.2. *Let $M \subset \mathbb{R}^s$ be a submanifold, $p \in M$ a point and (U_p, φ) a local chart, then*

$$T_pM = \langle \frac{\partial\varphi^{-1}}{\partial x_1}(\varphi(p)), \dots, \frac{\partial\varphi^{-1}}{\partial x_m}(\varphi(p))\rangle \subset \mathbb{R}^s.$$

Proof. Immediate from the definition. □

For hypersurfaces, that is manifold M embedded in $\mathbb{R}^{m+1}$ and defined by a single equation, there is an even easier way to express the tangent space at a point.

Lemma 1.7.3. *Let $M = \{F(x_1, \dots, x_{m+1}) = 0\} \subset \mathbb{R}^{m+1}$ be a submanifold and $(a_1, \dots, a_{m+1}) \in M$ a point then*

$$T_{(a_1,\dots,a_{m+1})}M = (\frac{\partial F}{\partial x_1}(a_1, \dots, a_{m+1}), \dots, \frac{\partial F}{\partial x_{m+1}}(a_1, \dots, a_{m+1}))^\perp.$$

Proof. Let $f : J \to M$ be a curve through p. Then $f(t) = (x_1(t), \dots, x_{m+1}(t))$ and by construction $F(x_1(t), \dots, x_{m+1}(t)) = 0$, for any $t \in J$. This yields

$$\frac{dF}{dt} = \sum_i \frac{\partial F}{\partial x_i} x_i' = 0,$$

that is to say the tangent vector $f'(0)$ is orthogonal to the vector of partial derivative. □

Remark 1.7.4. The assumption that M is a submanifold guaranties that

$$(\frac{\partial F}{\partial x_1}(a_1, \dots, a_{m+1}), \dots, \frac{\partial F}{\partial x_{m+1}}(a_1, \dots, a_{m+1})) \neq 0.$$

The vector $(\frac{\partial F}{\partial x_1}(a_1, \dots, a_{m+1}), \dots, \frac{\partial F}{\partial x_{m+1}}(a_1, \dots, a_{m+1}))$ is called the gradient of F or after a normalization the normal vector to M, see also page 36 and Chapter 3.

This allows to compute easily the tangent space of $S^n = \{x_1^2 + \ldots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$. For any $p = (x_{1p}, \ldots, x_{n+1p}) \in S^n$ we have

$$\frac{\partial F}{\partial x_i}(p) = 2x_{ip}.$$

This shows $T_pS^n = (x_{1p}, \ldots, x_{n+1p})^{\perp}$.

We are looking at T_pM as a vector subspace of $\mathbb{R}^s$, therefore it has a natural vector space structure. On the other hand the choice of a local chart (U_p, φ) defines an isomorphism between T_pM and $\mathbb{R}^m$. Therefore T_pM inherits a vector space structure independently of its immersion in $\mathbb{R}^s$. Furthermore given a differentiable function $F : M \to N$ it is well defined and linear, the map

$$DF_p : T_pM \to T_{F(p)}N$$

that associates the tangent vector of f to the tangent vector of $F \circ f$. This suggests that it should be possible to define the tangent space in an abstract way, disregarding any embedding in $\mathbb{R}^s$.

Our next aim is to give a different, somehow more abstract but fruitful in the long, point of view on the construction of the tangent space.

Let M be a m-manifold. Let $p \in M$ be a point. A smooth function in p is the data of a pair (f, U_p) with $p \in U_p$ and $f \in C^\infty(U_p)$. We will say that two smooth functions $(f, U_p) \sim (g, V_p)$ are equivalent if there exists an open neighborhood $W_p \subseteq U_p \cap V_p$ such that $f_{|W_p} = g_{|W_p}$. Let

$$C^\infty(M)_p := \cup_{U \ni p} C^\infty(U)/\sim .$$

It is easy, see the exercises, to show that $C^\infty(M)_p$ is a $\mathbb{R}$-vector space and has a $\mathbb{R}$-algebra structure induced by the multiplication of functions.

Definition 1.7.5. A **derivation** in p is a linear application $X : C^\infty(M)_p \to \mathbb{R}$ that satisfies the following requirement

$$X(fg) = f(p)X(g) + g(p)X(f)$$

known as Leibniz rule.

Remark 1.7.6. The derivations are elements in $C^\infty(M)_p^*$ that satisfies the Leibniz rule. The latter is clearly preserved by linear combinations. Therefore the derivations in p are a vector subspace in $C^\infty(M)_p^*$. Further note that, again by Leibniz rule, if $f \in C^\infty(M)$ is constant that $X(f) = 0$.

Definition 1.7.7. Let $D(M)_p \subset C^\infty(M)_p^*$ be the vector space spanned by derivations in p.

Our first aim is to compute the dimension of $D(M)_p$. Fix a local chart (U_p, φ) with $\varphi(p) = 0$ and $\varphi(q) = (x_1(q), \ldots, x_m(q))$. For any $f \in C^\infty(M)$ we define $f^* = f \circ \varphi^{-1}$, we may also assume without loss of generality that $f(p) = 0$. Fix an open ball centered at 0, $B \subset \varphi(U_p)$. For any point $(x_1, \ldots, x_m) \in B$ we have

$$f^*(x_1, \ldots, x_m) - f^*(0) = \int_0^1 \frac{d}{dt} f^*(tx_1, \ldots, tx_m)dt = \sum x_i \int_0^1 \frac{\partial f^*}{\partial x_i}(tx)dt$$

going back to M we may define functions $g_i \in C^\infty(f^{-1}(B))$ such that

$$f(q) = \sum x_i(q)g_i(q), \text{ for all } q \in \varphi^{-1}(B), \tag{2}$$

with

$$g_i(q) = \int_0^1 \frac{\partial f^*}{\partial x_i}(t\varphi(q))dt \tag{3}$$

In particular

$$g_i(p) = (\frac{\partial f^*}{\partial x_i})(0).$$

Let $X \in D(M)_p$ be a derivation then applying Leibniz rule and equations (2) (3) we have

$$X(f) = X(\sum x_i(p)g_i(p)) = \sum x_i(p)X(g_i) + \sum(\frac{\partial f^*}{\partial x_i})(0)X(x_i) = \sum(\frac{\partial f^*}{\partial x_i})(0)X(x_i)$$

DEFINITION 1.7.8. Let $\partial_i :=: \frac{\partial}{\partial x_i} \in D(M)_p$ be the derivation given by

$$f \mapsto (\frac{\partial f^*}{\partial x_i})(0),$$

Then we may rephrase the above equation as

$$X = \sum_{i=1}^{m} X(x_i)\partial_i, \tag{4}$$

REMARK 1.7.9. Note that $X(x_i) \in \mathbb{R}$ is a real number. Hence any derivation is a real linear combination of the ∂_i's. It is easy to see that $\{\partial_i\}$ are linearly independent in $D(M)_p$, consider for instance the projection on one coordinate.

Therefore we have proved that $D(M)_p$ is a vector space of dimension m and (∂_i) is a basis. We are ready to prove the following proposition.

PROPOSITION 1.7.10. *Let $M \subset \mathbb{R}^s$ be a m-manifold and $p \in M$ a point then $D(M)_p$ is a vector space of dimension m. Moreover there is a natural identification between T_pM and $D(M)_p$ given by*

$$v = (a_1, \ldots, a_m) \mapsto \sum a_i\partial_i.$$

PROOF. We already observed that both T_pM and $D(M)_p$ are vector spaces of dimension m. Fix a local chart (U_p, φ), with $\varphi(p) = 0$ and a basis (e_i) of $T_pM = \mathbb{R}^m$. Let $x_i : \mathbb{R} \to \mathbb{R}^m$ be the function

$$x_i(t) = te_i,$$

then the vector e_i is the tangent vector of $\varphi^{-1} \circ x_i$. We may then define the linear isomorphism

$$\chi : T_pM \to D(M)_p$$

given by

$$\chi(e_i) = \partial_i.$$

□

DEFINITION 1.7.11 (**Tangent space to a manifold**). Let M be a m-manifold and $p \in M$ a point. The tangent space in p is the vector space of derivation in the point p.

$$T_pM := D(M)_p$$

Let us start immediately to gain something from the abstract viewpoint. We define the differential of a morphism from the derivation point of view. Let $F : M \to N$ be a morphism, then by composition we have

$$F^* : C^\infty(N) \to C^\infty(M)$$

given by

$$F^*(f) = f \circ F.$$

Then to a derivation $X \in D(M)_p = T_pM$ we associate the derivation $DF_p(X)$ given by

$$DF_p(X)(f) = X(F^*(f)).$$

DEFINITION–LEMMA 1.7.12. Let $F : M \to N$ be a smooth map then the **rank** of F at the point p is $\dim Im(DF_p) \subset T_{F(p)}N$.

We got a new definition of the differential map

$$DF_p : T_pM \to T_pN.$$

It is immediate, and left to the reader, to prove that the two definition of differential agree. The latter allows a straight forward proof of the chain rule for differentials.

It is also possible to use the differential to define the tangent vector of a curve in a point avoiding coordinates or local charts.

DEFINITION 1.7.13. Let $f : J \to C \subset M$ be a curve with $f(0) = p$. Then the **tangent vector** of C in p is

$$Df_0(1).$$

REMARK 1.7.14. It is easy to see that this is equivalent to our previous geometric version of tangent vectors. From now on we will identify the three descriptions of the tangent space and we will use the one that suites more in the specific contest we are working on. See the exercise at the end of the chapter for a fourth one.

PROPOSITION 1.7.15. *Let M, N and S be manifolds and $F : M \to N$, $G : N \to S$ morphisms. Then*

$$D(G \circ F)_p = DG_{F(p)} \circ DF_p.$$

PROOF.

$$D(G \circ F)_p(X)(f) = X((G \circ F)^*(f)) = X(f \circ G \circ F) =$$

$$= DF_p(X)(f \circ G) = DG_{F(p)}(DF_p(X))(f) = DG_{F(p)} \circ DF_p(X)(f)$$

□

COROLLARY 1.7.16. *Let $F : M \to N$ be a diffeomorphism, then $DF_p : T_pM \to T_{F(p)}N$ is a linear isomorphism and*

$$(DF_p)^{-1} = D(F^{-1})_{F(p)}.$$

In particular for any local chart (U_p, φ) the differential $D\varphi_q^{-1}$ is an isomorphism for any $q \in U_p$.

REMARK 1.7.17. It is important to remember that inverse function theorem is a local inverse of Corollary 1.7.16. The isomorphism $D\varphi_q$ produces an explicit basis for T_qM. Let $\varphi(q) = (x_1(q), \ldots, x_m(q))$ we already observed that $(\frac{\partial}{\partial x_i}(\varphi(q)))$

is a basis for $T_{\varphi(q)}\mathbb{R}^m$ for any $q \in U_p$. Set (dx_i) the canonical dual basis. That is to say $dx_i(\frac{\partial}{\partial x_i}) = \delta_{ij}$ In this notation we have

$$T_q M = \langle \frac{\partial \varphi^{-1}}{\partial x_i} \rangle,$$

and $D\varphi^{-1}_{\varphi(p)}$ is given by sending $\frac{\partial}{\partial x_i} \in D(\mathbb{R}^m)_{\varphi(p)}$ to $\frac{\partial \varphi^{-1}}{\partial x_i} =: \frac{\partial}{\partial x_i} \in D(M)_p$. That is to say

$$D\varphi^{-1}_{\varphi(p)} = (dx_1, \ldots, dx_m).$$

The construction of derivation $D(M)_p$ is essentially based on the algebra structure of $C^\infty(M)$. More generally we may define.

Definition 1.7.18. Let A be an algebra over a field k. A **derivation** of A is a k-linear mapping $D : A \to A$ that satisfies the Leibniz rule

$$D(fg) = fD(g) + gD(f).$$

We will be mainly concerned with the special case of the real algebra $C^\infty(M)$. Let M be a m-manifold.

Definition 1.7.19. A **vector field** X on $C^\infty(M)$ is a derivation of $C^\infty(M)$.

Remark 1.7.20. For any point $p \in M$ we may associate to a vector field X a derivation (tangent vector) in p

$$X(p) : C^\infty(M)_p \to \mathbb{R}$$

given by

$$X(p)(f) = X(f)(p).$$

Definition 1.7.21. Let $\mathcal{X}(M)$ be the set of vector fields on M.

Then for (U_p, φ) a local chart $\partial_i \in \mathcal{X}(U_p)$ is the vector field associated to the partial derivation with respect to the i$^{\text{th}}$ coordinate that is

$$\partial_i(f) := \frac{\partial(f \circ \varphi^{-1})}{\partial x_i},$$

and equation (4) gives

$$X(p) = \sum X(p)(x_i)\partial_i(p),$$

and $X(p)(x_i)$ is a smooth function in p.

The set $\mathcal{X}(M)$ has a natural structure of $C^\infty(M)$-module given by

$$fX : g \mapsto fX(g), \quad X + Y : g \mapsto X(g) + Y(g).$$

The Leibniz rule forces $X(f) = 0$ for any constant f, and, via restriction and Lemma 1.3.7, from a vector field on M we may induce a vector field $X(U)$ on any open submanifold U and viceversa, maybe at the expense of shrinking U, see also the proof of Proposition 1.7.25.

Further note that, due to Leibniz rule, in general the composition of two vector fields is not a vector field.

Example 1.7.22. Consider $X, Y \in \mathcal{X}(\mathbb{R}^3)$ given in the canonical base by

$$X = y\partial_x - x\partial_y, \quad Y = z\partial_y - y\partial_z,$$

Then $XY(xy) = yz$ on the other hand $XY(x) = XY(y) = 0$.

It is then quite amazing that it is always well defined the so called **Lie bracket**, the bilinear form

$$[\cdot,\cdot] : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$

given by $[X, Y] = XY - YX$.

It is immediate to see that, if $\dim M > 0$, the space $\mathcal{X}(M)$ has an infinite dimensional $\mathbb{R}$-vector space structure and a $C^\infty(M)$-module structure. Note that for any open chart (U, φ) the basis $(\frac{\partial}{\partial x_i}(q))$ is defined on every point $q \in U$. This induces a natural map

$$U \times \mathbb{R}^m \to \cup_{p \in U} T_p U,$$

and shows that $\mathcal{X}(U)$ is finitely generated as $C^\infty(U)$-module. In the next section we will discover how to glue this local identifications to produce the tangent bundle. Moreover with the bracket operation $[\cdot,\cdot]$ it acquires the structure of a Lie algebra. We are not going to develop this theory, the interested reader can look at [**5**].

From the geometric point of view a vector field is the assignment of a tangent vector in any point of M, and this assignment varies in a smooth way. This suggests the possibility to consider the set of all tangent spaces of a manifold M and consider it with a differentiable structure. Let us work out the details.

1.7.1. Tangent bundle.

Let M be a m-manifold

Definition 1.7.23. The tangent bundle of M is the set

$$TM := \cup_{p \in M} T_p M$$

By definition TM has a natural projection $\pi : TM \to M$ sending $v \in T_p M$ to $p \in M$. Our aim is to show that TM has a natural structure of $2m$-manifold and π is a smooth map of constant rank m.

Let us start considering $W \subset \mathbb{R}^m$ an open subset. Then for any $p \in W$ we have $T_p M = \mathbb{R}^m$ and we have a natural basis in $T_p M$ corresponding to the vectors associated to the lines $p + te_i$, where $(e_1, \ldots, e_m)$ is a basis in $\mathbb{R}^m$. In the vector fields notation we are choosing the m linearly independent vector fields $\partial_i \in \mathcal{X}(W)$ and use them to define bases of $T_p M$ for any $p \in M$. This shows that in this case we have a natural identification

$$TW = W \times \mathbb{R}^m \subset \mathbb{R}^{2m},$$

given by

$$(p, \sum v_i \partial_i(p)) \mapsto (x_1(p), \ldots, x_m(p), v_1, \ldots, v_m)$$

Next, with the help of local charts we want to glue this local description through the manifold. Let (U_p, φ) be a local chart, then $\varphi : U_p \to W \subset \mathbb{R}^m$ is a diffeomorphism and the differential

$$D\varphi_q : T_q M \to T_{\varphi(q)} W = \mathbb{R}^m$$

is a linear isomorphism for any $q \in U_p$ given by $D\varphi_q = (dx_1(q), \ldots, dx_m(q))$, where $\{d(x_i)(q)\}$ is the canonical dual basis in $(T_q M)^*$, see Remark 1.7.17. Therefore we have a bijection

$$D\varphi : TU_p \to TW = W \times \mathbb{R}^m$$

given by

$$D\varphi(x, v) = (\varphi(x), (dx_1(x)(v), \ldots, dx_m(x)(v)).$$

The next step is to define the right topology on TM in such a way that $D\varphi$ is an homeomorphism.

To do this consider a differentiable structure on M and for any local chart (U, φ) we define on TU the quotient topology, say $\mathcal{U}_\varphi$ induced by $D\varphi^{-1}$. Then $\mathcal{B} = \cup \mathcal{U}_\varphi$ is a basis for a topology and we denote by $\mathcal{U}$ the associated topology on TM. From now on we let $TM = (TM, \mathcal{U})$ be the topological tangent bundle of M. It is easy to see that TM is Hausdorff, has a countable base of open sets and using the morphisms $D\varphi$ we have that it is a $2m$-dimensional topological variety. To conclude we need to check condition DS2. Let (U, φ) and (V, ψ) be local charts of M and $W_1 = \varphi(U \cap V)$, $W_2 = \psi(U \cap V)$. Then we have the differentials

$$D\varphi : T(U \cap V) \to W_1 \times \mathbb{R}^m$$

and

$$D\psi : T(U \cap V) \to W_2 \times \mathbb{R}^m.$$

Note that by construction $D\psi \circ D\varphi^{-1} = D(\psi \circ \varphi^{-1})$ therefore the coordinate change is given by

$$(\psi \circ \varphi^{-1}, D(\psi \circ \varphi^{-1})) : W_1 \times \mathbb{R}^m \to W_2 \times \mathbb{R}^m, \tag{5}$$

where $D(\psi \circ \varphi^{-1})$ is the Jacobian of $\psi \circ \varphi^{-1}$. Therefore the coordinate change is smooth and we have proven condition DS2. Moreover the map π is locally at any point p the projection on the first m coordinates. We have proved the following.

Proposition 1.7.24. *Let M and N be manifolds, and $F : M \to N$ be a smooth morphism. Then TM and TN are manifolds of dimension $2 \dim M$ and $2 \dim N$, respectively, the map $\pi : TM \to M$ is smooth of constant rank m, and $DF : TM \to TN$, defined as*

$$DF(x, v) = DF_x(v),$$

is a smooth morphism.

Let $\sigma : M \to TM$ be a section of the natural projection $p : TM \to M$. That is $\pi \circ \sigma = id_M$. Then σ associates to any point in M a tangent vector (derivation). This reminds as the construction of vector fields.

Proposition 1.7.25. *There is a bijection between sections of the map $\pi : TM \to M$ and $\mathcal{X}(M)$.*

Proof. Let $\sigma : M \to TM$ be a section. Then σ is locally defined by a form

$$\sum a_i(x_1, \ldots, x_m)\partial_i,$$

with a_i differentiable functions. In particular it defines a derivation on $C^\infty(M)$. On the other hand let X be a derivation on $C^\infty(M)$. We already observed that for any $p \in M$ we have a well defined tangent vector $X(p)$. To conclude we have to show that these vectors glue together to give a section. This is clearly a local question.

Let (U_p, φ) be a chart with $p \in U_p$. In general $C^\infty(U_p)$ is not a subalgebra of $C^\infty(M)$ therefore we cannot simply consider X as a vector field on U_p. On the other hand we have a way to associate a vector field $X(U_p) \in \mathcal{X}(U_p)$ from X that locally behaves like X.

Indeed by Lemma 1.3.7 there is a smooth function, $h \in C^\infty(M)$ that is identically 1 on K_p and identically zero outside V_p where K_p is compact and $V_p \subset U_p$ is open. Then for any $f \in C^\infty(U_p)$ we may define

$$X(U_p)(f)(x) = \begin{cases} X(hf)(x) & x \in V_p \\ 0 & x \in U_p \setminus V_p \end{cases}$$

We know that if f is locally zero then $X(f)$ is locally zero. Thus, by Lemma 1.3.10, we may check the requirement for $X(U_p)$ and restrict to the chart U_p where a vector field is given by the equation we already exploited, see Remark 1.7.20. This shows that locally to any vector field we are able to associate a unique section and allows to conclude. □

EXAMPLE 1.7.26. Consider $M = \mathbb{R}^3 \setminus \{0\}$. Fix coordinates (x, y, z) on $\mathbb{R}^3$ and the canonical basis (∂_i) for each $T_pM \simeq \mathbb{R}^3$. Let $X : M \to TM$ be the vector field defined as

$$X((x, y, z)) = -G(x/r^3, y/r^3, z/r^3),$$

with $r = \sqrt{x^2 + y^2 + z^2}$. This is the gravitational field, that is the opposite of the gradient of gravitational potential (the minus sign is not always used), of an object of unit mass at $(0, 0, 0)$, where G is the Gravitational constant.

REMARK 1.7.27. Note that in general vector fields are not preserved by smooth morphisms. As a simple example let $X : \mathbb{R}^3 \to T\mathbb{R}^3 = \mathbb{R}^3 \times \mathbb{R}^3$ be given by $X(x, y, z) = ((x, y, z), (z, x, y))$. Consider the projection $\pi_z : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$\pi_z(x, y, z) = (x, y).$$

Then the $\pi_* X$ is not well defined on $\mathbb{R}^2$. It is easy to see that if $F : M \to N$ is a diffeomorphism then the vector fields on M and N are in bijection via $DF : TM \to TN$.

Let $N \subset M$ be a submanifold, then it is immediate that $TN \subset TM$ is a submanifold. In particular any vector field $X \in \mathcal{X}(M)$ such that $X(p) \in T_pN$ for any $p \in N$ defines a **restricted vector field** $X_N \in \mathcal{X}(N)$.

We are now in the condition to prove (for compact manifolds) a stronger version of Whitney Theorem. We already know that any compact manifold can be embedded in $\mathbb{R}^N$, cfr. Proposition 1.6.9. We aim to improve this result giving an upper bound to N.

THEOREM 1.7.28 (Whitney embedding Theorem). *Let $M \subset \mathbb{R}^s$ be a m-submanifold. Then there is an embedding $f : M \to \mathbb{R}^{2m+1}$.*

PROOF. If $s \leq 2m + 1$ there is nothing to prove. Assume that $s > 2m + 1$. To conclude it is enough to prove that there is an embedding $f : M \to \mathbb{R}^{s-1}$. The idea is to use a linear projection. Let $x \in \mathbb{R}^s$ be a point. Let $H \subset \mathbb{R}^s \setminus \{x\}$ be a hyperplane. Let $L = x + dirH$ and assume that $L \cap M = \emptyset$. The latter can always be achieved by applying for instance the diffeomorphism $F : \mathbb{R}^s \to \mathbb{R}^s$ that maps $x \mapsto \frac{x}{1+||x||}$. Then define the projection from x onto H

$$\pi_x : \mathbb{R}^s \setminus L \to H = \mathbb{R}^{s-1},$$

given by

$$\pi_x(y) = \langle x, y \rangle \cap H.$$

We want to show that there are points $x \in \mathbb{R}^s$ such that $\pi_{x|M}$ is an embedding. The map π_x is open and M is a submanifold therefore it is enough to prove that

a) $\pi_{x|M}$ is injective

b) $\pi_{x|M}$ has constant rank m.

Let us first be concerned with a). Let $y \in \mathbb{R}^{s-1}$ be a point. Then $\pi_x(y)^{-1}$ is a line through x. Therefore $\pi_{x|M}$ is injective if and only if there are no lines through x intersecting M in two distinct points. Let us consider the abstract $(2m+1)$-manifold $Y := ((M \times M) \setminus \Delta) \times \mathbb{R}$. Let $g : Y \to \mathbb{R}^s$ be defined as

$$g(p, q, t) = p + (q - p)t.$$

Then g is well defined and smooth.

Claim 1.7.29. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map, with $n \leq m$. Assume that A has measure 0 then $f(A)$ has measure 0.

Proof of the claim. If $n = m - a$ consider the map $\tilde{f} : \mathbb{R}^{n+a} \to \mathbb{R}^m$ given by $\tilde{f}(x_1, \ldots, x_n, y_1, \ldots, y_a) = f(x_1, \ldots, x_m)$, and the subset $\tilde{A} = A \times (0, \ldots, 0)$. Then $\tilde{f}(\tilde{A}) = f(A)$ and $\tilde{A}$ has measure zero. Hence it is enough to prove the statement for $n = m$. By hypothesis we may cover A with countably many closed balls. Hence we may assume that $A \subset \overline{B}$ for some closed ball. The function f is smooth. Then there is a constant $M > 0$ such that for all $x, y \in \overline{B}$ we have

$$||f(x) - f(y)|| \leq M||x - y||.$$

Hence if we cover A with countably many disks of volume bounded by ϵ the image $f(A)$ is covered by a countable collection of balls of total volume $M\epsilon$. $\square$

Let $\{U_i, \varphi_i\}$ be a DS of Y, then the claim applied to $g \circ \varphi_i^{-1}$ shows that $g(Y)$ is of measure zero (recall that we always have countably many local charts). In particular there is a point $x \in \mathbb{R}^s$ that such that $x \notin g(Y)$ and therefore the projection from x satisfies condition a).

We are left to control point b). Note that $ker(D\pi_x)_y = \mathrm{dir}\langle x, y\rangle$ for any y. Therefore $\pi_{x|M}$ has constant rank m as long as $\dim(T_yM \cap \langle x, y\rangle) = 0$ for any $y \in M$, or equivalently $y + T_yM \not\ni x$. Let us consider TM, the tangent bundle of M. We know it is a $2m$-manifold and let $h : TM \to \mathbb{R}^s$ be the map given by $h(y, v) = y + v$. Then condition b) is satisfied for any $z \in \mathbb{R}^s \setminus g(TM)$. We have, by Claim 1.7.29, that $h(TM)$ is of measure zero therefore we may find points $z \in \mathbb{R}^s \setminus \{g(Y) \cup h(TM)\}$ such that π_z satisfies both a) and b) and allow to conclude the proof. $\square$

Remark 1.7.30. The really amazing result is that one can drop the dimension by 1. That is any m-manifold admits an embedding in $\mathbb{R}^{2m}$. This is really hard and completely out of reach for us. The idea is to project into $\mathbb{R}^{2m}$ producing only isolated double points (keep in mind our construction) and then apply delicate arguments to remove self intersections, see [**4**]

Claim 1.7.29 is the baby version of Sard's theorem stating that the image of critical points of a smooth map has measure zero. Let me spend few words on it. Let $f : X \to Y$ be a smooth map. We say that $x \in X$ is a **critical point** if $rkDf_x < \dim Y$. Let $C \subset X$ be the set of critical points. Then Sard's Theorem guaranties that $f(C)$ is of measure 0. If you recall our construction of submanifolds induced by smooth maps, see Proposition 1.6.7, Sard's theorem says that, outside of a measure zero set, fibers of a smooth surjective map are submanifolds. That is any time we have positive dimensional fibers almost all of them are nice manifolds. On the other hand note that measure zero set may well be dense therefore maybe in any neighborhood you have some fiber which is not a manifold.

1.8. Exercises

Exercise 1.8.1. Show that constant functions, the identity morphism, and the inclusion $S^{n-1} \hookrightarrow \mathbb{R}^n$ are smooth morphisms.

Exercise 1.8.2. Show that $GL(n)$ is a manifold, ore generally prove that for any k the set of matrices of rank at least k is a manifold in $M(n)$.Show that multiplication, inversion and sum of matrices are smooth functions on $M(n)$.

Exercise 1.8.3. Show that a continuous map $F : M \to N$, between manifolds is smooth if and only if for any open $W \subset N$ and $f \in C^\infty(W)$ we have $f \circ F \in C^\infty(F^{-1}(W))$.

Exercise 1.8.4. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $t \mapsto t^3$. Show that it is an homeomorphism but not a diffeomorphism.

Exercise 1.8.5. Prove that when M is a compact manifold any morphism in a Hausdorff space is proper.

Exercise 1.8.6. Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Show that $DA_0 = A$.

Exercise 1.8.7. Prove Definition-Lemma 1.7.12.

Exercise 1.8.8 (Germ of a function). Let $p \in M$ be a point. A differentiable function in p is the data of a pair (f, U_p) with $p \in U_p$ and $f \in C^\infty(U_p)$. We will say that two smooth functions $(f, U_p) \sim (g, V_p)$ are equivalent if there exists an open neighborhood $W_p \subseteq U_p \cap V_p$ such that $f_{|W_p} = g_{W_p}$. Let

$$C^\infty(M)_p = \cup_{U \ni p} C^\infty(U)/\sim .$$

Show that:

- $C^\infty(M)_p$ has an algebra structure
- the restriction $C^\infty(M) \to C^\infty(M)_p$ is surjective
- the kernel of the restriction map is given by smooth functions that vanish in a neighborhood of p.

Exercise 1.8.9. (submanifolds) Show that S^n is a submanifold of $\mathbb{R}^{n+1}$.

Show that the map $F : \mathbb{R}^3 \to \mathbb{R}$ given by

$$F(x, y, z) = (a - \sqrt{x^2 + y^2})^2 + z^2$$

has constant rank 1 at any point of $F^{-1}(b^2)$, for $a > b > 0$. Conclude that the torus in $\mathbb{R}^3$ is a submanifold.

Let $C_t := (x + y = t, x^2 + y^2 = 1) \subset \mathbb{R}^3$ say for which values of t C_t is a submanifold.

Assume that $N \subset M$ is a submanifold and let $i : N \to M$ be the natural inclusion as sets. Show that i is differentiable.

Show that $X \subset N$ and $Y \subset M$ are submanifolds then $X \times Y \subset N \times M$ is a submanifold.

Let $F : \mathbb{R}^3 \to \mathbb{R}^4$ be given by $F(x, y, z) = (x^2 - y^2, xy, xz, yz)$. Show that F induces an embedding of $\mathbb{P}^2_{\mathbb{R}}$ into $\mathbb{R}^4$. In particular $\mathbb{P}^2_{\mathbb{R}}$ is a submanifold of $\mathbb{R}^4$.

Show that $SL_n(\mathbb{R}) \subset GL_n(\mathbb{R})$ is a submanifold.

Show that $O(n) \subset GL_n(\mathbb{R})$ is a submanifold, describe $T_{Id}O(n)$.

Exercise 1.8.10. Compute the tangent space at any point of: S^2, a cylinder, a plane, the torus add more.

EXERCISE 1.8.11. Let $p \in M$ be a point of a manifold. Let $\mathcal{C}_p$ be the set of curves through p, that is smooth maps $f : J \to M$ with $f(0) = p$. We say that $f, g \in \mathcal{C}_p$ are **tangent at** p if for some local chart (U_p, φ)

$$(\varphi \circ f)'(0) = (\varphi \circ g)'(0).$$

A **tangent vector** at p is an equivalence class of curves with respect to being tangent at p. Prove that this is well defined, equivalent to our previous definitions, and $T_pM = \mathcal{C}_p/\sim$

EXERCISE 1.8.12. (Zariski tangent space) Let $\mathbf{m}_p := \{f \in C^\infty(M) | f(p) = 0\}$. Observe the following:

- $\mathbf{m}_p$ is an ideal in $C^\infty(M)$,
- let $\mathbf{m}_p^2 \subset \mathbf{m}_p$ be the ideal given by functions that vanish of order at least 2 in p. For any $f \in \mathbf{m}_p^2$ and any derivation $X \in D(M)_p$, $X(f) = 0$,
- any derivation $X \in D(M)_p$ defines a linear functional $f_X : \frac{\mathbf{m}_p}{\mathbf{m}_p^2} \to \mathbb{R}$,
- $\frac{\mathbf{m}_p}{\mathbf{m}_p^2}$ is a vector space of dimension m.

Conclude that $D(M)_p$ is isomorphic to $(\frac{\mathbf{m}_p}{\mathbf{m}_p^2})^*$.

EXERCISE 1.8.13. Let $X, Y \in \mathcal{X}(M)$ be vector fields. Show that the Lie bracket $[X, Y] = XY - YX$ is always a vector field.

EXERCISE 1.8.14. Let M be a connected manifold and $F : M \to N$ a smooth morphism. Show that F is constant if and only if $DF \equiv 0$.

EXERCISE 1.8.15. Show that $TS^1 \simeq S^1 \times \mathbb{R}^1$ and $TS^3 \simeq S^3 \times \mathbb{R}^3$ (hint: note that on these spheres it is possible to define 1, respectively 3, vector fields that are linearly independent at any point). It is out of our reach to prove that S^7 has the same property and these are the unique spheres having trivial tangent bundle. This is related to existence of the octonions, like the two examples where related to the existence of the complex field and quaternions.

EXERCISE 1.8.16. Let $Y \subset X$ be a submanifold. Prove that $TY \subset TX$ is a submanifold.

EXERCISE 1.8.17. Let $X \subset \mathbb{R}^N$ be a submanifold. Prove that TX is a submanifold of $\mathbb{R}^{2N}$.

CHAPTER 2

Vector bundles

In this chapter we prove Frobenius Theorem about integrability of vector fields and use this construction to motivate and study vector bundles on manifolds.

2.1. One parameter vector fields

Let M be a manifold. We already encountered vector fields, they can be either seen as derivations on $C^\infty(M)$ or sections of the tangent bundle, recall Proposition 1.7.25. In this section we want to focus on a slightly different perspective. Let $C \subset M$ be a submanifold of dimension 1, that is a curve on M. Then through any point $p \in C$ we have a tangent space $T_pC \subset T_pM$ and also the tangent vector induced by a parametrization of C. Let now X be a vector field. Then at any point $p \in M$, $X(p) \in T_pM$ is a tangent vector. Restrict to a local chart (U_p, φ) then $X = \sum a_i(x_1, \dots, x_m)\partial_i$, and consider a differentiable function $f : J \to \varphi(U_p)$, with $f(t) = (x_1(t), \dots, x_m(t))$. When we think of f as a curve in U_p, the function $f'(t) := \frac{df}{dt}(t) = Df_t(1)$ describes its tangent vectors. It is natural to ask whether there is such a f with $f'(t) = X(f(t))$. In other words we are asking for the existence of a curve whose tangent vectors are described by the vector field. Such a curve is called an integral curve of the vector field X.

Definition 2.1.1. Let X be a vector field on M. A curve $f : J \to M$ is an **integral curve** of X if for any $t \in J$, $f'(t) = Df_t(1) = X_{f(t)}$.

By definition integral curves are solutions of the following equation

$$Df_t(1) = X(f(t)). \tag{6}$$

In a chart (U_p, φ), with $\varphi(p) = (0, \dots, 0)$ we have $\varphi \circ f(t) = (x_1(t), \dots, x_m(t))$, $D\varphi(X) = \sum a_i(x_1, \dots, x_m)\partial_i$, therefore Equation (6) translates in the following system of ordinary differential equations

$$\frac{dx_i(t)}{dt} = a_i(x_1, \dots, x_m),$$

together with initial condition $(\varphi \circ f)(0) = (0, \dots, 0)$. Therefore, thanks to Cauchy existence result, integral curves always exists locally.

Remark 2.1.2. Note that even if integral curves always exists they do not need to be submanifolds. For example, consider $M = S^1 \times S^1 \subset \mathbb{R}^2_{(x_1,x_2)} \times \mathbb{R}^2_{(y_1,y_2)}$. Fix any irrational number a, the integral manifold of the non-vanishing vector field $X_a = (x_2\partial_{x_1} - x_1\partial_{x_2}) + a(y_2\partial_{y_1} - y_1\partial_{y_2})$ is dense in M.

This allows to give a geometric point of view on differential equations. Moreover we may also consider vector fields depending on a parameter t.

DEFINITION 2.1.3. Let M be a m-manifold and $J \subset \mathbb{R}$ an interval, with $0 \in J$. A **one parameter vector field** is a map

$$X : M \times J \to TM,$$

such that for any $\bar{t} \in J$, the assignment $X(\bullet, \bar{t})$ is a vector field on M. A curve $f(t)$ is **integral** for X if

$$\frac{df(t)}{dt} = X(f(t), t),$$

for any $t \in J$. Let us indicate with $\mathcal{X}(M)_J$ the set of one parameter vector fields on M defined on $M \times J$.

EXAMPLE 2.1.4. This is an evolution of Example 1.7.26. Consider $M = \mathbb{R}^3$. Fix coordinates (x, y, z) on $\mathbb{R}^3$ and the canonical basis $(\partial_i(p))$ for each $T_pM \simeq \mathbb{R}^3$. Let $f : \mathbb{R} \to \mathbb{R}^3$, with $f(t) = (x(t), y(t), z(t))$ be a smooth function with $f(0) = (0, 0, 0)$. Let $X : M \setminus f(\mathbb{R}) \times \mathbb{R} \to T(M \setminus f(\mathbb{R}))$ be the vector field defined as

$$X((x, y, z), t) = -G((x - x(t))/r(t)^3, (y - y(t))/r(t)^3, (z - z(t))/r(t)^3),$$

with $r(t) = \sqrt{(x - x(t))^2 + (y - y(t))^2 + (z - z(t))^2}$. This is the gravitational field of an object of unit mass that moves along the curve $f(t)$. We had to exclude $f(\mathbb{R})$ to ensure that X is well defined on the manifold.

PROPOSITION 2.1.5. *Let $X \in \mathcal{X}(M)_J$ be a one parameter vector field. Then there is an open subset $W_{M\times\{0\}} \subset M \times J$ and a smooth function $G : W \to M$, such that*

- *for any $\overline{x} \in M$ the curve $f(t) := G(\overline{x}, t)$ is integral for X,*
- *$G(x, 0) = x$*

PROOF. Fix $p \in M$ and let (U_p, φ) be a local chart, after maybe shrinking U_p, we may assume, by Cauchy theorem, that the solution exists and is unique in $U_p \times (-\delta_p, \delta_p)$, for some $\delta_p > 0$. That is there exists a function $G_p : U_p \times (-\delta_p, \delta_p) \to M$ with the required properties. We have now to glue these local solutions. This can be done thanks to the uniqueness part of Cauchy theorem. To conclude observe that $\cup_{p\in M} U_p \times (-\delta_p, \delta_p)$ is an open neighborhood of $M \times \{0\}$. $\square$

DEFINITION 2.1.6. The function G produced in proposition 2.1.5 is called the **flow of the one parameter vector field**.

REMARK 2.1.7. Note that the flow is such that $G(\overline{x}, t)$ is an integral curve of the vector field for any t, and $G(x, 0) = x$. Hence we may rewrite the differential equation of the field via the following equations of the flow:

$$DG_{(x,0)}(\partial_i) = \partial_i, \quad DG_{(x,t)}(\frac{d}{dt}) = X(x, t) \tag{7}$$

If $W \supset M \times (-\delta, \delta)$ and $s \in (-\delta, \delta)$, then by uniqueness we have

$$G(p, s) = G(G(p, s), 0). \tag{8}$$

Let $\theta_s : M \to M$ be defined as

$$\theta_s(p) := G(p, s).$$

Then by Equations (7), (8) we have that θ_s is a local diffeomorphism. Moreover θ_s is a bijection by uniqueness of solutions. Therefore θ_s is a diffeomorphism and it is homotopy equivalent to the identity. In other words the flow describes the manifold M has a dynamical system whose points are moved according to the one parameter vector field. This opens our arguments to dynamics on the manifolds. We refer the interested reader to [**1**].

If M is compact we only need a finite number of local charts, then there is a positive δ such that the flow is defined on $M \times (-\delta, \delta)$, but something even better is at hand.

PROPOSITION 2.1.8. *Let M be a compact manifold and $X(p,t) \in \mathcal{X}(M)_{\mathbb{R}}$ a one parameter vector field then there is a flow $G : M \times \mathbb{R} \to M$. Moreover*

$$G(p, s_1 + s_2) = G(G(p, s_1), s_2),$$

therefore there is a morphism of groups

$$\mathbb{R} \to Diff(M, M),$$

all diffeomorphisms built in this way are homotopy to the identity. Finally this produces a map

$$\mathcal{X}(M)_{\mathbb{R}} \to \mathrm{Hom}(\mathbb{R}, Diff(M, M)).$$

PROOF. Let $f : (a, b) \to M$ be an integral curve and assume that $f(0) = p$. For the first statement it is enough to show that we may prolong f to an integral curve $\tilde{f} : (a - \delta, b + \delta) \to M$, such that $f(t) = \tilde{f}(t)$ for $t \in (a, b)$. M is compact therefore the flow $G : M \times (-\delta, \delta) \to M$ is well defined, for some $\delta > 0$. Choose $\bar{t} \in (a, a + \delta)$ and define the integral curve $G(f(\bar{t}), t)$ on $(t - \delta, t + \delta)$. By Equation (8) we have

$$G(p, t + \bar{t}) = G(f(\bar{t}), t),$$

hence by uniqueness of solution it has to agree with f and it prolongs it.

Since the flow $G : M \times \mathbb{R} \to M$ is well defined then, again by uniqueness of solutions we have that

$$G(p, s_1 + s_2) = G(G(p, s_1), s_2).$$

□

On non compatc manifold it is in general not true that a vector field $X \in \mathcal{X}(M)_{\mathbb{R}}$ defines a flow on $M \times \mathbb{R}$. This naturally leads to the following definition

DEFINITION 2.1.9. Let $X \in \mathcal{X}(M)_{\mathbb{R}}$ be a one paramenter vector field. Then X is called complete if there exists a flow $G : M \times \mathbb{R} \to M$ associated to X.

As observed before to a complete vector field is associated a one paramenter group of diffeomorphism homotopically equivalent to the identity.

REMARK 2.1.10. Note that the theory of one paramenter vector fields contains that of vector fields, simply defining $X(p, t) = X(p)$, for any $p \in M$.

2.2. Frobenius Theorem

It is quite natural in our set up to ask for integral submanifolds of higher dimension. That is we talked about integral curves associated to a vector field on a manifold M, but what happens if we choose two or more vector fields? Is it possible to "integrate" them? In other words is it possible to describe submanifolds $N \subset M$ such that at any point T_pN is spanned by the chosen vector fields?

Let us start with a simple example. Let $W \subset \mathbb{R}^3$ be open and consider a system of partial differential equations

$$\partial z/\partial x = g(x, y, z), \quad \partial z/\partial y = h(x, y, z).$$

Given $(a,b,c) \in W$, a solution, if any, will be a function $z = f(x,y)$ such that

$$c = f(a,b),\ f_x(x,y) = g(x,y,f(x,y)),\ f_y(x,y) = h(x,y,f(x,y)).$$

From a geometric point of view if we let let $F(x,y,z) = z - f(x,y)$ then $V := \{F = 0\}$ is a surface in $W \subset \mathbb{R}^3$. Recall that $(\partial_x, \partial_y, \partial_z)$ is a basis for derivations on W, hence, by Lemma 1.7.3, we have

$$T_{(x,y,z)}V = (\partial_x F, \partial_y F, \partial_z F)^\perp = (-f_x, -f_y, 1)^\perp = \langle \partial_x + g(x,y,z)\partial_z, \partial_y + h(x,y,z)\partial_z \rangle.$$

In other words if we consider the two vector fields X and Y, given by

$$X = \partial_x + f_x\partial_z,\ Y = \partial_y + f_y\partial_z$$

then V is an integral submanifold for $\{X,Y\}$. Note further that for this particular choice of vector fields, since $f_{xy} = f_{yx}$ we have

$$[X,Y] = XY - YX = 0$$

This shows that, in this set up, our initial question has a necessary condition, namely $[X,Y] = 0$, and it reflects the independence on the order of partial derivatives. It is therefore easy to guess that some condition on integrability are needed in this more general framework. It is time to introduce some definitions.

Definition 2.2.1. Let M be a manifold, a **distribution** D **of rank** k is the assignment of a subspace $D_p \subset T_pM$ such that:

a) $\dim D_p = k$ for any $p \in M$,
b) for any $p \in M$ there is a chart (U_p, φ) and k vector fields $\{X_1, \ldots, X_k\} \subset \mathcal{X}(U_p)$ such that for any $q \in U_p$ $D_p = \langle X_1(q), \ldots, X_k(q) \rangle$. Such a set $\{X_1, \ldots, X_k\}$ is called a **local basis** at q

We say that a vector field $Y \in \mathcal{X}(M)$ belongs to the distribution D,

$$Y \in D,$$

if for any $p \in M$ $Y(p) \in D_p$. That is to say that $Y = \sum c_i X_i$ with $c_i \in C^\infty(U_p)$ for any local basis. A distribution is said **involutive** if for any pair of vector fields $X, Y \in D$ we have $[X,Y] \in D$. A distribution is **integrable** at p if there exists an open $W_p \subset M$ and a submanifold $F \ni p$ such that for any $q \in W \cap F$

$$T_qF = D_q,$$

such a F is called a **leaf** of the distribution. Integrable distributions are also called **foliations**.

Example 2.2.2. Let $M = \mathbb{R}^{n+k}$ and $D_p := \{\partial_i(p)\}_{i=1,\ldots n} \subset T_pM$. Let D be the distribution defined by the D_p. Then D is clearly involutive and the leaves of D are the fibers of the projection onto the last k coordinates $\pi : M \to \mathbb{R}^k$. Despite this may seem a very special case we will prove that any foliation is locally of this type.

Remark 2.2.3. The notion of integrable distribution extends that of integral curve. Note that a distribution of rank 1 is a single vector field, hence is always involutive since $[X,X] = XX - XX = 0$. The result on integral curves in the preceding section can be rephrased saying that a rank 1 distribution is always involutive and a foliation.

We aim to study foliations. The first step is to prove that Example 2.2.2 locally describes any rank 1 foliation. The following is just a rephrasing of the existence of integral curves with a local change of variables.

LEMMA 2.2.4. *Let $D = \{X\}$ be a rank 1 distribution, i.e. a non vanishing vector field, on M. Let $p \in M$ be a point then there is a local chart (U_p, φ) such that for any $q \in \mathrm{U}_p$, $X(q) = \partial_1(q)$.*

PROOF. The statement is local therefore we may assume, after shrinking M, that $M \simeq B_\epsilon(0) \subset \mathbb{R}^m$, moreover we may assume that $X(0) = \partial_1(0)$. Let $X = \sum_1^m a_i \partial_i$, with a_i smooth functions, $a_1(q) \neq 0$, for all $q \in M$, and $a_i(0) = 0$ for $i \geq 2$. Consider the following system of ordinary differential equations

$$\frac{dx^i}{dx_1} = \frac{a_i(x_1, \ldots, x_m)}{a_1(x_1, \ldots, x_m)} \text{ for } i = 2, \ldots, m. \tag{9}$$

Then for any $(z_2, \ldots, z_m)$ the system has a unique solution

$$x^i = x^i(x_1, \ldots, x_m),$$

with initial data

$$x^i(0, z_2, \ldots, z_m) = z_i, \tag{10}$$

for $i = 2, \ldots, m$. Moreover the x^i are smooth functions in the variables $(x_1, z_2, \ldots, z_m)$. Consider the following sistem

$$x_1 = z_1,\ x_2 = x^2(z_1, \ldots, z_m), \ldots,\ x_m = x^m(z_1, \ldots, z_m).$$

By equations (9), (10) and construction the Jacobian $(\partial x^i / \partial z_j)$ evaluated in $z_1 = 0$ is the identity therefore by the inverse function theorem we may change coordinates from $(x_1, \ldots, x_m)$ to $(z_1, \ldots, z_m)$ in a neighborhood of the origin. In these coordinates, by Equation (9), we may rewrite

$$X = \sum a_i \frac{\partial}{\partial x_i} = \sum (a_1 \frac{\partial x_i}{\partial z_1}) \frac{\partial}{\partial x_i} = a_1 \frac{\partial}{\partial z_1}.$$

To conclude it is then enough to normalize the first coordinate with

$$x^1(z_1, \ldots, z_n) := \int_0^{z_1} \frac{dt}{a_1(t, z^2, \ldots, z^m)}.$$

$\square$

THEOREM 2.2.5 (Frobenius Theorem). *Let M be a m-manifold and D a distribution of rank k. Then D is integrable if and only if it is involutive.*

One direction of the Frobenius is clear. If D is integrable then the vector fields $X, Y \in D$ belong to $TF \subset TM$ therefore $[X, Y] \in TF = D$. To prove Frobenius Theorem we start with a local version of it.

PROPOSITION 2.2.6. *Let D be an involutive distribution of rank k on M. Let $p \in M$ be a point, then there is a local chart (U_p, φ) such that for all $q \in U_p$ we have*

$$D_q = \langle \partial_1(q), \ldots, \partial_k(q) \rangle.$$

PROOF. Let $\{X_1, \ldots, X_k\}$ be a local basis for the distribution, after eventually shrinking the open neighborhood of p. We prove the Proposition by induction on k. The first step is Lemma 2.2.4. Then we may assume the Proposition is true for distributions of rank $k-1$. By Lemma 2.2.4 we have $M \simeq B_\epsilon(0) \subset \mathbb{R}^m$ and $X_k = \partial_k$. Define, for $j \leq k-1$ the vector fields

$$Y_j = X_j - X_j(x_k)X_k,$$

then $Y_j(x_k) = 0$, for $j \leq k-1$ and $X_k(x_k) = 1$. Moreover by definition

$$D = \langle Y_1, \ldots, Y_{k-1}, X_k \rangle,$$

and evaluating the bracket on x_k we see that

$$0 = [Y_i, Y_j](x_k) = (\sum b_{ijh}Y_h)(x_k) + a_{ij}X_k(x_k) = a_{ij}$$

hence

$$D_Y = \langle Y_1, \ldots, Y_{k-1} \rangle$$

is involutive. By induction hypothesis we have a coordinate system, say $(y_1, \ldots, y_m)$ such that

$$\{\frac{\partial}{\partial y_i}\}_{i=1,\ldots,k-1} = D_Y.$$

Since $\frac{\partial}{\partial y_i}$, for $i = 1, \ldots, k-1$, is a linear combination of Y_j, for $i = 1, \ldots, k-1$ we still have

$$\frac{\partial}{\partial y_i}(x_k) = 0,$$

for $i = 1, \ldots, k-1$. Let

$$[\frac{\partial}{\partial y_i}, X_k](x_k) = (\sum_1^{k-1} c_{ikh}\frac{\partial}{\partial y_h} + c_iX_k)(x_k) = c_i, \tag{11}$$

on the other hand

$$[\frac{\partial}{\partial y_i}, X_k](x_k) = \frac{\partial}{\partial y_i}(X_k(x_k)) + X_k(\frac{\partial}{\partial y_i}(x_k)) = \frac{\partial}{\partial y_i}(1) + X_k(0) = 0$$

then, as before, we get $c_i = 0$. That is

$$[\frac{\partial}{\partial y_i}, X_k] = \sum_1^{k-1} c_{ikh}\frac{\partial}{\partial y_h}. \tag{12}$$

Since since $(\frac{\partial}{\partial y_j})$ is a local basis we have $X_k = \sum_1^n b_j\frac{\partial}{\partial y_j}$ and

$$[\frac{\partial}{\partial y_i}, X_k] = [\frac{\partial}{\partial y_i}, \sum_1^n b_j\frac{\partial}{\partial y_j}] = \sum_1^n \frac{\partial b_j}{\partial y_i}\frac{\partial}{\partial y_j},$$

then plugging in Equation (12) we get

$$\frac{\partial b_j}{\partial y_i} = 0,$$

for $i \leq k-1$ and $k \leq j \leq n$. That is $b_j = b_j(y_k, \ldots, y_m)$ for $j \geq k$. Let $Y_k = \sum_{j=k}^m b_j\frac{\partial}{\partial y_j}$, then

$$D = \{Y_1, \ldots, Y_k\}.$$

Moreover by Lemma 2.2.4 there is a coordinate change, $(y_1, \ldots, y_m)$ to $(z_1, \ldots, z_m)$ such that

$$y_i = z_i, \text{ for } i = 1, \ldots, k-1,$$

and

$$Y_k = \frac{\partial}{\partial z_k}.$$

Hence in this coordinate system $Y_i = \frac{\partial}{\partial z_i}$, For $i = 1, \ldots k$. $\square$

We are now in the condition to conclude Frobenius Theorem.

Proof of Frobenius Theorem. We need to produce the leaves of a rank k distribution D. Fix $p \in M$ a point. Then by Proposition 2.2.6 there is a local chart (U_p, φ) such that $D = \langle \partial_i \rangle$, for $i \leq k$. Let $\pi : \mathbb{R}^m \to \mathbb{R}^{m-k}$ the projection onto the last $m-k$ coordinates then $\pi \circ \varphi$ is a smooth function of constant rank k and

$$\ker(D(\pi \circ \varphi)_q) = D_q,$$

for any $q \in U_p$. Therefore for any $(z_{k+1}, \ldots, z_m) \subset \operatorname{im}(\pi \circ \varphi)$, the subset $(\pi \circ \varphi)^{-1}(z_{k+1}, \ldots, z_m)$ is a k-manifold and it is the required leaf. $\square$

Remark 2.2.7. With some more effort, but no new ideas, one can prove that the leaf passing through a point p is

$$F_p := \{q \in M | \text{there exists a piece-wise smooth integral curve of } D \text{ joining } p \text{ and } q\}.$$

We may use Frobenius Theorem to produce a new point of view on coordinates.

Corollary 2.2.8. *Let M be a manifold assume that $\{X_1, \ldots, X_m\}$ are vector fields such that $[X_i, X_j] = 0$ for any pair i, j and $\{X_1(p), \ldots, X_m(p)\}$ is a local basis for T_pM. Then the X_i define local coordinates in a neighborhood of p.*

Proof. By hypothesis $\{X_1, \ldots, X_m\}$ is a distribution of rank m in a neighborhood of p and by Frobenius it is integrable. Moreover, following the proof of Proposition 2.2.6, this yields a coordinate change such that $X_i = \frac{\partial}{\partial z_i}$, for $i = 1, \ldots, m$. $\square$

The above Corollary shifts the attention from coordinates to vector fields. This is sometimes useful when treating special structures, coming from theoretical descriptions, where it is difficult or even not possible to introduce explicit local coordinates.

2.3. Vector bundles

We already realized how useful could be the Tangent bundle of a manifold. Let M be a m-manifold and $\{U_i, \varphi_i\}$ a DS, then

$$TM_{|U_i} \simeq \mathbb{R}^m \times U_i.$$

In particular locally any manifold of dimension m has isomorphic tangent bundle and the geometry of M encoded in TM only depends on the way we glue together these pieces.

This suggests the possibility to define in an abstract way some gluing condition and attach to a manifold M various type of objects like TM. Before plunging in the abstract description let us work out a special example.

2.3.1. Cotangent bundle. Let M be a manifold of dimension m and $f \in C^\infty(M)$ a smooth function. Then we have $f : M \to \mathbb{R}$ and $Df : TM \to \mathbb{R}$. In particular for any $p \in M$ let

$$df(p) := Df_p : T_pM \to \mathbb{R}.$$

Then $df(p)$ is a linear map, that is a linear functional on T_pM. Therefore we may consider

$$df(p) \in T_pM^*.$$

As we did for the tangent bundle we define the set

$$TM^* = \cup_{p\in M} T_pM^*,$$

there is a natural projection $\pi : TM^* \to M$, and df is just a section of π. As we did for TM let us work out a DS to produce a manifold.

Let $\{U_i, \varphi_i\}$ be a DS on M, with local coordinates $(x_1(p), \dots, x_m(p))$. Then, keep in mind Remark 1.7.17, define

$$(dx_1(p), \dots, dx_m(p))$$

the dual basis of T_pM^*. It is worthwhile to spend a couple of lines on this dual basis.

Remark 2.3.1. We know that $TU_i = U_i \times \mathbb{R}^m$ and $\{\partial_1, \dots, \partial_m\}$ are vector fields such that for any $p \in U_i$, the set $\{\partial_1(p), \dots, \partial_m(p)\}$ is a basis of T_pM. Therefore we may define

$$dx_i : U_i \to U_i \times (\mathbb{R}^m)^*$$

as

$$dx_i(p)(\partial_j(p)) = \delta_{ij}.$$

The dx_i are sections of the map

$$\pi : \cup_{p\in U_i} T_pM^* \to U_i$$

and $\{dx_1(p), \dots, dx_m(p)\}$ is a basis of T_pM^* for any $p \in U_i$.

Since U_i is a local chart it is easy to see that dx_i are smooth morphisms with the usual DS of the product. This also offers a closer look on the differential of a function $f \in C^\infty(U_i)$. For any vector $X(p) \in T_pM$ the element $dx_i(p)$ assigns a number $dx_i(p)(X(p))$ that is the i^{th} component of $X(p)$ in the base $\{\partial_i(p)\}$. For $f \in C^\infty(U_i)$ we have by definition

$$\frac{\partial f}{\partial x_i}(p) = Df_p(\partial_i(p)).$$

Hence we may rewrite df in the local base $\{dx_i\}$ as

$$df = \sum_i \frac{\partial f}{\partial x_i} dx_i.$$

Since $df(p)$ is a linear form on T_pM we may apply it to a vector field $X \in \mathcal{X}(M)$. We defined $df(p) = Df_p$ hence we have

$$df(p)(X_p) = X_p(f),$$

for $X_p \in D(M)_p$ a derivation. This shows that we may apply df to a vector field $X \in \mathcal{X}(U_i)$ to get a an element in $C^\infty(U)$

$$df(X)(p) := X(p)(f).$$

Thus we may see df as a linear approximation of f in the direction of $X(p)$ In particular in the local expression we found we have

$$df(X)(p) = \sum_i \frac{\partial f}{\partial x_i}(p) dx_i(p)(X(p)).$$

Let us go back to the DS. Note that for any smooth function $F : M \to N$ we have the differential $DF : TM \to TN$. Since $DF_p : T_pM \to T_{F(p)}N$ is a linear map we have the transposed linear map $DF_p^* : T_{F(p)}N^* \to T_pM^*$ where

$$DF_p^*(h)(v) = h(DF_p(v)),$$

and, with the choice of canonical dual basis, DF_p^* is given by the transpose matrix of DF_p. Hence fix a local chart $\varphi_i : U_i \to V$ and dual basis $\{dx_i\}$. Then we define a local chart on TU_i^* by

$$(\varphi_i, (D(\varphi_i^{-1})^t)) : TU_i^* \to U_i \times \mathbb{R}^m.$$

In particular, recalling Equation (5) at page 20, the change of coordinates is given by

$$(\varphi_i \circ \varphi_j^{-1}, (D(\varphi_i^{-1} \circ \varphi_j)^t)).$$

We proved the following

Proposition 2.3.2. *Let M be a m-manifold. Then the* **cotangent bundle** *TM^* is a $2m$-manifold and $\pi : TM^* \to M$ is a smooth map.*

As in the Tangent bundle case, sections of cotangent bundle have a geometric meaning.

Definition 2.3.3. A section of $\pi : TM^* \to M$ is called a **differential 1-form**. The space of differential 1-forms is called $\Omega^1(M)$.

Remark 2.3.4. One forms are given, locally, by

$$\sum a_j(x_1, \ldots, x_m) dx_j,$$

for $a_j \in C^\infty(U_i)$. In particular for any $f \in C^\infty(M)$ we may write

$$df = \sum \frac{\partial f}{\partial x_i} dx_i \in \Omega^1(M)$$

this defines the (external) differentiation

$$d : C^\infty(M) \to \Omega^1(M).$$

The image of this map is the set of **exact forms**.

Differential 1-forms, and their friends k-forms obtaining wedging the former, are related to integration on manifolds and Riemannian geometry, see [**2**] for an excellent introduction.

An important, and quite surprising, difference between vector fields and 1-forms is the behaviour with respect to morphisms. We observed that in general it is not possible to define a vector field through a morphism, recall Remark 1.7.27. On the other hand let $F : M \to N$ be a morphism and $\alpha \in \Omega^1(N)$ a 1-form. Then we may define

$$F^*\alpha(v) = \alpha(DF(v)),$$

for $v \in T_pM$, and it is a straightforward check, left to the reader, that $F^*\alpha$ is a 1-form. This produces the pull-back map for 1-forms

$$F^* : \Omega^1(N) \to \Omega^1(M).$$

Further note that this operation commutes with differentiation of functions, that is $d(F^*(f)) = F^*(df)$, where $F^*(f) := f \circ F$.

Via the pull-back it is possible to produce 1-forms on submanifolds of a manifold M. Indeed let $N \subset M$ be a submanifold and $\alpha \in \Omega^1(M)$. Then $i^*\alpha \in \Omega^1(N)$, where $i : N \to M$ is an embedding. In particular via the 1-forms of $\mathbb{R}^N$ we produce 1-forms on a submanifold $N \subset \mathbb{R}^N$. In general the behaviour of $i^*\alpha$ form may be different from that of α.

Example 2.3.5. Let $M \subset \mathbb{R}^N$ be a submanifold, and $\alpha = dx_1 \in \Omega^1(\mathbb{R}^N)$. Then α is never zero, that is $\alpha(p)$ is not the zero form for any $p \in \mathbb{R}^N$. On the other hand if $q \in M$ is such that $T_qM \subset (1, 0, \dots, 0)^\perp$, then $i^*\alpha(p)$ is zero. This suggests that pull-back form may be used to study the geometry of submanifolds.

2.3.2. Vector bundles. It is time to provide an abstract description, and hence a generalization, of the bundles we introduced so far.

Definition 2.3.6. Let M and F be manifolds. A (smooth) **fibration** on M with fiber F is

a) a manifold E
b) a morphism $\pi : E \to M$
c) an open covering $\{U_i\}$
d) diffeomorphisms $f_i : \pi^{-1}U_i \to U_i \times F$ such that the following diagrams commute

$$\begin{tikzcd} \pi^{-1}U_i \arrow[r, "f_i"] \arrow[d, "\pi"'] & U_i \times F \arrow[dl] \\ U_i & \end{tikzcd} .$$

The diffeomorphisms f_i are called **trivializations**. We may, and will, assume that $\{U_i, \varphi_i\}$ are a DS on M.

Note that the diffeomorphism f_i forces $\pi^{-1}(x) \simeq F$ for any $x \in M$. Moreover we have the **transition function** $f_{ij} = f_i \circ f_j^{-1}$ that are diffeomorphisms on $U_{ij} \times F$, where $U_{ij} = U_i \cap U_j$. In particular for any $x \in U_{ij}$ the map $f_{ij|\{x\}\times F}$ is a diffeomorphism of F. The commutation in d) forces also the following **cocycle conditions**

$$f_{ij} = f_{ji}^{-1}, \quad f_{ij} \circ f_{jk} = f_{ik}.$$

A **section** of a fibration $\pi : E \to M$ is a smooth map $s : M \to E$ such that $\pi \circ s = id_M$.

We will not develop the theory of fibrations in full generality, for this the interested reader may refer to [**3**]. We restrict to vector bundles where both the fibers and the diffeomorphisms are particularly simple.

Definition 2.3.7. A rank k (real) **vector bundle** is a fibration $\pi : E \to M$ with $F \simeq \mathbb{R}^k$ and diffeomorphism

$$f_{ij|\{x\}\times\mathbb{R}^k} \in GL(k, \mathbb{R}), \text{ for any } x \in U_{ij}.$$

Remark 2.3.8. The manifold $M \times \mathbb{R}^k$ is naturally a vector bundle, called the trivial vector bundle. We may use the DS $\{(U_i, \varphi_i)\}$ to define a DS on $M \times \mathbb{R}^k$ via

$$(\varphi_i, id_{\mathbb{R}^k}) \circ f_i : \pi^{-1}(U_i) \to \mathbb{R}^m \times \mathbb{R}^k.$$

TM and TM^* are m-vector bundles with trivialization given, respectively, by $(\varphi_i, D\varphi_i)$ and $(\varphi_i, (D\varphi_i^{-1})^t)$.

Examples of fibrations, different from vector bundles, are:

- any diffeomorphism is a fibration with fiber a connected 0-manifold,
- the antipodal map $a : S^n \to \mathbb{P}^n_{\mathbb{R}}$ as a fibration with $F = \{p, -p\}$,
- the Hopf fibration

$$h : S^3 \to S^2 \quad (a, b, c, d) \mapsto (a^2 + b^2 - c^2 - d^2, 2(ad + bc), 2(bd - ac)).$$

an S^1 fibration over S^2. A way to see this is to consider it on complex numbers, there it can be defined as $h(z_0, z_1) = (|z_0|^2 - |z_1|^2, 2z_0\overline{z}_1)$, where we realize $S^3 := \{(z_0, z_1) \in \mathbb{C}^2 ||z_0|^2 + |z_1|^2 = 1\}$ and $S^2 := \{(x, w) \in \mathbb{R} \times \mathbb{C} ||w|^2 + x^2 = 1\}$. Then it is not difficult to see that $h(z_0, z_1) = h(z_2, z_3)$ if and only if there is a $\lambda \in \mathbb{C}$ with $|\lambda|^2 = 1$ such that $z_2 = \lambda z_0$ and $z_3 = \lambda z_1$. This shows that the fibers are S^1. With more effort one can prove that it is a fibration.

We may see Hopf fibration in the light of projective spaces as follows. Let $\pi : \mathbb{C}^2 \setminus 0 \to \mathbb{P}(\mathbb{C}^2) = \mathbb{P}^1_{\mathbb{C}}$ be the quotient map induced by the equivalence relation $z \sim w$ if and only if there is a $\lambda \in \mathbb{C}^*$ such that $z = \lambda w$. Recall that $\mathbb{P}^1_{\mathbb{C}} \cong S^2$ and let $X = \{(z_0, z_1) \in \mathbb{C}^2 ||z_0|^2 + |z_1^2| = 1\} \subset \mathbb{C}^2 \setminus \{0\}$, then $X \cong S^3$ and $\pi_{|X} : S^3 \to S^2$ is the Hopf map.

Note that the geometric information carried by a vector bundle are all encoded in its transition functions.

There are some operations we may perform on vector bundles.

2.3.2.1. *Restriction.* Let $\pi : E \to M$ be a k-vector bundle and $N \subset M$ a submanifold. Then $\pi_{|\pi^{-1}(N)} : \pi^{-1}(N) \to N$ is a k-vector bundle on N and the trivialization functions are exactly the same fiberwise.

2.3.2.2. *Product.* Let $E \to M$ and $G \to N$ be rank a and b vector bundles. Then $E \times G \to M \times N$ is naturally a $(a + b)$-vector bundle. When $M = N$ we may go a bit further. Let $\Delta \subset M \times M$ be the diagonal. Then it is easy to see that $\Delta \simeq M$, by projection on one of the factors, therefore we may define, by restriction, $E \times G$ as a rank $(a+b)$ vector bundle on M, this is usually called either the product vector bundle or the direct sum.

2.3.2.3. *Dual, tensor, wedge, sym.* All standard operations on vector spaces can be carried out on vector bundles. We already encountered the dual during the construction of the cotangent bundle. In a similar fashion we may define $E \otimes G$, $\bigwedge^r G$ and $Sym^s(E)$ using as transition functions the corresponding matrices.

2.3.2.4. *Morphisms.* Let $\pi_E : E \to M$ and $\pi_G : G \to N$ be two rank a and b vector bundles. Let $h : E \to G$, be a smooth map such that it induces a smooth function $\tilde{h} : M \to N$, that is $\tilde{h} \circ \pi_E = \pi_G \circ h$. Then $h_x := h_{|F_x} : F_x \simeq \mathbb{R}^a \to F_{\tilde{h}(x)} \simeq \mathbb{R}^b$ is a map for any $x \in M$.

Definition 2.3.9. We say that h is a **vector bundle morphism** if h_x is a linear map for any $x \in M$ and we will say it is ***-jective** if h_x is *-jective. The map h is a vector bundle isomorphism if $\tilde{h}$ is a diffeomorphism and h_x is a linear isomorphism, for any $x \in M$.

REMARK 2.3.10. The differential of a smooth function $Df : TM \to TN$ is a vector bundle morphism. Given a manifold M and two vector bundles $E \to M$ and $G \to M$ of rank a and b. A vector bundle morphism that commutes with id_M is simply given by a smooth function $\psi : M \to M_{a,b}(\mathbb{R})$.

Note that for any vector bundle morphism $h(F_x) \subset F_{\tilde{h}(x)}$ is a vector subspace.

The sum and scalar multiplication on a vector bundle E are vector bundle morphisms.

DEFINITION 2.3.11. Let $h : E/M \to G/M$ be an injective vector bundle morphism inducing the identity, then $h(E) \subset G$ may be seen in a natural way as a subvector bundle A **vector subbundle** is the image of an injective vector bundle morphism, that induces the identity on the base.

Let $E \subset G$ be a vector subbundle of rank $a \leq b$. Then it is natural to consider its quotient Q. Fiberwise the associated vector space is just $Q_x = G_x/E_x$. To define it globally observe that $E \subset G$ is given by a smooth function $q : M \to M_{a,b}(\mathbb{R})$ and for any $x \in M$ the matrix $q(x)$ has a independent columns. Since we may work locally we assume that the first a columns are independent on $W \subset M$ and therefore we may identify Q_x with $\{x_1 = \ldots = x_a = 0\} \subset \mathbb{R}^b$, define locally $Q = \cup Q_x$ together with a map

$$W \times \mathbb{R}^{b-a} \to Q \quad (p, (x_1, \ldots, x_{b-a})) \mapsto (p, (0, \ldots, 0, x_1, \ldots, x_{b-a})).$$

This defines the **quotient bundle**. There is a quotient bundle that is particularly interesting for us. Let $X \subset M$ be a submanifold. Then we have the inclusion embedding $i : X \to M$ that gives as a bundle morphism $Di : TX \to TM$, it is easy to check that it is an injective morphism and moreover if we take the restriction $Di(TX)_{|X}$ we may look at it as a subbundle of $TM_{|X}$. Therefore we have a well defined quotient

$$NX := TM_{|X}/Di(TX),$$

the **normal bundle** of X in M. Note that NX is a vector bundle of rank $m - \dim X$.

REMARK 2.3.12. We can now reinterpret the notion of distribution. A rank k distribution D on a manifold M is a vector subbundle $E \subset TM$ of rank k. The integrability condition is just to say that for any point $p \in M$ there is k-submanifold $N_p \subset M$ such that $D_{|N_p} = TN$.

2.4. Exercises

EXERCISE 2.4.1. Let $X_1 = y^2\partial_x$ and $X_2 = x^2\partial_y$ be two vector field on $\mathbb{R}^2$. Prove that X_1 and X_2 are complete but $X_1 + X_2$ is not complete.

EXERCISE 2.4.2. Let $\{X_1, \ldots, X_s\}$ be a local basis for a distribution D. Prove that D is involutive if $[X_i, X_j] \in D$.

EXERCISE 2.4.3. Compute the Lie bracket $[\cdot, \cdot]$ for the following vector fields on $\mathbb{R}^3$: $y\partial_x - x\partial_y$, $z\partial_y - y\partial_z$, $\partial_x + \partial_y + \partial_z$.

EXERCISE 2.4.4. Determine which of the following local bases produce an integrable distribution on an open subset of $\mathbb{R}^3$ and when it is integrable determione the leaves:

- $\{\partial_x + \partial_y, \partial_z\}$
- $\{5\partial_x, 7\partial_z\}$

- $\{\partial_x - \partial_y, \partial_z - \partial_x, \partial_y - \partial_x\}$
- $\{\partial_x + y\partial_z, \partial_y\}$
- $\{y\partial_x, x\partial_y\}$

EXERCISE 2.4.5. Let $\mathcal{D}$ be the distribution on $\mathbb{R}^3$ associated to the local basis $\{x_1\partial_2 - x_2\partial_1, \partial_3\}$. Prove that it is integrable and find the leaf of the foliation.

EXERCISE 2.4.6. Let $F : M \to N$ be a surjective map of constant rank. Show that for any $p \in N$ the sets $F^{-1}(p)$ are the leaves of a foliation.

EXERCISE 2.4.7. Show that the leaves of a foliation are submanifolds.

EXERCISE 2.4.8. Let $F : \mathbb{R}^3 \to \mathbb{R}$ be the map $F(x, y, z) = x^2 - y^2 + z^2$ show that it is of constant rank 1 on $\mathbb{R}^3 \setminus \{0\}$ and determine the associated foliation.

EXERCISE 2.4.9. Show that on $\mathbb{R}$ any 1-form is exact. Produce a non exact one form on S^1.

EXERCISE 2.4.10. Let $\pi : E \to M$ be a rank k-vector bundle. Assume that there are k sections $\{s_1, \dots s_k\}$ such that $\{s_1(x), \dots, s_k(x)\}$ are linearly independent for any $x \in M$. Prove that $E = M \times \mathbb{R}^k$.

EXERCISE 2.4.11. Let $\pi : E \to M$ be a rank a vector bundle. Let $f : N \to M$ be a smooth map. Let

$$f^*E := \{(x, v) \in N \times E | f(x) = \pi(v)\} \subset N \times E$$

be the **pull-back** vector bundle. Prove that the projection on the first factor has a natural structure of vector bundle and the projection on the second factor produces a commutative diagram

$$\begin{tikzcd} f^*E \arrow[r, "F"] \arrow[d, "\pi_1"'] & E \arrow[d, "\pi"] \\ N \arrow[r, "f"] & M \end{tikzcd} .$$

Show that if E is trivial then f^*E is trivial.

EXERCISE 2.4.12. Let $G := \{([x], v) \in \mathbb{P}^1_{\mathbb{R}} \times \mathbb{R}^2 | v \in x\}$, prove that it is a manifold. Prove that the canonical projection on the first factor is a vector bundle. Prove that it is not the trivial vector bundle. (same for $\mathbb{P}^n_{\mathbb{R}}$)

EXERCISE 2.4.13. Determine the normal bundle of a plane $P \subset \mathbb{R}^3$ and of $S^2 \subset \mathbb{R}^3$. Observe that $N_{S^2/\mathbb{R}^3}$ is the trivial bundle (this is the condition of orientability) recall that TS^2 is not trivial, how could it be ?

CHAPTER 3

Gauss map

In this chapter we will study the Gauss map of surfaces in $\mathbb{R}^3$.

3.1. Surfaces in $\mathbb{R}^3$

Let $S \subset \mathbb{R}^3$ be a submanifold of dimension 2. Let $\{U_i, \varphi_i\}$ be a DS on S. For any $p \in U_i$ we have a well defined tangent space T_pS and its orthogonal complement $T_pS^\perp$. The map φ_i^{-1} induces local coordinates (u, v) and a base $(X_u(p), X_v(p))$ for any T_pS. Where X_u, X_v are the vector fields associated to ∂_u and ∂_v respectively, recall Lemma 1.7.3. To uniformize our notation with those classically used for surfaces in $\mathbb{R}^3$ we define $\mathbf{x}(u, v) := \varphi_i^{-1}$, $\mathbf{x}_u := X_u$, $\mathbf{x}_v := X_v$, hence

$$T_{(u,v)}S = \langle \mathbf{x}_u, \mathbf{x}_v \rangle.$$

Define

$$N(q) = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{||\mathbf{x}_u \wedge \mathbf{x}_v||}$$

the **normal versor**. This produces a smooth morphism **the Gauss map**

$$N : U_i \to S^2.$$

Remark 3.1.1. Recall that T_pS is independent on the local chart chosen. On the other hand the choice of N is not canonical. We could have chosen

$$\frac{\mathbf{x}_v \wedge \mathbf{x}_u}{||\mathbf{x}_u \wedge \mathbf{x}_v||}$$

instead. That is the normal versor is defined only up to a sign. This is related to the problem of orientability of S. One says that a surface is **orientable** if the Gauss map is defined on all of S. That is it is possible to glue the Gauss maps on local charts to produce a well defined map $N : S \to S^2$. Think at the Möbius band, maybe some Escher picture, to understand the geometric meaning of this notion. Altrenatively this is equivalent to have a trivial Normal bundle.

From now on to simplify the treatment we will assume that $S \subset \mathbb{R}^3$ is an orientable surface, that is there is a well defined Gauss map $N : S \to S^2$. The local description shows that N is differentiable and the differential is $DN : TS \to TS^2$, with

$$DN_p : T_pS \to T_{N(p)}S^2.$$

Note that for any point of S^2 the tangent space is $T_pS^2 = \langle p \rangle^\perp$, recall exercise 1.8.10. Therefore we may interpret DN_p as a linear self map on T_pS. As such we may consider it is a way to measure the way the tangent space is varying in a neighborhood of a point.

For curves the variation of tangent direction is measured by the curvature, a number or a 1x1 matrix. Here we have a two dimensional vector space to control therefore we need a 2x2 matrix. This is what DN_p is devoted to.

From now on we will always consider

$$DN_p : T_pS \to T_pS$$

as a linear endomorphism of T_pS. Our first computation is the following.

LEMMA 3.1.2. *Let $DN_p : T_pS \to T_pS$, then $DN_p(\mathbf{x}_u) = N_u(p)$ and $DN_p(\mathbf{x}_v) = N_v(p)$.*

PROOF. Let $\alpha(t)$ be an integral curve of $\mathbf{x}_u$ with $\alpha(0) = p$. Then

$$DN_p(\mathbf{x}_u) = DN_p(\alpha'(0)) = \frac{d}{dt}N(\alpha(t))_{|t=0} = N_u(p),$$

and similarly for $\mathbf{x}_v$. □

Since a linear map is determined by the image of a basis Lemma 3.1.2 is the local way to determine the differential of the Gauss map. Let us start computing it in some special cases

3.1.0.1. *DN_p of a sphere.* Let $S \subset \mathbb{R}^3$ be a sphere centered at the origin of radius r. We already know that for any point $p \in S$ $T_pS = \langle p \rangle^\perp$ that is

$$N(p) = \frac{p}{||p||} = \frac{1}{r}p,$$

and

$$DN_p = \frac{1}{r}Id.$$

3.1.0.2. *DN_p of a cylinder.* Let $S = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 = r^2\}$, then we may parametrize it via $\varphi^{-1} : \mathbb{R}^2 \to S$ with $\varphi^{-1}(u, v) = (r \cos u, r \sin u, v)$

$$N(x, y, z) = \frac{(x, y, 0)}{r},$$

and

$$DN_p = \begin{bmatrix} \frac{1}{r} & 0 \\ 0 & 0 \end{bmatrix}.$$

See more examples in the exercises at the end of the chapter.

DEFINITION 3.1.3. A linear endomorphism $f : \mathbb{R}^s \to \mathbb{R}^s$ is called **self-adjoint** if for any pair of vectors $v, w \in \mathbb{R}^s$

$$\langle f(v) \cdot w \rangle = \langle v \cdot f(w) \rangle,$$

where $\langle \cdot \rangle$ is the euclidean scalar product.

LEMMA 3.1.4. *DN_p is self-adjoint*

PROOF. To prove the claim it is enough to prove that

$$\langle DN_p(\mathbf{x}_u) \cdot \mathbf{x}_v \rangle = \langle \mathbf{x}_u \cdot DN_p(\mathbf{x}_v) \rangle.$$

Since N is orthogonal to $\mathbf{x}_v$ and $\mathbf{x}_u$ we have

$$0 = \partial_u \langle N \cdot \mathbf{x}_v \rangle = \langle N_u \cdot \mathbf{x}_v \rangle + \langle N \cdot \mathbf{x}_{uv} \rangle$$

and

$$0 = \partial_v \langle N \cdot \mathbf{x}_u \rangle = \langle N_v \cdot \mathbf{x}_u \rangle + \langle N \cdot \mathbf{x}_{vu} \rangle.$$

Therefore we conclude by Lemma 3.1.2 and equality of mixed partials. □

REMARK 3.1.5. Let us recall some important facts of self adjoint operators from linear algebra. Let $A : \mathbb{R}^s \to \mathbb{R}^s$ be a self-adjoint linear operator

a) on any hortonormal base A is represented by a symmetric matrix
b) if a versor v maximize the quantity $\langle A(v)\cdot v\rangle$ then v is an eigenvector of A
c) A can be diagonalized by a hortonormal basis.

Let us see what this means for DN_p. First consider the bilinear symmetric form

$$B : T_pS \times T_pS \to \mathbb{R}$$

given by

$$B(v, w) = \langle v \cdot DN_p(w)\rangle,$$

and the associated quadratic form

$$Q(w) = B(w, w).$$

DEFINITION 3.1.6. In the above notation the **second fundamental form** of the surface S at the point p is

$$II_p(v) = -Q(v) = -\langle v \cdot DN_p(v)\rangle,$$

for $v \in T_pS$.

This lemma explains the minus sign and sheds some light on the matter, I hope.

LEMMA 3.1.7. *Let $\alpha(t)$ be a regular curve with $\alpha(0) = p$ and $\alpha'(0) = v$. Then*

$$II_p(v) = k_\alpha(0)\langle n_\alpha(0) \cdot N(p)\rangle,$$

where k_α and n_α are, respectively, curvature and normal versor of α.

PROOF. We may assume that α is parametrized by arc length. Since

$$\langle \alpha'(t) \cdot N(\alpha(t))\rangle = 0,$$

derivation with respect to t yields

$$0 = \langle \alpha(t)'' \cdot N(\alpha(t)\rangle + \langle \alpha'(t) \cdot DN_{\alpha(t)}(\alpha'(t))\rangle,$$

to conclude apply Frenet formulas and our definition

$$k_\alpha(0)\langle n_\alpha(0) \cdot N(p)\rangle - II_p(v) = 0.$$

□

Let α be, as in Lemma, parametrized by arc length, then α' and N are hortogonal versors at any point of $\alpha(t)$. Therefore we may associate a hortonormal moving frame

$$(\alpha', N, \alpha' \wedge N).$$

Since $|\alpha'| = 1$, then α' is hortogonal to α'' and

$$k_\alpha n_\alpha = \alpha'' = k_n N + k_g \alpha' \wedge N.$$

Since the basis is hortonormal we have

$$k_\alpha^2 = k_n^2 + k_g^2,$$

further note that $k_n(0) = \langle \alpha''(0) \cdot N(p)\rangle = II_p(\alpha'(0))$.

DEFINITION 3.1.8. k_n is called the **normal curvature**, k_g is called the **geodesic curvature**.

REMARK 3.1.9. A **geodesic** is a curve with zero geodesic curvature, that is a curve whose normal is parallel to the normal of the surface at any point. We are not going to explore it, but geodesics are local minimum of distance, that is the curve of minimal distance between two points in a neighborhood of a surface.

In this way it is easy to derive Meusnier Theorem

COROLLARY 3.1.10 (Meusnier Theorem). *Let $\alpha(t)$ be a regular curve on a surface S, with $\alpha(0) = p$ and $\alpha'(0) = v$. Then the normal curvature $k_n(0)$ depends only on v.*

PROOF. In our notation we have $k_n(0) = k_\alpha(0)\langle n_\alpha(0)\cdot N(p)\rangle = II_p(v)$ □

Let us go a bit further.

LEMMA 3.1.11. *Let $p \in S$ be a point, T_pS the tangent space and $H \ni p$ a plane. If $T_pH \neq T_pS$, then $C := H \cap S$ is a submanifold in a neighborhood of p and $T_pC = T_pH \cap T_pS$. In particular if H is parallel to $N(p)$ the resulting manifold $C := H \cap S$ is called a* **normal section**.

PROOF. We may assume that $H = (z = 0) \ni p = (0, 0, 0)$. Let $h : \mathbb{R}^3 \to \mathbb{R}$ be the projection on the z coordinate. By hypothesis we have $T_xS \not\subset H$. Let $i : S \to \mathbb{R}^3$ be the inclusion map, then $h \circ i : S \to \mathbb{R}$ is a differentiable map of constant rank 1 in a neighborhood of p. Therefore $(h \circ i)^{-1}(0) = H \cap S$ is a 1-manifold in a neighborhood of p and T_pC is the kernel of $D(h \circ i)_p$ that is $T_pC = (z = 0 \cap T_pS)$. □

Let $C = H \cap S$ be a normal section at p. Then $T_pC = T_pH \cap T_pS$ and we may choose a local parametrization by arc length, $\alpha(t)$, with $\alpha(0) = p$ and $n_\alpha(0) = N(p)$. This yields

$$k_n = k_\alpha = II_p(\alpha'(0)).$$

In particular all normal curvature, i.e. the second fundamental form, are encoded in normal sections.

Let $S^1 \subset T_pS$ be the set of versors, and $k_n : S^1 \to \mathbb{R}^1$ the map given by

$$k_n(v) = II_p(v).$$

Since S^1 is compact there is a maximum, say k_1, for $k_n(S^1)$. Let v_1 be such that $k_n(v_1) = k_1$, and v_2 an orthogonal versor. Then by Remark 3.1.5 v_1 is an eigenvector and we may diagonalize DN_p on the basis (v_1, v_2). On the orthonormal basis (v_1, v_2) the matrix of DN_p is given by

$$\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix},$$

that is for $w = av_1 + bv_2 \in T_pS$ we have

$$II_p(w) = a^2k_1 + b^2k_2.$$

When we restrict to versors $v \in S^1$ there is a θ such that $v = \cos\theta v_1 + \sin\theta v_2$. Therefore we have the **Euler formula**

$$-k_n(v) = \cos\theta^2 k_1 + \sin\theta^2 k_2, \tag{13}$$

and k_1, k_2 are maximum and minimum of normal curvatures.

DEFINITION 3.1.12. When $k_1 \neq k_2$ the eigenvalues k_1 and k_2 of DN_p are called **principal curvatures** and the eigenversors v_1, v_2 are called **principal directions**. The leaf of a principal direction distribution is called **line of curvature**.

A 2x2 matrix has not many invariants.

DEFINITION 3.1.13. The **Gaussian** curvature at the point p is $K(p) := k_1 k_2 = \det DN_p$, the **mean curvature** is $H(p) = \frac{k_1+k_2}{2} = \frac{Trace(DN_p)}{2}$.

REMARK 3.1.14. By the examples we already worked out we have:
3.1.0.3. *Sphere.* A radius r sphere has $k_1 = k_2 = 1/r$, therefore $K(p) = 1/r^2$ and $H(p) = 1/r$,
3.1.0.4. *Cylinder.* A radius r cylinder has $k_1 = 1/r$, $k_2 = 0$, therefore $K(p) = 0$ and $H(p) = \frac{1}{2r}$,
3.1.0.5. *Plane.* A plane has $k_1 = k_2 = 0$ therefore $K(p) = H(p) = 0$.

As we will see in a while the sign and vanishing of $K(p)$ has a geometric meaning.

DEFINITION 3.1.15. Let $p \in S$ be a point. We say that p is
elliptic if $K(p) > 0$
hyperbolic if $K(p) < 0$
parabolic if $K(p) = 0$
umbilical if $k_1 = k_2$
planar if $DN_p \equiv 0$.

The Gaussian curvature encodes both global and local geometric properties of the surface.

PROPOSITION 3.1.16. *Let $p \in l \subset S$ be a smooth point on a line l, then $K(p) \leq 0$.*

PROOF. Let H be a plane containing l and normal to S at p, then

$$0 = k_l = k_n(v),$$

where $v = T_p l$. Therefore p cannot be elliptic. □

Umbilical points can be easily found as follows

LEMMA 3.1.17. *Let $S \subset \mathbb{R}^3$ be a surface then the set of umbilical points is given by the equation $H^2 - K = 0$, for H and K the mean and Gaussian curvature.*

PROOF. The equation $H^2 - K = 0$ translates, in terms of principal curvatures, as

$$(k_1 + k_2)^2 - 4k_1k_2 = (k_1 - k_2)^2 = 0.$$

Hence it defines the set of points where $k_1 = k_2$. □

Note that by Equation (13) at hyperbolic points there are exactly two directions $u_1, u_2 \in T_pS$ such that $k_n(u_i) = 0$. Moreover $K(p) < 0$ then there is a neighborhood U_p of hyperbolic points. This allows to define the asymptotic curves.

DEFINITION 3.1.18. Let $p \in S$ be a hyperbolic point and u_1, u_2 such that $k_n(u_i) = 0$. Then v_i are called **asymptotic directions**. The leaf of the distribution of an asymptotic direction is called **asymptotic line**.

REMARK 3.1.19. Note that thanks to Frobenius Theorem we know that both lines of curvature (for non umbilical points) and asymptotic lines (for hyperbolic points) exist. These pairs of vector fields are always linearly independents and may be used to define a local parametrization of the surface S. Note that lines of curvature are mutually orthogonal, this is in general not the case for asymptotic lines.

The first global result is the following.

PROPOSITION 3.1.20. *Let $S \subset \mathbb{R}^3$ be a connected surface all of whose points are umbilical then S is contained in either a sphere or a plane.*

PROOF. Since connected manifolds are also path-connected it is enough to prove the statement on a neighborhood of any point. Let $U_p \subset S$ be a local chart with coordinates $\mathbf{x}(u, v)$. Then for any $v = a_1\mathbf{x}_u + a_2\mathbf{x}_v \in T_qS$ we have

$$DN_q(v) = \lambda(q)v,$$

for some smooth map $\lambda : U_p \to \mathbb{R}$, that is

$$N_u a_1 + N_v a_2 = \lambda(a_1\mathbf{x}_u + a_2\mathbf{x}_v).$$

Hence $N_u = \lambda\mathbf{x}_u$ and $N_v = \lambda\mathbf{x}_v$ and differentiating with mixed derivatives we get

$$\lambda_u\mathbf{x}_v - \lambda_v\mathbf{x}_u = 0.$$

The latter forces $\lambda_u = \lambda_v = 0$ and λ is therefore constant. If $\lambda \equiv 0$ then N is constant and it is easy to see, by derivation, that

$$\langle \mathbf{x}(u, v) \cdot N\rangle = \text{constant},$$

hence U_p is contained in the plane $p + N(p)^{\perp}$.

To conclude let $\lambda \neq 0$ then, again by derivation,

$$\mathbf{x}(u, v) - \frac{1}{\lambda}N = \text{constant}.$$

Then U_p is contained in the sphere of radius $1/|\lambda|$ centered in $p - \frac{1}{|\lambda|}N$. □

PROPOSITION 3.1.21. *Let $S \subset \mathbb{R}^3$ be a compact connected orientable surface with $K(p) \neq 0$ for any $p \in S$. Then S is diffeomorphic to the sphere.*

PROOF. By hypothesis the Gauss map $N : S \to S^2$ is well defined and since $K(p) \neq 0$ it is a local diffeomorphism. To conclude we need to prove that it is bijective. We already observed that N is an open map therefore $N(S)$ is open in S^2 and since S is compact $N(S)$ is also closed. This shows that N is surjective. Then N is a covering and since S^2 is simply connected $\sharp N^{-1}(x) = 1$ for any $x \in S^2$. □

Next we show that a compact surface always possesses elliptic points.

PROPOSITION 3.1.22. *Let $S \subset \mathbb{R}^3$ be a compact surface. Then there is $p \in S$ with $K(p) > 0$. In particular in Proposition 3.1.21 we have $K(p) > 0$ for any $p \in S$. In particular there are not smooth compact surfaces of negative curvature at any point.*

PROOF. Let $x \in S$ such that $||x|| \geq ||p||$ for any $p \in S$. Then the norm function $f(x) = ||x||$ has a maximum at x, therefore $T_xS = \langle x\rangle^{\perp}$. Then any normal section is a plane curve C with x maximum for the norm function. This shows that k_n has a fixed sign and therefore p is elliptic. □

REMARK 3.1.23. In Proposition 3.1.22 the compactness assumption is needed, think for instance to a plane. It is far more complicate, but possible, to produce examples of non smooth compact surfaces for which all smooth points have negative curvature. A less sophisticated example is given by smooth non compact surfaces of constant negative curvature, see Exercise 3.3.11.

3.1.1. Local equations of DN_p. Let now $\mathbf{x} := \varphi_i^{-1} : \mathbb{R}^2 \to U_p \subset S$ be a coordinate chart, with $\langle \mathbf{x}_u, \mathbf{x}_v \rangle = T_{\mathbf{x}(u,v)}S$, then

$$N(u,v) = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{||\mathbf{x}_u \wedge \mathbf{x}_v||}.$$

Recall that by Lemma 3.1.2 $N_u = DN(\mathbf{x}_u)$ and $N_v = DN(\mathbf{x}_v)$. Let $\alpha : (-\epsilon, \epsilon) \to \mathbb{R}^2$ be a curve with $\alpha(0) = p$, then we may consider $\beta = \mathbf{x} \circ \alpha$ to get $\beta(t) = \mathbf{x}(u(t), v(t))$. In this notations

$$II(\beta') = -\langle DN(\beta')\cdot\beta' \rangle = -(u')^2\langle N_u\cdot\mathbf{x}_u \rangle - 2u'v'\langle N_u\cdot\mathbf{x}_v \rangle - (v')^2\langle N_v\cdot\mathbf{x}_v \rangle \tag{14}$$

Note that deriving $\langle N(u,v)\cdot\mathbf{x}_u(u,v) \rangle = 0$ and $\langle N(u,v)\cdot\mathbf{x}_v(u,v) \rangle = 0$ we get

$$\langle N_u\cdot\mathbf{x}_u \rangle = -\langle N\cdot\mathbf{x}_{uu} \rangle,\ \langle N_u\cdot\mathbf{x}_v \rangle = \langle N_v\cdot\mathbf{x}_u \rangle = -\langle N\cdot\mathbf{x}_{uv} \rangle,\ \langle N_v\cdot\mathbf{x}_v \rangle = -\langle N\cdot\mathbf{x}_{vv} \rangle.$$

Therefore Equation (14) takes the form

$$II(\beta') = (u')^2\langle N\cdot\mathbf{x}_{uu} \rangle + 2u'v'\langle N\cdot\mathbf{x}_{uv} \rangle + (v')^2\langle N\cdot\mathbf{x}_{vv} \rangle \tag{15}$$

Let us now briefly recall the first fundamental form of a surface. Let $S \subset \mathbb{R}^3$ be a surface and $p \in S$ a point. Then the ordinary scalar product of $\mathbb{R}^3$ restricts to a scalar product on T_pS, and as usual, we may use it to determine lenghts and areas via integration. This defines the **first fundamental form** of S applied to the vector β' simply as $I(\beta') = \langle \beta'\cdot\beta' \rangle$ and therefore on the local base $\mathbf{x}_u$, $\mathbf{x}_v$ it reads

$$I(\beta') = (u')^2\langle \mathbf{x}_u\cdot\mathbf{x}_u \rangle + 2u'v'\langle \mathbf{x}_u\cdot\mathbf{x}_v \rangle + (v')^2\langle \mathbf{x}_v\cdot\mathbf{x}_v \rangle.$$

Classically, Gauss notation, all these have the following names.

DEFINITION 3.1.24.
$E = \langle \mathbf{x}_u\cdot\mathbf{x}_u \rangle$, $F = \langle \mathbf{x}_u\cdot\mathbf{x}_v \rangle$, $G = \langle \mathbf{x}_v\cdot\mathbf{x}_v \rangle$
$e = -\langle N_u\cdot\mathbf{x}_u \rangle = \langle N\cdot\mathbf{x}_{uu} \rangle$, $f = -\langle N_u\cdot\mathbf{x}_v \rangle = \langle N\cdot\mathbf{x}_{uv} \rangle$, $g = -\langle N_v\cdot\mathbf{x}_v \rangle = \langle N\cdot\mathbf{x}_{vv} \rangle$.

A direct computation furnishes the so called **Weingarten equations**. We have

$$\begin{bmatrix} N_u \\ N_v \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^t \begin{bmatrix} \mathbf{x}_u \\ \mathbf{x}_v \end{bmatrix},$$

for (a_{ij}) the matrix representing DN with respect to the basis $(\mathbf{x}_u, \mathbf{x}_v)$. Taking the scalar product yields

$$-\begin{bmatrix} e & f \\ f & g \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^t \begin{bmatrix} E & F \\ F & G \end{bmatrix},$$

that is

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^t = -\begin{bmatrix} e & f \\ f & g \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1}.$$

Finally we derive the equations

$$a_{11} = \frac{fF - eG}{EG - F^2},\ a_{12} = \frac{gF - fG}{EG - F^2}$$

$$a_{21} = \frac{eF - fE}{EG - F^2},\ a_{22} = \frac{fF - gE}{EG - F^2}$$

and also the expressions of Gaussian and mean curvature

$$K = \frac{eg - f^2}{EG - F^2},\ H = -\frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2}.$$

Recall that $H^2 - K = \frac{(k_1 - k_2)^2}{4}$ therefore we also have the expression for the principal curvatures

$$k_{1,2} = H \pm \sqrt{H^2 - K}.$$

Local expressions are useful do study the local behaviour of surfaces.

PROPOSITION 3.1.25. *Let $p \in S$ be an elliptic point. Then there is a neighborhood $U_p \subset S$ such that $U_p \cap (p + T_pS) = \{p\}$, that is the surface is, locally, on one side of the tangent space.*

Let $p \in S$ be a hyperbolic point then for any neighborhood U_p the surface is on both side of the plane $p + T_pS$.

PROOF. Let $\mathbf{x}(u, v)$ be a parametrization with $p = (0, 0, 0) = \mathbf{x}(0, 0)$ and $T_pS = (z = 0) \subset \mathbb{R}^3$ with coordinate (x, y, z). Fix $N(0) = (0, 0, 1)$, then the behaviour of the point, with respect to the tangent space, is dictated by the z coordinate.

By Taylor's formula we have

$$\mathbf{x}(u, v) = \mathbf{x}_u(0, 0)u + \mathbf{x}_v(0, 0)v + \frac{1}{2}(\mathbf{x}_{uu}(0, 0)u^2 + 2\mathbf{x}_{uv}(0, 0)uv + \mathbf{x}_{vv}(0, 0)v^2) + o(2),$$

thus using Equation (15) we find

$$\langle \mathbf{x}(u, v) \cdot N(0, 0) \rangle = \frac{1}{2} II_{(0,0)}(\mathbf{x}_u(0, 0)u + \mathbf{x}_v(0, 0)v) + o(2).$$

In particular the sign of the z coordinate depends only on the sign of $II_{(0,0)}$. So for an elliptic point the sign is constant and never vanishes in a neighborhood, while for a hyperbolic point it changes. $\square$

REMARK 3.1.26. Note that for neither parabolic nor planar points there is anything like this. The cylinder has all points on one side of the tangent space. Plane has all points on the tangent space. While for "monkey saddle"

$$(u, v) \mapsto (u, v, u^3 - 3v^2u),$$

$(0, 0)$ is a planar point and points are on both sides. Similar examples for parabolic points can be described with revolution surfaces.

3.2. Ruled surfaces

In this section we are interested in surfaces covered by lines.

DEFINITION 3.2.1. A one parameter family of lines is a pair of smooth maps $\alpha : I \to \mathbb{R}^3$ and $\tau : I \to \mathbb{R}^3$ together with the map

$$\mathbf{x} : I \times J \to \mathbb{R}^3$$

given by

$$(u, v) \mapsto \alpha(u) + v\tau(u).$$

Assume that $0 \in J$, $|\tau(u)| = 1$, for any $u \in I$ then the image $S := \mathbf{x}(I \times J)$ is called a **ruled surface**. The (portion of) lines $\mathbf{x}_{|\{v\}\times J}$ are called the **rulings** of S while $\mathbf{x}_{|I\times\{0\}}$ is called a **directrix** of S. The surface S is said to be ruled by the map $\mathbf{x}$.

REMARK 3.2.2. The simplest examples of ruled surfaces are:

- (portion of) plane, with α the constant map in a point $p \in H$, and τ any curve in H
- cone, again α constant in a point p and τ any curve,
- cylinder α any and τ constant.

Note that we do not ask S to be smooth and in general it is not. On the other hand if we assume $\tau(u)' \neq 0$ then we may assume that $|\tau(u)| = 1$. Hence we have $\langle \tau(u) \cdot \tau'(u) \rangle = 0$. To check the singularities let us compute $\mathbf{x}_u = \alpha' + v\tau'$ and $\mathbf{x}_v = \tau$. The singular locus is given by points where $\mathbf{x}_u \wedge \mathbf{x}_v = 0$ and, where smooth, the tangent space is

$$T_{\mathbf{x}(u,v)} S = \langle \alpha' + v\tau', \tau \rangle.$$

The directrix of S is clearly non unique, but there is a special ones that contains all singular points of S. Let $\gamma_\lambda(u) = \alpha(u) + \lambda(u)\tau(u)$ be a directrix. We want to choose λ in such a way that γ' is orthogonal to τ' for any u. By definition we have

$$\gamma' = \alpha' + \lambda'\tau + \lambda\tau',$$

since $\langle \tau(u) \cdot \tau'(u) \rangle = 0$ it is enough to choose

$$\lambda(u) = -\frac{\langle \alpha(u)' \cdot \tau'(u) \rangle}{\langle \tau(u)' \cdot \tau'(u) \rangle}$$

The **line of striction** is then

$$\beta(u) = \alpha(u) - \frac{\langle \alpha(u)' \cdot \tau'(u) \rangle}{\langle \tau(u)' \cdot \tau'(u) \rangle} \tau(u),$$

it is easy to check that β is independent from the directrix choosen. Next we want to prove that all singularities of S are in $\beta(I)$. To do this let us reparametrize the surface S with $\mathbf{x}(u, v) = \beta(u) + v\tau(u)$, then

$$\mathbf{x}_u = \beta' + v\tau', \quad \mathbf{x}_v = \tau$$

Then

$$\langle \mathbf{x}_u \cdot \tau' \rangle = v,$$

hence the singular points are contained in $v = 0$ that is the directrix β.

We already gave examples of smooth ruled surface of non cylindrical type, for instance the plane.

EXAMPLE 3.2.3. A more intersting example is the ruled quadric $S := \{xy = z\} \subset \mathbb{R}^3$. Note that the lines $l = (x = z = 0)$ and $r = (x = 1, y = z)$ are contained in S and do not intersect. Moreover any plane H containing l intersects r in a point and is such that $H \cap S = l \cup m_H$ for some line m_H. This shows that S is ruled by the lines m_H. As an exercise one can write down the parametrization explicitly.

Let $p \in S$ be a smooth point of a ruled surface. We already know by Proposition 3.1.16 that $K_p \leq 0$. Moreover by Exercise 3.3.4 the Gaussian curvature vanishes only if the tangent plane is constant along the line.

DEFINITION 3.2.4. A **developable surface** is a ruled surface with fixed tangent plane along the ruling, away from the line of striction.

In particular developable surfaces have zero Gaussian curvature.

EXAMPLE 3.2.5. Keeping in mind Remark 3.2.2 we may easily write down two examples of developable surfaces

3.2.0.1. *Cylinders.* $\tau(u) = v$ constant

3.2.0.2. *Cones.* $\alpha(u) = p$ constant

There is a third one which is a bit less immediate.

3.2.0.3. *Tangent developable.* Let $\alpha : I \to \mathbb{R}^3$ be a smooth curve parametrized by arc length with non vanishing curvature. Then the developable surface S associated to α is the ruled surface given by the parametrization $h : I \times \mathbb{R} \to \mathbb{R}^3$, with

$$h(u, v) = \alpha(u) + v\alpha'(u).$$

In particular S is smooth away from the directrix $\alpha(I)$. The points of $\alpha(I)$ can be either smooth or non smooth points of S. In particular along a ruling, away from the directrix, we have

$$T_{\mathbf{x}(\overline{u},v)}S = \langle \alpha'(\overline{u}), \alpha'(\overline{u}) + v\alpha''(\overline{u})\rangle = \langle \alpha'(\overline{u}), \alpha''(\overline{u})\rangle.$$

Hence the tangent space is constant along the ruling and $K \equiv 0$ away from the directrix.

We aim to prove the converse of this statement that is

THEOREM 3.2.6. *A surface S with zero Gaussian curvature and no planar points is a developable surface.*

REMARK 3.2.7. Developable surfaces are isometric to a plane, that is can be developed on a plane.

Even if it is not strictly necessary we take this theorem as an excuse to introduce a global point of view on the Gauss mapping using projective geometry. Our next task is therefore to develop the theory of projective spaces to give a proof of Theorem 3.2.6.

3.3. Exercises

EXERCISE 3.3.1. Compute the image of the Gauss map for the following surfaces:

- $S = \{x^2 + y^2 + z^2 = r^2\}$ (sphere)
- $S = \{x^2 + y^2 = r^2\}$ (cylinder)
- $S = \{z = x^2 - y^2\}$ (hyperbolic paraboloid)

EXERCISE 3.3.2. Let $S \subset \mathbb{R}^3$ be a surface, $p \in S$ a point, and $C \subset S$ a curve through p. Observe that $T_pC \subset T_pS$ and conclude that $\{(x, y, z) \subset \mathbb{R}^3 | x(x^2 + y^2 + z^2 - 1) = 0\}$ is not a submanifolds of $\mathbb{R}^3$

EXERCISE 3.3.3. Compute the differential of the Gauss map for the following surfaces:

- $S = \{z = x^2 - y^2\}$
- $S = \{z = ax^2 + by^2\}$
- $S = \{z = 0\}$

EXERCISE 3.3.4. Let $l \subset S$ be a line. Show that the points of l are parabolic or planar if and only if T_pS is constant in the direction of l.

EXERCISE 3.3.5. Let S be a surface of revolution of a curve α parametrized by arclenght, show that the circles of revolution and the curves α are lines of curvature.

EXERCISE 3.3.6. Let S be the surface of revolution around the z axis, of the curve $\alpha(t) = (x(t), 0, z(t))$, assume that α is parametrized by the arclength. Show that $K = -\frac{x''}{x}$ and prove that circles of revolution and the curves α are lines of curvature.

EXERCISE 3.3.7. Let S be the helycoid given by the following parametrization

$$\mathbf{x}(u, v) = (u \cos v, u \sin v, av),$$

for some $a \neq 0$. Determine K_p for any $p \in S$.

EXERCISE 3.3.8. Let $S = \{z = xy^2\} \subset \mathbb{R}^3$ show that S is a submanifold and $(0, 0, 0)$ is a planar point.

EXERCISE 3.3.9. Let $S = \{xyz = 1\} \subset \mathbb{R}^3$ show that S is a submanifold and determine the type of the points $p \in S$.

EXERCISE 3.3.10. Let $q \in k[x, y, z]$ be a polynomial of degree 2 and $S = \{q(x, y, z) = 0\} \subset \mathbb{R}^3$. Prove that if S is a submanifold $K(p)K(q) \geq 0$ for any pair of points $p, q \in S$.

EXERCISE 3.3.11. Lets now investigate a very interesting surface, called the pseudosphere. It is the surface of revolution obtained by rotating the tractrix about the x-axis, and so it is parametrized by

$$\mathbf{x}(u, v) = (u - tanh(u), sech(u) \cos v, sech(u) \sin v),$$

for $u > 0, v \in [0, 2\pi)$. Note that the circles (of revolution) are lines of curvature and the various tractrices are lines of curvature. In the plane of one tractrix, say t the surface normal and the curve normal agree. Prove that the curvature of the tractrix is $\frac{1}{sinh(u)}$ and $N(p) = -n_t$ therefore $k_1 = -\frac{1}{sinh(u)}$ Prove that the normal curvature of the circle is $sinhu$ (hint: to do this observe that $k_n = k \cos\theta = cosh(u)tanh(u) = sinh(u)$)

With a different approach either observe or recall that

$$(e^{-s}, tanh^{-1}(\sqrt{1 - e^{-2s}}) - \sqrt{1 - e^{-2s}})$$

is the arc lenght parametrization of the tractrix on $\mathbb{R}^{>0}$ and conclude by the above exercise.

EXERCISE 3.3.12. Give examples of smooth developable surfaces and of singular developable surfaces.

EXERCISE 3.3.13. Show that the line of striction is unique.

CHAPTER 4

Projective geometry

Still to come

4.1. Developable surfaces 2

The aim of this section is to prove Theorem 3.2.6 and give a classification of developable surfaces as an exercise. For this reason we will mix projective and differential geometry. Indeed the proof of this theorem without projective geometry is quite subtle since from the local point of view of differential geometry it is not easy to detect lines. On the other hand looking from the projective point of view we may assoiate to a tangent plane its projectivization and consider it as a line in $\mathbb{P}^2$ and therefore a point in $(\mathbb{P}^2)^*$.

To take avdantage from this let $(\mathbb{P}^2_{\mathbb{R}})^*$ be the dual projective space and $S \subset \mathbb{R}^3$ a smooth surface. We may consider the map

$$\gamma : S \to (\mathbb{P}^2_{\mathbb{R}})^*$$

mapping $p \in S$ to $[T_pS]$. Note that T_pS is a 2-dimensional vector space in $\mathbb{R}^3$ and therefore defines uniquely a projective line in $\mathbb{P}^2_{\mathbb{R}}$ hence a point in $(\mathbb{P}^2_{\mathbb{R}})^*$. Note that $\mathbb{P}^2_{\mathbb{R}}$ has a natural DS inherited by the local charts $\varphi_i : U_i \to \mathbb{R}^2$, where $U_i := \{[x_0, x_1, x_2] | x_i \neq 0\}$. Via this it is easy to see that γ is a differentiable map. The map γ is a different way to describe the Gauss map we introduced in § 3.1 for orientable surfaces

$$N : S \to S^2,\ p \mapsto N(p).$$

This time orientability is not required since we use the tangent space instead of the normal vector.

The next ingredient we need is the notion of radiality, that is a differentiable way to recognize lines.

Definition 4.1.1. A smooth map $f : J \to \mathbb{R}^n$ is **radial** if it exists a smooth map $g : J \to \mathbb{R}$ and a fixed vector $v \in \mathbb{R}^n$ such that

$$f(t) = g(t)v,$$

for any $t \in J$.

Remark 4.1.2. The image of a radial function is (a portion of) a line through the origin. Let $H : J \to \mathbb{R}^n$ be a parametrization of (a portion of) a line then H is in general not radial but its derivative is.

Clearly derivatives of radial functions are radial functions. If f is radial then $f' = g'v$ therefore if $g \neq 0$ on the interval J then

$$f' = \frac{g'}{g} f. \tag{16}$$

Next we want to prove that Equation (16) characterizes radial functions.

LEMMA 4.1.3. *Let $f : J \to \mathbb{R}^n$ and $\alpha : J \to \mathbb{R}$ be smooth functions. Assume that $f' = \alpha f$. Then f and any derivative of f are radial.*

PROOF. If $f \equiv 0$ there is nothing to prove. Then we may assume that $0 \in J$ and $f(0) = v \neq 0$. We have an ODE system

$$f' = \alpha f \tag{17}$$

with initial conditions $f(0) = v$.

Let $x' = \alpha x$ be the differential equation with initial value $x(0) = 1$. Then by Cauchy there is a unique solution, say $g : J \to \mathbb{R}$.

Then f and gv satisfy the ODE system (17) with the same initial condition. Therefore by uniqueness $f(t) = g(t)v$.

To conclude observe that for a radial function f we have $f' = g'v$, therefore $f'' = g''v$. □

We are ready to prove our main Theorem.

THEOREM 4.1.4. *Let $S \subset \mathbb{R}^3$ be a surface. Assume that $K(p) \equiv 0$ for any $p \in S$ and p is not planar. Then there exists an open $U_p \subset S$ such that U_p is ruled by the fibers of the Gauss map γ and it is developable.*

PROOF. Let us fix a local parametrization $\mathbf{x}(u, v) : B \to S$ with $\mathbf{x}(0, 0) = p$. Assume that p is a parabolic, not planar point. Then, up to shrinking S, we may assume that $D\gamma$ has constant rank 1. Therefore by the constant rank Theorem 1.4.2 there is a choice of coordinates with

a) $I \times J$, with $I = (-\epsilon, \epsilon)$ and $J = (-\delta, \delta)$, $\mathbf{x} : I \times J \to U_p \subset S$,
b) $\gamma(\mathbf{x}(u, v)) = \gamma(\mathbf{x}(u, 0)) = q_u$ for any $u \in I$.

Let

$$\tilde{\gamma} : I \times J \to \mathbb{R}^3 \setminus \{0\}$$

given by

$$\tilde{\gamma}(u, v) = \mathbf{x}_u \wedge \mathbf{x}_v.$$

Then $\tilde{\gamma}(u, v)$ is orthogonal to $T_{\mathbf{x}(u,v)}S$ and

$$\langle \tilde{\gamma} \cdot \mathbf{x}_u \rangle = \langle \tilde{\gamma} \cdot \mathbf{x}_v \rangle = 0, \tag{18}$$

for any $(u, v) \in I \times J$. Moreover by construction $\tilde{\gamma}(u, v) \in \langle \tilde{\gamma}(u, 0) \rangle$ for any $u \in I$. Therefore there is a smooth map $\lambda : I \times J \to \mathbb{R} \setminus \{0\}$ such that

$$\tilde{\gamma}(u, v) = \lambda(u, v)\tilde{\gamma}(u, 0),$$

in other words $\tilde{\gamma}$ is radial as a function of v, and there is a smooth function $\theta : J \to \mathbb{R}$ such that

$$\tilde{\gamma}_v = \theta \tilde{\gamma}. \tag{19}$$

The next step is to relate the radiality of $\tilde{\gamma}_v$ with the existence of (part of) lines in $\mathbf{x}(I \times J)$.

For this observe that by Lemma 4.1.3, $\mathbf{x}(\{u_0\} \times J)$ is a (part of a) line if and only if $\mathbf{x}_v$ is radial. That is $\mathbf{x}_{vv} = \mu \mathbf{x}_v$, for some function μ.

The Gauss map is a second order operator therefore it is perfectly suited to study $\mathbf{x}_{vv}$.

CLAIM 4.1.5. For any $(u, v) \in I \times J$ we have $\langle \mathbf{x}_v \cdot \tilde{\gamma}_u \rangle = 0$

PROOF. By definition $\langle\tilde{\gamma}\cdot\mathbf{x}_u\rangle = 0$ therefore taking derivative with respect to v, Equation (19) we get

$$0 = \langle\tilde{\gamma}_v\cdot\mathbf{x}_u\rangle + \langle\tilde{\gamma}\cdot\mathbf{x}_{uv}\rangle = \langle\theta\tilde{\gamma}\cdot\mathbf{x}_u\rangle + \langle\tilde{\gamma}\cdot\mathbf{x}_{uv}\rangle = \langle\tilde{\gamma}\cdot\mathbf{x}_{uv}\rangle, \tag{20}$$

where the latter equality is again the fact that $\tilde{\gamma}$ is orthogonal to the tangent space. Then by Equation (20) we conclude that

$$0 = \langle\tilde{\gamma}\cdot\mathbf{x}_v\rangle_u = \langle\tilde{\gamma}_u\cdot\mathbf{x}_v\rangle.$$

□

To conclude we prove that $\mathbf{x}_v$ is radial. For this it is enough to prove that there is a smooth function μ such that

$$\mathbf{x}_{vv} = \mu\mathbf{x}_v.$$

Note that $\tilde{\gamma}$ and $\tilde{\gamma}_u$ are linearly independent since there are not planar points. Moreover by definition and the Claim $\mathbf{x}_v \neq 0$ and $\mathbf{x}_v \in \langle\tilde{\gamma}, \tilde{\gamma}_u\rangle^{\perp}$. Hence to conclude it is enough to prove that $\mathbf{x}_{vv} \in \langle\tilde{\gamma}, \tilde{\gamma}_u\rangle^{\perp}$.

Arguing as in the Claim we take derivatives

$$0 = \langle\tilde{\gamma}\cdot\mathbf{x}_v\rangle_v = \langle\tilde{\gamma}\cdot\mathbf{x}_{vv}\rangle$$

$$0 = \langle\tilde{\gamma}_u\cdot\mathbf{x}_v\rangle_v = \langle\tilde{\gamma}_{uv}\cdot\mathbf{x}_v\rangle + \langle\tilde{\gamma}_u\cdot\mathbf{x}_{vv}\rangle.$$

To conclude observe that

$$\langle\tilde{\gamma}_{uv}\cdot\mathbf{x}_v\rangle = \langle\theta_u\tilde{\gamma} + \theta\tilde{\gamma}_u\cdot\mathbf{x}_v\rangle = 0$$

therefore

$$\langle\tilde{\gamma}_{uv}\cdot\mathbf{x}_v\rangle = 0.$$

□

REMARK 4.1.6. Note that despite the use of dual projective plane seems quite limited it is crucial to produce lines on the surface via the lift of the Gauss map.

We proved the result assuming that S does not contain any planar point. The assumption is necessary since one could glue smoothly a portion of a plane to a portion of a cone/cylinder to get smooth surfaces of constant Gaussian curvature that do not have the required local behaviour in a neighborhood of the junction points.

For this let $k : (-\epsilon, \epsilon) \to \mathbb{R} \setminus \{0\}$ and $\tau : (-\epsilon, \epsilon) \to \mathbb{R}$ be smooth functions with

$$\tau((-\epsilon, 0]) = 0, \ \tau(0, \epsilon) \subset \mathbb{R} \setminus \{0\}.$$

The by the classification theorem of differentiable curves there is a curve α that has curvature k and torsion τ (if you never heard about it, it is just the Cauchy existence and uniqueness plus the fact that Frenet ODE system has anti-symmetric coefficient matrix). Since the torsion vanishes on $(-\epsilon, 0]$ there is a plane H such that $\alpha((-\epsilon, 0]) \subset H$. For the same reason $\alpha((0, \epsilon))$ is not contained in any plane. Let $p \in H$ be a point. Then the cone with vertex p over $\alpha(-\epsilon, \epsilon)$ is the example we are looking at. The points on the line $\langle\alpha(0), p\rangle$ are smooth planar points and in any neighborhood we have both planar and parabolic non planar points. Locally the Gauss map has a single two dimensional fiber, the plane H, and the rulings of the cone outside H. In particular the neighborhoods of these points are not ruled by fibers of the Gauss map.

4.2. Exercise

EXERCISE 4.2.1. Prove that any developable surface with no planar points is one of the three types described by Example 3.2.5.

CHAPTER 5

Exams

Here is the list of 18+6 questions you have to know. In any written exam you have to answer to 3 in the first group and 1 in the second.

First group

(1) Definition of differentiable structure and differentiable m-manifold
(2) Statement of rank Theorem
(3) Definition of n-submanifold and m-manifold
(4) Definition of embedding
(5) Definition of derivation and tangent space at a point p of a manifold M
(6) Definition of differential of a map $\varphi : M \to N$ between manifolds
(7) Definition of vector field
(8) Definition of flow of a one parameter vector field.
(9) Definition of distribution
(10) Statement of Frobenius Theorem
(11) Definition of fibration and transition functions
(12) Definition of vector bundle
(13) Definition of vector bundle morphism
(14) Definition of normal bundle of a submanifold $X \subset M$
(15) Definition of self-adjoint linear endomorphism
(16) Definition of normal curvature and statement of Meusnier Theorem
(17) Definition of principal curvatures, principal directions, Gaussian curvature and mean curvature
(18) Definition of ruled surface and developable surface.

Second group

(1) Prove that if $F : N \to M$ is a map of constant rank k, and $q \in F(N)$ is a point, then $F^{-1}(q)$ is a $(n-k)$-submanifold.
(2) State and prove the chain rule for maps between manifolds.
(3) Prove that DN_p is self-adjoint.
(4) Let $\alpha : J \to S$ be a curve prove, that $II_p(\alpha') = k_\alpha \langle n_\alpha \cdot N \rangle$, and conclude Meusnier Theorem.
(5) Prove that if $p \in l \subset S$ is a point on a line in a smooth surface S then $K(p) \leq 0$.
(6) Prove that tangent developable are developable surfaces.

Versione italiana

Primo gruppo

(1) Definizione di struttura differenziabile (DS) e manifold.
(2) Enunciato del teorema del rango costante.
(3) Definizione di manifold e submanifold.
(4) Definizione di embedding.
(5) Definizione di derivazione e di spazio tangente in un punto ad un manifold.
(6) Definizione di differenziale di un morfismo $\varphi : M \to N$ tra manifolds.
(7) Definizione di campo di vettori.
(8) Definizione di flusso per un campo di vettori ad un paramtro.
(9) Definizione di distribuzione.
(10) Enunciato del Teorema di Frobenius e delle nozioni di integrabilità e involutività.
(11) Definizione di fibrazione e di funzioni di transizione.
(12) Definizione di fibrato vettoriale.
(13) Definizione di morfismo tra fibrati vettoriali.
(14) Definizione di fibrato normale di un submanifold $X \subset M$.
(15) Definizione di operatore lineare autoaggiunto.
(16) Definizione di curvatura normale e enunciato del Teorema di Meusnier.
(17) Definizione di curvature principali, direzioni principali, curvatura Gaussiana e curvatura media.
(18) Definizione di superficie rigata e superficie sviluppabile.

Secondo gruppo

(1) Si mostri che se $F : N \to M$ è un morfismo di rango costante k e $q \in F(N)$ è un punto allora $F^{-1}(q)$ è un $(n-k)$-submanifold.
(2) Enunciare e dimostrare la regola del differenziale di funzioni composte.
(3) Si mostri che DN_p è auto aggiunto.
(4) Sia $\alpha : J \to S$ una curva. Si mostri che $II_p(\alpha') = k_\alpha \langle n_\alpha \cdot N \rangle$, e si dimostri il Teorema di Musnier.
(5) Si mostri che se $p \in l \subset S \subset \mathbb{R}^3$ è un punto su una retta allora $K(p) \leq 0$.
(6) Si mostri che la sviluppabile delle tangenti è una superficie sviluppabile.

Bibliography

[1] Arnold, Vladimir I. *Ordinary differential equations.* Translated from the Russian by Roger Cooke. Second printing of the 1992 edition. Universitext. Springer-Verlag, Berlin, 2006. ii+334 pp. ISBN: 978-3-540-34563-3; 3-540-34563-9

[2] Boothby, William M. *An introduction to differentiable manifolds and Riemannian geometry.* Second edition. Pure and Applied Mathematics, 120. Academic Press, Inc., Orlando, FL, 1986. xvi+430 pp. ISBN: 0-12-116052-1; 0-12-116053-X

[3] Husemoller, Dale *Fibre bundles.* Third edition. Graduate Texts in Mathematics, 20. Springer-Verlag, New York, 1994. xx+353 pp. ISBN: 0-387-94087-1

[4] Milnor, John *Lectures on the h-cobordism theorem.* Notes by L. Siebenmann and J. Sondow Princeton University Press, Princeton, N.J. 1965 v+116 pp.

[5] Warner, Frank W. *Foundations of differentiable manifolds and Lie groups* Scott, Foresman and Co., Glenview, Ill.-London, 1971. viii+270 pp.